Conformal asymptotic analysis

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2.1.1 Minkowski spacetime

Minkowski space $\mathbb{M}$ is $\mathbb{R}^4$ endowed with the Minkowski metric, whose expression in Cartesian coordinates is given by (the speed of light being taken equal to 1)

$$\eta = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \,. \tag{2.1}$$

Another useful expression of the metric η is in terms of spherical coordinates. It is particularly useful in order to perform an explicit conformal compactification. Is it a straightforward calculation to show that

$$\eta = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\omega^2 \,,\ \mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \,, \tag{2.2}$$

where the spherical coordinates (r, θ, φ) are related to (x, y, z) by

$$x = r\sin\theta\cos\varphi \,,\ y = r\sin\theta\sin\varphi \,,\ z = r\sin\theta \,.$$

The metric $\mathrm{d}\omega^2$ defined in (2.2) is the euclidian metric on the 2-sphere.

The Minkowski metric acts on vectors at a point or vector fields on $\mathbb{M}$ as follows

$$V = V^0\frac{\partial}{\partial t} + V^1\frac{\partial}{\partial x} + V^2\frac{\partial}{\partial y} + V^3\frac{\partial}{\partial z} \,,\ W = W^0\frac{\partial}{\partial t} + W^1\frac{\partial}{\partial x} + W^2\frac{\partial}{\partial y} + W^3\frac{\partial}{\partial z} \,,$$
$$\eta(V, W) = \eta_{ab}V^aW^b = V^0W^0 - V^1W^1 - V^2W^2 - V^3W^3 \,,$$
$$\eta(V, V) = (V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 \,. \tag{2.3}$$

Remark 2.1. *Note that the tangent space to $\mathbb{M}$ at a given point p is $\mathbb{R}^4$ endowed with the Minkowski metric, but as a vector space. Minkowski space has the structure of an affine space. The tangent space at any given point will be referred to as Minkowski vector space. We shall see in the next chapter that it is the model for the tangent space to any spacetime.*

We see that for each point $p \in \mathbb{M}$, (2.3) distinguishes three disjoint classes of tangent vectors.

Definition 2.1. *Let $p \in \mathbb{M}$, a vector $V \in T_p\mathbb{M}$ is said to be*

- *spacelike if $\eta(V,V) < 0$ (the projection of V on the space directions is longer than its time component),*
- *null if $\eta(V,V) = 0$ (the time and space parts of the vector are of equal length),*
- *timelike if $\eta(V,V) > 0$ (the time part of the vector is longer than its space part),*
- *causal (or also non-spacelike) if $\eta(V,V) \geq 0$.*

Remark 2.2. *This gives us a local classification of curves (at least differentiable) as timelike, spacelike or null according to the classification of their tangent vector at a point.*

Remark 2.3. *Let us consider on $\mathbb{M}$ the trajectory of a particle whose "experience" of time is described by the variable t. This is a curve $\gamma(t) = (t, x(t), y(t), z(t))$. Its tangent vector is*

$$\dot\gamma(t) = \frac{\partial}{\partial t} + \dot x(t)\frac{\partial}{\partial x} + \dot y(t)\frac{\partial}{\partial y} + \dot z(t)\frac{\partial}{\partial z}$$

and

$$\eta(\dot\gamma(t), \dot\gamma(t)) = 1 - \dot x(t)^2 - \dot y(t)^2 - \dot z(t)^2\,.$$

In the framework of classical mechanics, the vector

$$V(t) = \dot x(t)\frac{\partial}{\partial x} + \dot y(t)\frac{\partial}{\partial y} + \dot z(t)\frac{\partial}{\partial z}$$

is understood as describing the speed of the particle at time t. At a given time t, we know that the particle goes faster than, slower than, or at the speed of light, depending whether $|V(t)|^2 = \dot x(t)^2 + \dot y(t)^2 + \dot z(t)^2 > 1$, $|V(t)|^2 < 1$ or $|V(t)|^2 = 1$. However there is nothing unique about the choice of time parameter t, it is relative to the observer. A change of time parameter t will change the value of the time component of $\dot\gamma$ and the length of the space part of the tangent vector will then need to be compared to some quantity other than 1 (in fact the length of the time part) to compare the speed of the particle with that of light. As a matter of fact, even the notion of time and space part is not well defined, many other choices are possible corresponding to different choices of coordinates.

In relativity, the notion that replaces that of speed vector is that of 4-velocity vector, it is $\dot\gamma(t)$, the tangent vector to the trajectory of the particle. This is still a non unique notion since its "length" changes with a change of parameter of the curve. Its direction however is an intrinsic notion. And this gives us an intrinsic way of comparing the speed of a particle with that of light : a particle at a given point moves faster than, slower than, or at the speed of light depending whether the tangent vector field to its trajectory at that point (measured for any choice of parameter that is not singular at that point) is spacelike, timelike or null.

A massive particle will move along a timelike curve, a massless particle will move along a null curve.

Definition 2.2. *Given $p \in \mathbb{M}$, the set of null vectors in $T_p\mathbb{M}$ is the cone*

$$C_p = \left\{ V = V^0 \frac{\partial}{\partial t} + V^1 \frac{\partial}{\partial x} + V^2 \frac{\partial}{\partial y} + V^3 \frac{\partial}{\partial z}\,;\ (V^0)^2 = (V^1)^2 + (V^2)^2 + (V^3)^2 \right\}.$$

It is called the lightcone at p.

There are some useful orthogonality properties between vectors in the spacelike, timelike and lightlike cases. They are worth writing and proving in details since the orthogonality for an indefinite symmetric 2-form is less intuitive than for a positive definite one. First, let us introduce some notations that will be used extensively in the following proofs. Let $U \in T_p\mathbb{M}$, we denote

$$U = U^0 \partial_t + U',$$

where U' is the projection of U on the spatial directions, i.e.

$$U' = U^1 \partial_x + U^2 \partial_y + U^3 \partial_z\,.$$

We shall also denote $|U'|$ the euclidian norm of U'

$$|U'|^2 = |U^1|^2 + |U^2|^2 + |U^3|^2\,.$$

Let $U, V \in T_p\mathbb{M}$, we denote by $\langle U', V' \rangle$ the euclidian inner product of U' and V' :

$$\langle U', V' \rangle = U^1V^1 + U^2V^2 + U^3V^3\,.$$

Proposition 2.1 (Orthogonal to a timelike vector)**.** *Let T be a timelike vector at a point p and $V \in T_p\mathbb{M}$ such that $\eta(T, V) = 0$, then V is spacelike or zero.*

Proof. We assume that $V \neq 0$. We know that T is timelike, i.e.

$$|T^0| > |T'|\,.$$

Moreover,

$$\eta(T, V) = T^0V^0 - \langle T', V' \rangle = 0\,.$$

This implies in particular that $V' \neq (0, 0, 0)$, otherwise the equality above would imply also that $V_0 = 0$ ad this would contradict $V \neq 0$. In addition, it follows that

$$|V^0| = \frac{\langle T', V' \rangle}{|T^0|} \leq \frac{|T'||V'|}{|T^0|} < |V'|\,.$$

This concludes the proof. □

Remark 2.4. *This means that the orthogonal in $T_p\mathbb{M}$ to a timelike vector at p for the metric η is a hyperplane in $T_p\mathbb{M}$ containing only spacelike vectors.*

A vector orthogonal to a spacelike vector is not necessarily timelike, a simple example is given by the vectors ∂_x and ∂_y, but if we restrict ourselves to a plane spanned by a timelike and a spacelike vector, then the result becomes true.

Proposition 2.2. *Consider at a point p in $\mathbb{M}$ a spacelike vector V and a timelike vector T. Let W be a vector in the plane spanned by T and V and that is orthogonal to V, i.e. $\eta(W,V) = 0$, then W is timelike or zero.*

Proof. The restriction of η to the plane spanned by T and V is a quadratic form whose matrix in the basis $\{T, V\}$

$$A := \begin{pmatrix} \eta(T,T) & \eta(T,V) \\ \eta(T,V) & \eta(V,V) \end{pmatrix}$$

is real symmetric and has negative determinant

$$\det A = \eta(T,T)\eta(V,V) - \eta(T,V)^2 \, .$$

Hence A has one positive and one negative eigenvalue. In the basis $\{V \, , \, W\}$ (assuming of course $W \neq 0$), the matrix of the quadratic form is diagonal since $\eta(V,W) = 0$. Since $\eta(V,V) < 0$ and the determinant of the matrix must still be strictly negative, it follows that $\eta(W,W) > 0$, i.e. W is timelike. □

When looking at the space of vectors orthogonal to a null vector field, the situation gets more unusual.

Proposition 2.3. *Let V be a non-zero null vector at a point p in $\mathbb{M}$. The subspace of $T_p\mathbb{M}$ of vectors orthogonal to V contains V ; except for the straight line generated by V, it is entirely composed of spacelike vectors ; it is the hyperplane tangent to the light-cone containing V.*

Proof. The fact that V is orthogonal to itself is trivial since V is assumed to be null. The vector V can be decomposed as follows

$$V = V^0 \partial_t + V' \, .$$

We can find two linearly independent vectors U and W in the hyperplane spanned by ∂_x, ∂_y, ∂_z which are orthogonal to V' for the euclidian inner product on $\mathbb{R}^3$. Then U, V, W are three linearly independent vectors orthogonal to V and which consequently span the hyperplane orthogonal to V. Moreover they are mutually orthogonal and since V is null and U and W are spacelike, it follows that any linear combination of the three is spacelike unless it is parallel to V. □

Definition 2.3. *Let S be a $\mathcal{C}^1$ hypersurface in $\mathbb{M}$. We say that S is :*

- *spacelike if its normal vector at each point is a timelike vector, this means that its tangent plane at each point is entirely composed of spacelike vectors ;*
- *null if its normal vector at each point is a null vector, this means that its tangent plane at each point is composed of spacelike vectors and one null direction given by the normal vector ;*
- *achronal or weakly spacelike if its normal vector at each point is a causal vector ;*
- *timelike if its normal vector at each point is a spacelike vector, this means that its tangent plane at each point is generated by one timelike and two spacelike vectors ;*

2.1.2 Spacetime, connection, curvature

Our framework will be a *spacetime* which we shall define as follows

Definition 2.4. *A space-time is Lorentzian manifold of dimension* 4 *(a pair* $(\mathcal{M}, g)$ *where* $\mathcal{M}$ *is a* 4*-dimensional manifold and* g *is a Lorentzian metric on* $\mathcal{M}$*, i.e. a symmetric* 2*-form on* $\mathcal{M}$ *of signature* $+ - --$*) that is orientable.*

Recall that a tensor bundle on $\mathcal{M}$ is a multiple tensor product of the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T^*\mathcal{M}$. We will need to differentiate tensor fields (sections of tensor bundles), for this we need a connection, we will use the Levi-Civita connection.

The Levi-Civita connection on Lorentzian manifold is defined exactly as in the Riemannian case, in fact the definition and uniqueness of the Levi-Civita connection are independent of the signature of the metric.

Definition 2.5. *A connection* ∇_a *is an extension of the differential to arbitrary tensor fields, such that :*

1. *it is linear from any tensor bundle* F *of given valence to* $T^*\mathcal{M} \otimes F$ *;*
2. *it satisfies the Leibniz rule.*

Theorem 2.1. *There exists a unique connection* ∇_a *such that :*

1. *it is torsion-free, meaning that* $[\nabla_a\, ,\ \nabla_b]\, f = 0$ *for any scalar field* f*, where* $[\nabla_a\, ,\ \nabla_b]$ *is the commutator of* ∇_a *and* ∇_b*,* $[\nabla_a\, ,\ \nabla_b] = \nabla_a \nabla_b - \nabla_b \nabla_a$ *;*
2. *it commutes with the metric, i.e.* $\nabla_a g_{bc} = 0$ *and* $\nabla_a g^{bc} = 0$*.*

It is called the Levi-Civita connection. In a local coordinate basis $\partial_{\mathbf{a}}$*,* $\mathrm{d}x^{\mathbf{a}}$*, denoting*

$$\nabla_{\mathbf{a}} = \nabla_{\partial_{\mathbf{a}}}\, ,$$

its action on vector fields, 1*-forms and general tensor fields is given by :*

$$\begin{aligned} \mathrm{d}x^{\mathbf{b}}(\nabla_{\mathbf{a}} V) &= \partial_{\mathbf{a}} V^{\mathbf{b}} + \Gamma_{\mathbf{ac}}{}^{\mathbf{b}} V^{\mathbf{c}}\, , \\ (\nabla_{\mathbf{a}} \omega)(\partial_{\mathbf{b}}) &= \partial_{\mathbf{a}} \omega_{\mathbf{b}} - \Gamma_{\mathbf{ab}}{}^{\mathbf{c}} \omega_{\mathbf{c}}\, . \end{aligned}$$

$$\begin{aligned} \nabla_{\mathbf{a}} K^{\mathbf{i}_1 \ldots \mathbf{i}_p}{}_{\mathbf{j}_1 \ldots \mathbf{j}_q} = \partial_{\mathbf{a}} K^{\mathbf{i}_1 \ldots \mathbf{i}_p}{}_{\mathbf{j}_1 \ldots \mathbf{j}_q} &- \Gamma_{\mathbf{a}\mathbf{j}_1}{}^{\mathbf{b}} K^{\mathbf{i}_1 \ldots \mathbf{i}_p}{}_{\mathbf{b} \ldots \mathbf{j}_q} - \ldots - \Gamma_{\mathbf{a}\mathbf{j}_q}{}^{\mathbf{b}} K^{\mathbf{i}_1 \ldots \mathbf{i}_p}{}_{\mathbf{j}_1 \ldots \mathbf{b}} \\ &+ \Gamma_{\mathbf{ab}}{}^{\mathbf{i}_1} K^{\mathbf{b} \ldots \mathbf{i}_p}{}_{\mathbf{j}_1 \ldots \mathbf{j}_q} + \ldots + \Gamma_{\mathbf{ab}}{}^{\mathbf{i}_p} K^{\mathbf{i}_1 \ldots \mathbf{b}}{}_{\mathbf{j}_1 \ldots \mathbf{j}_q}\, . \end{aligned} \tag{2.4}$$

where the Christoffel symbols $\Gamma_{\mathbf{ab}}{}^{\mathbf{c}}$*, are defined by*

$$\Gamma_{\mathbf{ab}}{}^{\mathbf{c}} = \frac{1}{2} g^{\mathbf{cd}} \left(\partial_{\mathbf{a}} g_{\mathbf{bd}} + \partial_{\mathbf{b}} g_{\mathbf{ad}} - \partial_{\mathbf{d}} g_{\mathbf{ab}} \right) \tag{2.5}$$

and satisfy

$$\Gamma_{\mathbf{ab}}{}^{\mathbf{c}} = \Gamma_{(\mathbf{ab})}{}^{\mathbf{c}}\, .$$

Remark 2.5. *Note that (2.4) is a horrible abuse of notation, the left-hand side really denotes the component of the intrinsic object* $\nabla_a K^{i_1 \dots i_p}{}_{j_1 \dots j_q}$ *but a rigourous notation would be too heavy and it is usual to resort to this shortened (but always bear it in mind, wrong) notation.*

Remark 2.6. *It is important to note that the Christoffel symbols* $\Gamma_{\mathbf{ab}}{}^{\mathbf{c}}$ *are not a tensor field : it is very easy to see that they depend on the choice of local coordinates (see exercise 2.2). However, the connection is an intrinsic object independent of the coordinate system.*

Proposition 2.4. *When the commutator of two covariant derivatives acts on tensor fields of arbitrary valence, it involves another tensor field : the Riemann curvature tensor* R_{abcd}*. More precisely,*

$$\begin{aligned}[\nabla_a\,,\ \nabla_b]\, K^{i_1 \dots i_p}{}_{j_1 \dots j_q} &= R_{abc}{}^{i_1} K^{c \dots i_p}{}_{j_1 \dots j_q} + \dots + R_{abc}{}^{i_p} K^{i_1 \dots c}{}_{j_1 \dots j_q} \\ &\quad - R_{abj_1}{}^{d} K^{i_1 \dots i_p}{}_{d \dots j_q} - \dots - R_{abj_q}{}^{d} K^{i_1 \dots i_p}{}_{j_1 \dots d}\,. \end{aligned} \tag{2.6}$$

In a local coordinate basis, its expression in terms of the Christoffel symbols is given by

$$R_{\mathbf{abc}}{}^{\mathbf{d}} = \partial_{\mathbf{b}}\left(\Gamma_{\mathbf{ac}}{}^{\mathbf{d}}\right) - \partial_{\mathbf{a}}\left(\Gamma_{\mathbf{bc}}{}^{\mathbf{d}}\right) + \Gamma_{\mathbf{bc}}{}^{\mathbf{e}}\Gamma_{\mathbf{ae}}{}^{\mathbf{d}} - \Gamma_{\mathbf{ac}}{}^{\mathbf{e}}\Gamma_{\mathbf{be}}{}^{\mathbf{d}}\,. \tag{2.7}$$

Theorem 2.2. *The Riemann tensor has the following symmetries :*

1. $R_{(ab)cd} = 0$ *;*
2. $R_{ab(cd)} = 0$ *;*
3. $R_{[abc]}{}^{d} = 0$ *; it is the first Bianchi identity which, using* $R_{(ab)cd} = 0$*, becomes*

$$R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d} = 0\,;$$

4. $\nabla_{[a} R_{bc]d}{}^{e} = 0$ *(second Bianchi identity).*

Definition 2.6. *We define some important curvature quantities that are special parts of the full Riemann tensor :*

- *the Ricci tensor* R_{ab} *is the trace of the Riemann tensor in its second and fourth indices*

$$R_{ab} := R_{acb}{}^{c} = g^{cd} R_{acbd}\,;$$

- *the scalar curvature* R *is the trace of the Ricci tensor*

$$R := R_a{}^a = g^{ab} R_{ab}$$

  and it is often denoted by Scal_g *;*

- *the Einstein tensor* G_{ab} *is defined as*

$$G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}\,;$$

- *the Weyl tensor C_{abcd} is the trace-free part of the Riemann tensor*

$$C_{abcd} = R_{abcd} - \frac{1}{2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{1}{3}Rg_{a[c}g_{d]b}\,.$$

Proposition 2.5. *We have the following properties of the Ricci and Einstein tensors :*

1. $R_{ab} = R_{(ab)}$ *(which implies immediately* $G_{ab} = G_{(ab)}$*) ;*
2. $\nabla^a G_{ab} = 0$.

The **Einstein vacuum equations** that characterize the geometry of an empty universe are simply

$$G_{ab} = 0\,. \tag{2.8}$$

In the case of a universe containing energy or matter, the Einstein equations will become

$$G_{ab} = 8\pi T_{ab}$$

where T_{ab} is a tensor (referred to as the stress-energy tensor) describing the distribution of matter and energy in the universe.

Considered as an equation on the metric, Einstein's equations are a system of non linear second order partial differential equations. Taking the trace of G_{ab}, we obtain

$$G_a{}^a = R_a{}^a - \frac{1}{2}Rg_a{}^a = R - 2R = -R\,,$$

whence (2.8) is equivalent to

$$R_{ab} = 0\,. \tag{2.9}$$

Einstein vacuum spacetimes are also referred to as Ricci-flat spacetimes.

There is a modified version of the Einstein equation, due to Einstein himself in 1917, involving a constant Λ called the "cosmological constant". It has the following form

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}\,. \tag{2.10}$$

Einstein introduced this modification because the original form of the theory did not allow for a static universe (unless it is also flat), it had to be expanding or contracting. The cosmological constant induces a repulsive force which Einstein adjusted so that it would counterbalance gravitation exactly. His new version of the theory thus allows for a static universe : the Einstein cylinder which we shall encounter again later. The reason for this is probably partly religious but also a static universe was the commonly accepted picture at that time. This unfortunately prevented him from discovering the expansion of the universe which Hubble proved in 1929. He subsequently declared that this was his greatest mistake. It is interesting to notice that observations made from 1993 to 2005 show that the expansion of the universe is now faster than we would expect. A well accepted explanation is that a repulsive force induced by a cosmological constant is responsible for it : in the early stages of the universe, the expansion from the big bang was slowed down by gravity, but as the universe expanded, the effects of gravity weakened and this repulsive force (referred to as dark energy) accelerated the expansion. The universe would appear to have a small but strictly positive cosmological constant. It is regrettable that Einstein never knew that his greatest mistake was just another brilliant idea.

Taking the trace of (2.10), we see that in the vacuum case, i.e. for $T_{ab} = 0$, the cosmological constant is a multiple of the scalar curvature :

$$\Lambda = \frac{1}{4} R \, .$$

2.1.3 Causality

If $(\mathcal{M}, g)$ is a spacetime, then we can find in the neighbourhood of each point an orthonormal basis. In such a basis, the metric g is described by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

The tangent space at each point is therefore a copy of Minkowski vector space. This gives us natural definitions of timelike, spacelike, null and causal vectors and a similar classification for curves and hypersurfaces.

Definition 2.7. *Let $p \in \mathrm{M}$, a vector $V \in T_p\mathcal{M}$ is said to be*

- *spacelike if $g(V, V) < 0$,*
- *null if $g(V, V) = 0$,*
- *timelike if $g(V, V) > 0$,*
- *causal (or also non-spacelike) if $g(V, V) \geq 0$.*

The definitions of timelike, spacelike, etc... for curves and hypersurfaces follow exactly as they do in Minkowski space.

Definition 2.8. *A time orientation on a spacetime $(\mathcal{M}, g)$ is a globally defined nowhere vanishing continuous timelike vector field on $\mathcal{M}$. If a time orientation exists on $(\mathcal{M}, g)$, the spacetime is said to be time orientable.*

Definition 2.9. *Let $(\mathcal{M}, g)$ be a time orientable spacetime and T^a a time orientation. A causal vector V at a point is then said to be future oriented (resp. past oriented) if $g_{ab}V^aT^b > 0$ (resp. $g_{ab}V^aT^b < 0$).*

Proposition 2.6. *Let $(\mathcal{M}, g)$ be a time orientable spacetime on which we consider T^a and τ^a two time orientations. Then one of the two following assertions is true :*

(ı) for any causal vector V at a given point, the signs of $g_{ab}V^aT^b$ and $g_{ab}V^a\tau^b$ are the same ; the orientations are then said to be the same ;

($\imath$) for any causal vector V at a given point, the signs of $g_{ab}V^aT^b$ and $g_{ab}V^a\tau^b$ are opposite ; the orientations are then said to be opposite.

Proposition 2.7. *Let $(\mathcal{M}, g)$ be a time orientable spacetime. A spacelike vector has no time orientation. More precisely, given V a spacelike vector at a point p, there exist two choices T^a and τ^a of time orientation on $\mathcal{M}$ such that $g_{ab}V^aT^b > 0$ and $g_{ab}V^a\tau^b < 0$.*

An important notion is that of the domain of influence of a set :

Definition 2.10. *Let $(\mathcal{M}, g)$ be a time orientable spacetime on which a time orientation has been chosen. We consider a set A in $\mathcal{M}$. The future (resp. past) domain of influence of A in $(\mathcal{M}, g)$ is the set of points of $\mathcal{M}$ that can be reached from a point of A along a future (resp. past) oriented causal curve. These are often merely referred to as the future or the past of A. The domain of influence of A is the union of its future and past domains of influence.*

This is related to the concepts of Cauchy hypersurfaces and global hyperbolicity. Of all the equivalent definitions that have been proposed for a globally hyperbolic spacetime, the first one due to Leray, the clearest is certainly that which R.P. Geroch put forward in 1970 [10]. The fundamental definition is that of a Cauchy hypersurface.

Definition 2.11 (Cauchy hypersurface)**.** *Let $(\mathcal{M}, g)$ be a time orientable spacetime. A Cauchy hypersurface on $(\mathcal{M}, g)$ is a hypersurface Σ satisfying :*

1. *Σ is spacelike ;*
2. *every inextendible timelike curve intersects Σ at exactly one point (which entails in particular that the domain of influence of Σ is $\mathcal{M}$).*

We see that this is an adequate surface on which to impose initial data for covariant equations (a covariant equation on a Lorentzian space-time will necessarily be a generalization to the case of a curved spacetime of covariant equations on Minkowski space, which are hyperbolic equations), since they propagate the information at finite speed lower than or equal to the speed of light, the condition that the domain of influence of Σ should be the whole spacetime is exactly what ensures that by specifying some data on Σ, we have enough information to propagate the solution to the whole spacetime. Moreover, the second condition is here to guarantee that the information propagated along causal geodesics does not come back to a point where the solution is already determined, thus creating some possible incompatibility. A globally hyperbolic spacetime as defined by Geroch is simply a spacetime that admits a Cauchy hypersurface.

Definition 2.12. *A spacetime $(\mathcal{M}, g)$ is said to be globally hyperbolic if it admits a Cauchy hypersurface.*

So globally hyperbolic spacetimes are essentially the spacetimes for which the Cauchy problem makes sense. The spacetimes in which it is hardest to make any sense at all of the Cauchy problem are called totally vicious spacetimes, they are such that any point can be reached from any other point in the spacetime along a future oriented timelike curve. An example of a totally vicious part of a spacetime is the inner part of a rotating black hole.

In fact, global hyperbolicity has stronger consequences : the existence of a smooth time function t whose level hypersurfaces Σ_t are all Cauchy hypersurfaces and are diffeomorphic to a fixed 3-surface Σ. For a long time, the only available proof of this result was due to Geroch and his construction only guaranteed the existence of a continuous time function whose level hypersurfaces were homeomorphic to a fixed hypersurface. The work of Bernal and Sanchez [1, 2] proved that the time function can be chosen smooth when the metric is smooth. Their result in fact gives a $\mathcal{C}^k$ time function when the metric is $\mathcal{C}^k$. We will assume that the metric and the time function are $\mathcal{C}^\infty$ for simplicity, we will not consider here situations in which the precise regularity of the metric and the time function may be crucial.

2.1.4 Forms and conservation laws

Recall that the bundle of differential 1-forms on $\mathcal{M}$ is simply $\Lambda^1(\mathcal{M}) = T^*\mathcal{M}$ and the bundle of differential p-forms is the p^{th} exterior power of $\Lambda^1(\mathcal{M})$, i.e.

$$\Lambda^p(\mathcal{M}) = \underbrace{\Lambda^1(\mathcal{M}) \wedge \Lambda^1(\mathcal{M}) \wedge ... \wedge \Lambda^1(\mathcal{M})}_{p \text{ times}},$$

it is the totally skew-symmetric part of

$$\underbrace{T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes ... \otimes T^*\mathcal{M}}_{p \text{ times}}.$$

Definition 2.13 (Volume form)**.** *The volume-form on $(\mathcal{M}, g)$ is the 4-form e whose expression in a coordinate basis is given by (the ordering of coordinates being chosen in agreement with the orientation of $\mathcal{M}$)*

$$e = \sqrt{|g|}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 . \tag{2.11}$$

Equivalently, it is defined as follows : consider an ortonormal basis $\mathcal{B} = \{e_0, e_1, e_2, e_3\}$, for any set of 4 vectors $\{U, V, W, Z\}$, denoting U^0, U^1, U^2, U^3 the components of U in $\mathcal{B}$, etc..., we have

$$e_{abcd}U^aV^bW^cZ^d = \det \begin{pmatrix} U^0 & V^0 & W^0 & Z^0 \\ U^1 & V^1 & W^1 & Z^1 \\ U^2 & V^2 & W^2 & Z^2 \\ U^3 & V^3 & W^3 & Z^3 \end{pmatrix} . \tag{2.12}$$

We shall often simply denote the volume form dVol*, or* dVol_g *to make the relation to the metric explicit.*

The volume form has the following useful properties :

Proposition 2.8. *The volume form is covariantly constant, i.e.*

$$\nabla_i e_{abcd} = 0 .$$

Moreover,

$$e_{abcd}e^{pqrs} = -24 g_a^{[p} g_b^q g_c^r g_d^{s]} , \; e_{abcd}e^{pqrd} = -6 g_a^{[p} g_b^q g_c^{r]} , \; e_{abcd}e^{pqcd} = -4 g_a^{[p} g_b^{q]} ,$$
$$e_{abcd}e^{pbcd} = -6 g_a^p , \; e_{abcd}e^{abcd} = -24 , \; e_{ab}{}^{cd} e_{cd}{}^{pq} = -4 g_a^{[p} g_b^{q]} .$$

Proof. The proof of the covariant constancy follows easily from the expression of the volume form in terms of the spinorial symplectic forms (see [23], Vol. 1, p. 138, eq. (3.3.31)). The proof of the other properties is merely a matter of counting permutations. □

We shall essentially use differential 1-forms and differential 3-forms (often simply referred to as 1-forms and 3-forms) in the context of conservation laws (exact or approximate). Hence we will often make use of the Hodge duality and of Stokes' theorem.

Definition 2.14 (Hodge dual)**.** *Let $\omega \in \Gamma(\Lambda^p(\mathcal{M}))$, $0 \leq p \leq 4$, the Hodge dual of ω is the $(4-p)$-form defined by*

$$*\omega := \frac{1}{p!} e \underbrace{\lrcorner \dots \lrcorner}_{p \text{ times}} \omega \,, \tag{2.13}$$

where e is the volume-form on $(\mathcal{M}, g)$. More explicitely,

- *for a 0-form f*
$$(*f)_{abcd} = f e_{abcd} \,;$$
- *for a 1-form α*
$$(*\alpha)_{abc} = e_{abcd}\alpha^d \,;$$
- *for a 2-form β*
$$(*\beta)_{ab} = \frac{1}{2} e_{abcd}\beta^{cd} \,;$$
- *for a 3-form γ*
$$(*\gamma)_a = \frac{1}{6} e_{abcd}\gamma^{bcd} \,;$$
- *for a 4-form ϵ*
$$(*\epsilon) = \frac{1}{24} e_{abcd}\epsilon^{abcd} \,.$$

The Hodge star is an isomorphism between p-forms and $4-p$-forms, as the following property, which is a direct consequence of Proposition 2.8, shows :

Proposition 2.9. *For a p-form α, we have*

$$*(*\alpha) = (-1)^{p+1}\alpha \,.$$

Proof. This is obvious for a 0-form, let us check the property for the other types of forms.

- $p = 1$:

$$\begin{aligned} *(*\alpha)_a &= \frac{1}{6} e_{abcd} e^{bcdi} \alpha_i \\ &= -\frac{1}{6} e_{abcd} e^{bcdi} \alpha_i = -\frac{1}{6}(-6 g_a^i)\alpha_i = \alpha_a \,. \end{aligned}$$

- $p = 2$:

$$\begin{aligned}
(\alpha)_{ab} &= \frac{1}{2} e_{abcd} e^{cdij} \alpha_{ij} \\
&= \frac{1}{2} e_{abcd} \frac{1}{2} e^{ijcd} \alpha_{ij} \\
&= \frac{1}{4} (-4 g_a^i g_b^j) \alpha_{ij} \text{ since } \alpha \text{ is skew,} \\
&= -\alpha_{ab} \,.
\end{aligned}$$

- $p = 3$:

$$\begin{aligned}
(\alpha)_{abc} &= e_{abcd} \frac{1}{6} e^{dijk} \alpha_{ijk} \\
&= -\frac{1}{6} e_{abcd} \frac{1}{2} e^{ijkd} \alpha_{ijk} \\
&= -\frac{1}{6} (-6 g_a^{[i} g_b^j g_c^{k]}) \alpha_{ijk} \\
&= g_a^i g_b^j g_c^k \alpha_{ijk} \text{ since } \alpha \text{ is skew,} \\
&= \alpha_{abc} \,.
\end{aligned}$$

- $p = 4$. In this case we have $\alpha_{abcd} = f e_{abcd}$, hence

$$\begin{aligned}
(\alpha)_{abcd} &= e_{abcd} \frac{1}{24} e^{ijkl} \alpha_{ijkl} \\
&= \frac{1}{24} e_{abcd} e^{ijkl} f e_{ijkl} \\
&= \frac{1}{24} (-24) f e_{abcd} = -\alpha_{abcd} \,.
\end{aligned}$$

This proves the proposition. □

The Hodge $*$ operator has the following property, that entirely characterizes it :

Theorem 2.3. *For any two p-forms α, β, $1 \le p \le 3$,*

$$\alpha \wedge *\beta = (-1)^p \frac{(4-p)!}{4!} \langle \alpha \,,\ \beta \rangle_g \, e \,, \tag{2.14}$$

where

$$\langle \alpha \,,\ \beta \rangle_g = \alpha_{a_1 \ldots a_p} \beta^{a_1 \ldots a_p} \,.$$

Proof. We write the proof for each value of p.

- $p = 1$. Since the quantity $\alpha \wedge *\beta$ is a 4-form, it is necessarily a multiple of the volume form, all we need to do is work out the proportionality factor. We proceed as follows : since

$$\alpha \wedge *\beta = \alpha_{[a} \, (*\beta)_{bcd]} = f e_{abcd}$$

then

$$\alpha_{[a}\,(*\beta)_{bcd]}e^{abcd} = -24f\,.$$

We calculate

$$\begin{aligned}
\alpha_{[a}\,(*\beta)_{bcd]}e^{abcd} &= \alpha_a\,(*\beta)_{bcd}e^{abcd} \\
&= \alpha_a e_{bcdi}\beta^i e^{abcd} \\
&= -e_{ibcd}e^{abcd}\alpha_a\beta^i = 6g_i^a\alpha_a\beta^i = 6\alpha_i\beta^i\,.
\end{aligned}$$

Whence

$$(\alpha\wedge *\beta)_{abcd} = -\frac{1}{4}\alpha_i\beta^i e_{abcd}\,.$$

- $p = 2$. We proceed similarly :

$$\begin{aligned}
\alpha_{[ab}\,(*\beta)_{cd]}e^{abcd} &= \alpha_{ab}\,(*\beta)_{cd}e^{abcd} \\
&= \alpha_{ab}\frac{1}{2}e_{cdij}\beta^{ij}e^{abcd} \\
&= \frac{1}{2}e_{ijcd}e^{abcd}\alpha_{ab}\beta^{ij} \\
&= \frac{1}{2}(-4g_i^{[a}g_j^{b]})\alpha_{ab}\beta^{ij} \\
&= -2g_i^a g_j^b\alpha_{ab}\beta^{ij} \text{ since } \alpha \text{ is skew,} \\
&= -2\alpha_{ij}\beta^{ij}\,.
\end{aligned}$$

Whence

$$(\alpha\wedge *\beta)_{abcd} = \frac{1}{12}\alpha_{ij}\beta^{ij}e_{abcd}\,.$$

- $p = 3$:

$$\begin{aligned}
\alpha_{[abc}\,(*\beta)_{d]}e^{abcd} &= \alpha_{abc}\,(*\beta)_{d}e^{abcd} \\
&= \alpha_{abc}\frac{1}{6}e_{dijk}\beta^{ijk}e^{abcd} \\
&= -\frac{1}{6}e_{ijkd}e^{abcd}\alpha_{abc}\beta^{ijk} \\
&= -6(-\frac{1}{6})g_i^{[a}g_j^b g_k^{c]}\alpha_{abc}\beta^{ijk} \\
&= g_i^a g_j^b g_k^c\alpha_{abc}\beta^{ijk} \text{ since } \alpha \text{ is skew,} \\
&= \alpha_{ijk}\beta^{ijk}\,.
\end{aligned}$$

Whence

$$(\alpha\wedge *\beta)_{abcd} = -\frac{1}{24}\alpha_{ijk}\beta^{ijk}e_{abcd}\,.$$

This concludes the proof. □

If instead of taking the exterior product with another 1-form, we take the exterior derivative of the Hodge dual of a 1-form, we recover a 4-form which is $(-1/4)$ times the dual of the divergence of the associated vector field :

Proposition 2.10. *Let α be a 1-form, we have*

$$(\mathrm{d}(*\alpha))_{abcd} = -\frac{1}{4}\nabla_i \alpha^i e_{abcd} \,.$$

Proof. The proof is analogous to that of the case $p = 1$ in the above theorem, with just an additional ingredient : the covariant constancy of the volume form. We leave it as an exercise (see exercise 2.4). □

We now turn to conservation laws, i.e. to Stokes' theorem for the Hodge dual of a 1-form. Recall Stokes' theorem for a 3-form :

Theorem 2.4. *Let Ω a bounded open subset of $\mathcal{M}$ with piecewise $\mathcal{C}^1$ boundary S. Let $\omega \in \Gamma(\Lambda^3(\mathcal{M}))$, $\mathcal{C}^1$ on $\bar{\Omega}$, then*

$$\int_S \omega = \int_\Omega \mathrm{d}\omega \,.$$

In the case where ω is the Hodge dual of a 1-form α the above equality gives the Lorentzian generalization of the divergence theorem.

Theorem 2.5 (The divergence theorem)**.** *Let Ω a bounded open subset of $\mathcal{M}$ with piecewise $\mathcal{C}^1$ boundary S, l^a a vector field transverse to S and outgoing*[1]*, n^a a normal vector field to S such that $l_a n^a = 1$. Let α be a 1-form $\mathcal{C}^1$ on $\bar{\Omega}$, then*

$$\int_S \alpha_a n^a (l \lrcorner \mathrm{dVol}) = \int_\Omega \nabla_a \alpha^a \mathrm{dVol} \,.$$

Proof. We have essentially proved the result above. Take the 3-form ω to be

$$\omega = *\alpha \,,$$

then

$$\mathrm{d}\omega = -\frac{1}{4}\nabla_a \alpha^a \mathrm{dVol} \,.$$

Moreover, denoting by $n^\flat$ the 1-form $(n^\flat)_a = n_a$,

$$\begin{aligned}
\int_S *\alpha &= \int_S \langle l, n\rangle_g * \alpha \\
&= \int_S l \lrcorner (n^\flat \wedge *\alpha) - \underbrace{\int_S n^\flat \wedge (l \lrcorner * \alpha)}_{=0 \text{ since } n \perp S} \\
&= \int_S l \lrcorner (-\frac{1}{4}) \alpha_a n^a \mathrm{dVol} \,.
\end{aligned}$$

This concludes the proof. □

[1] The vector field l is chosen outgoing in order for the orientation of the 3-form $l \lrcorner \mathrm{dVol}$ on S to be consistent with the orientation of S. The theorem is true whether or not l is outgoing.

2.1.5 Symmetries, Killing vectors

Definition 2.15 (Killing vector field)**.** *On a space-time* $(\mathcal{M}, g)$*, a vector field* K *is said to be a Killing vector field if and only if its flow is an isometry for the metric* g*, i.e.*

$$\mathcal{L}_K g = 0\,,$$

which is equivalent to the Killing equation

$$\nabla^{(a} K^{b)} = 0\,,$$

i.e. the symmetric part of the covariant derivative of K *vanishes.*

Definition 2.16 (Stationarity, staticity)**.** *A spacetime is said to be stationary if it admits a global timelike Killing vector field. It is said to be static if it admits a global timelike Killing vector field that is orthogonal to a family of spacelike hypersurfaces (equivalently, orthogonal to a Cauchy hypersurface).*

As an example, the symmetry group of Minkowski spacetime (preserving the metric, orientation and time-orientation) is the Poincaré group. It is the 10-dimensional group generated by the four Cartesian coordinate translations, the three space rotations and the three boosts or hyperbolic rotations. The infinitesimal generators of these transformations provide the 10 independent Killing vector fields of Minkowski spacetime :

translations : ∂_t, ∂_x, ∂_y, ∂_z ;

rotations : $x\partial_y - y\partial_x$, $y\partial_z - z\partial_y$, $z\partial_x - x\partial_z$;

boosts : $x\partial_t + t\partial_x$, $y\partial_t + t\partial_y$, $z\partial_t + t\partial_z$, which are sometimes viewed as generating rotations in the planes (it, x), (it, y) and (it, z) (hyperbolic rotations).

2.2 A touch of functional analysis

In this section we study the equation

$$\frac{\partial \phi}{\partial t} = iA\phi\,, \tag{2.15}$$

on a Hilbert space $\mathcal{H}$ with inner product $\langle .\,,\ .\rangle$, where A is a self-adjoint operator on $\mathcal{H}$. *By Hilbert space, we mean separable Hilbert space.* Let us first recall some basic definitions.

Definition 2.17 (Adjoint)**.** *Let* A *be a densely defined operator on a Hilbert space* $\mathcal{H}$*, i.e. its domain*

$$D(A) = \{\phi \in \mathcal{H}\,;\ A\phi \in \mathcal{H}\}$$

is dense in $\mathcal{H}$*. The adjoint* A^* *of* A *is an operator on* $\mathcal{H}$ *whose domain is*

$$D(A^*) = \{\phi \in \mathcal{H}\,;\ \text{ the map } \psi \mapsto \langle A\psi\,,\ \phi\rangle \text{ extends as a bounded linear map on } \mathcal{H}\}$$

and it is defined by

$$\forall \phi \in D(A^*)\,,\ \forall \psi \in D(A)\,,\ \langle \psi\,,\ A^*\phi\rangle = \langle A\psi\,,\ \phi\rangle\,.$$

Definition 2.18 (Self-adjoint operator)**.** *A densely defined operator A on $\mathcal{H}$ is self-adjoint if $A = A^*$. This is equivalent to :*

1. *it is symmetric, i.e.*
$$\forall \phi\,,\ \psi \in D(A)\,,\ \langle A\phi\,,\ \psi\rangle = \langle \phi\,,\ A\psi\rangle\,,$$
2. $D(A^*) = D(A)$.

Remark 2.7. *Note that if A is a densely defined symmetric operator on $\mathcal{H}$, then $D(A) \subset D(A^*)$, so in order to prove that it is self-adjoint, it suffices to establish that $D(A^*) \subset D(A)$.*

Let us now consider the equation (2.15) and more particularly the Cauchy problem

$$(\mathcal{P})\ \left\{\begin{array}{l} \frac{\partial \phi}{\partial t} = iA\phi\,, \\ \phi(0) = \phi_0 \in \mathcal{H}\,. \end{array}\right.$$

The first thing we easily remark is that in the natural function space of solutions, we have uniqueness of solutions of the Cauchy problem. Let us be more explicit. A natural space in which to look for solutions of $(\mathcal{P})$ is

$$\mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})\,.$$

Indeed, we want to be able to apply A to ϕ at each time, so ϕ must be in $D(A)$ for all t. Note that in particular, this requires that $\phi_0 \in D(A)$. Also we want to take the value of ϕ at $t = 0$, so ϕ must be continuous in t. This gives us $\mathcal{C}(\mathbb{R}_t\,;\ D(A))$, which is embedded in $\mathcal{C}(\mathbb{R}_t\,;\ \mathcal{H})$; this is because we use on $D(A)$ the graph norm

$$\|\phi\|^2_{D(A)} = \|\phi\|^2_{\mathcal{H}} + \|A\phi\|^2_{\mathcal{H}}$$

which is always larger than the norm in $\mathcal{H}$. We also need to differentiate ϕ with respect to t, so we need it to be a differentiable function with values in $\mathcal{H}$, but then since we already have assumed $\phi \in \mathcal{C}(\mathbb{R}_t\,;\ D(A))$, the equation implies that

$$\frac{\partial \phi}{\partial t} = iA\phi \in \mathcal{C}(\mathbb{R}_t\,;\ \mathcal{H})$$

and therefore $\phi \in \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$.

Let us show the uniqueness of solutions of $(\mathcal{P})$ in $\mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$.

Proposition 2.11. *Let $\phi_0 \in D(A)$. If $\phi \in \mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$ is a solution of $(\mathcal{P})$, then it satisfies*

$$\|\phi(t)\|_{\mathcal{H}} = \|\phi_0\|_{\mathcal{H}}\ \forall t \in \mathbb{R} \tag{2.16}$$

and it is therefore unique.

Proof. Let us prove (2.16) :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle \phi\,,\ \phi\rangle &= \langle iA\phi\,,\ \phi\rangle + \langle \phi\,,\ iA\phi\rangle \\ &= i\langle A\phi\,,\ \phi\rangle - i\langle \phi\,,\ A\phi\rangle = 0 \end{aligned}$$

since A is self-adjoint. This establishes that any solution ϕ of $(\mathcal{P})$ in $\mathcal{C}(\mathbb{R}_t\,;\ D(A))\cap\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$ satisfies (2.16). Now let us consider ϕ and ψ two solutions of $(\mathcal{P})$ in $\mathcal{C}(\mathbb{R}_t\,;\ D(A))\cap\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$. Then $\psi-\phi$ is a solution of (2.15) in $\mathcal{C}(\mathbb{R}_t\,;\ D(A))\cap\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$ with initial data $\psi(0)-\phi(0)=\phi_0-\phi_0=0$. It satisfies

$$\|\psi(t)-\phi(t)\|_{\mathcal{H}}=\|\psi(0)-\phi(0)\|_{\mathcal{H}}=0\ \forall t\in\mathbb{R}\,.$$

Therefore we have $\psi=\phi$. □

The existence of solutions is then given by a classic theorem which is (the easy part of) Stone's theorem :

Theorem 2.6. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, there exists a family of operators denoted e^{itA}, $t\in\mathbb{R}$, such that*

1. *for all $t\in\mathbb{R}$, e^{itA} is a unitary operator on $\mathcal{H}$,*
2. *$e^{i0A}=\mathrm{Id}_{\mathcal{H}}$,*
3. *for all $t,s\in\mathbb{R}$, $e^{isA}e^{itA}=e^{i(t+s)A}$ (therefore in particular $e^{isA}e^{itA}=e^{itA}e^{isA}$),*
4. *for all $\phi\in\mathcal{H}$, $e^{itA}\phi\in\mathcal{C}(\mathbb{R}_t\,;\ \mathcal{H})$,*
5. *for all $\phi\in D(A)$, $e^{itA}\phi\in D(A)$ and*
$$\frac{e^{itA}\phi-\phi}{t}\to iA\phi \text{ as } t\to 0\,.$$

Remark 2.8. *Properties 1-4 are the definition of a strongly continuous 1-parameter group of unitary operators.*

In fact, this part of Stone's theorem is merely a direct consequence of the Borel functional calculus version of the spectral theorem. The difficult part of Stones theorem is the converse result, that any one parameter group of unitary operators is of the form e^{itA} where A is a self-adjoint operator on $\mathcal{H}$. The Borel functional calculus version of the spectral theorem is the following result, which we admit here (see M. Reed and B. Simon, vol. 2 [24], p. 262).

Theorem 2.7 (Spectral theorem : functional calculus form)**.** *Let A be a self-adjoint operator on a separable Hilbert $\mathcal{H}$, then there exists a unique map $\hat{\Phi}$ from bounded Borel functions on $\mathbb{R}$ into $\mathcal{L}(A)$ such that*

1. *$\hat{\Phi}$ is an algebraic $*$-homomorphism, i.e.*
   - $\Phi(f+g)=\Phi(f)+\Phi(g)$,
   - $\Phi(fg)=\Phi(f)\circ\Phi(g)$,
   - $\Phi(1)=\mathrm{Id}_{\mathcal{H}}$,
   - $\Phi(\bar{f})=(\Phi(f))^*$,
2. *$\hat{\Phi}$ is norm continuous, more precisely $\|\hat{\Phi}(f)\|_{\mathcal{L}(\mathcal{H})}\leq\|f\|_{L^\infty}$,*

3. *let* $\{h_n\}_{n\in\mathbb{N}}$ *be a sequence of bounded Borel functions such that for all* x, $h_n(x) \to x$ *as* $n \to +\infty$ *and for all* x *and* n $|h_n(x)| \leq |x|$, *then for any* $\psi \in D(A)$,

$$\lim_{n\to+\infty} \hat{\Phi}(h_n)\psi = A\psi \,,$$

4. *let* $\{h_n\}_{n\in\mathbb{N}}$ *be a sequence of bounded Borel functions converging pointwise towards* h *and such that the sequence* $\|h_n\|_{L^\infty}$ *is bounded, then* $\hat{\Phi}(h_n)$ *converges strongly towards* $\hat{\Phi}(h)$.

   In addition,

5. *if* $A\psi = \lambda\psi$ *then* $\hat{\Phi}(h)\psi = h(\lambda)\psi$,
6. *if* $h \geq 0$ *then* $\hat{\Phi}(h) \geq 0$.

Of course, in the case where A is bounded, we do not need the functional calculus to define e^{itA}, we can use the usual power series which converges in norm in $\mathcal{L}(\mathcal{H})$. When A is unbounded however, the functional calculus not only allows to define e^{itA} but also to verify easily its properties.

It is important to note that e^{itA} commutes with A on $D(A)$.

Proposition 2.12. *Let* $\phi \in D(A)$, *then* $e^{itA}A\phi = Ae^{itA}\phi$ *for all* $t \in \mathbb{R}$.

Proof. This is an immediate consequence of properties *3* and *5*. We have for all $t, s \in \mathbb{R}$

$$\begin{aligned} \frac{e^{i(t+s)A}\phi - e^{itA}\phi}{s} &= e^{itA}\frac{e^{i(s)A}\phi - \phi}{s} \to e^{itA}A\phi \text{ as } s \to 0\,, \\ &= \frac{e^{i(s)A}e^{itA}\phi - e^{itA}\phi}{s} \to Ae^{itA}\phi \text{ as } s \to 0\,. \quad \square \end{aligned}$$

An important consequence of this property is that the Cauchy problem for (2.15) will be well-posed in all the successive domains of A.

Theorem 2.8. *Let* $\phi_0 \in D(A^k)$, $k \in \mathbb{N}^*$, *then the solution of* (2.15) *in* $\mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})$ *satisfies*

$$\phi \in \bigcap_{p=0}^{k} \mathcal{C}^p(\mathbb{R}_t\,;\ D(A^{k-p}))\,.$$

Proof. For $k = 1$ there is nothing new. Let us assume $k \geq 2$, then the theorem is a direct consequence of the previous proposition. Indeed $A\phi_0 \in D(A^{k-1})$, with $k - 1 \geq 1$ and therefore

$$e^{itA}A\phi_0 \in \mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})\,.$$

But since A commutes with e^{itA} on $D(A)$, we get

$$e^{itA}A\phi_0 = Ae^{itA}\phi_0 = A\phi(t) \in \mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})\,.$$

This implies that

$$\phi \in \mathcal{C}(\mathbb{R}_t\,;\ D(A^2)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ D(A))\,. \tag{2.17}$$

And using the equation we also have

$$iA\phi = \frac{\partial \phi}{\partial t} \in \mathcal{C}(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{H})\,. \tag{2.18}$$

Putting (2.17) and (2.18), we get

$$\phi \in \mathcal{C}(\mathbb{R}_t\,;\ D(A^2)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ D(A)) \cap \mathcal{C}^2(\mathbb{R}_t\,,\ \mathcal{H})\,.$$

The theorem follows by induction. □

Remark 2.9. *In some important cases, $\mathcal{H}$ will be a distribution space ($\mathcal{H} = L^2(\mathbb{R}^n)$ for example), and A will be a differential operator. In such cases, it is often possible to gain the existence and uniqueness of solutions of* (2.15) *in $\in \mathcal{C}(\mathbb{R}_t\,;\ \mathcal{H})$ in the sense of distributions ; such "minimum regularity" solutions will still have their norm in $\mathcal{H}$ conserved throughout time.*

2.3 Wave equation, conformal invariance

Definition 2.19 (D'Alembertian)**.** *On a given spacetime $(\mathcal{M}, g)$ the D'Alembertian operator is defined by*

$$\Box_g = \nabla_a \nabla^a\,. \tag{2.19}$$

It is easy to check that in a local coordinate basis, its expression is given by

$$\Box_g = \frac{1}{\sqrt{|g|}} \partial_{\mathbf{a}} (\sqrt{|g|} g^{\mathbf{ab}} \partial_{\mathbf{b}})\,. \tag{2.20}$$

Definition 2.20 (Wave equation)**.** *The wave equation on a space-time $(\mathcal{M}, g)$ is the equation*

$$\Box_g \phi = 0$$

for a scalar field ϕ.

We now define the notion of conformal rescaling and we shall see shortly that the D'Alembertian has a role to play in such transformations.

Definition 2.21 (Conformal class)**.** *Consider a spacetime $(\mathcal{M}, g)$. We say that a metric $\hat{g}$ on $\mathcal{M}$ is conformally equivalent to g if there exists a positive nowhere vanishing smooth function Ω on $\mathcal{M}$ such that $\hat{g} = \Omega^2 g$. We also say that $\hat{g}$ is a conformal rescaling of g. The conformal class $[g]$ of g is the set of all metrics on $\mathcal{M}$ that are conformally equivalent to g.*

Under a conformal rescaling, the connection changes in a rather simple way. Recall the expression of the Christoffel symbols in a coordinate basis

$$\Gamma_{\mathbf{ab}}{}^{\mathbf{c}} = \frac{1}{2} g^{\mathbf{cd}} \left(\partial_{\mathbf{a}} g_{\mathbf{bd}} + \partial_{\mathbf{b}} g_{\mathbf{ad}} - \partial_{\mathbf{d}} g_{\mathbf{ab}} \right)\,,$$

then for a metric $\hat{g} = \Omega^2 g$, $\Omega > 0$ on $\mathcal{M}$ and smooth, we have the Christoffel symbols

$$\begin{aligned}
\hat{\Gamma}_{\mathbf{ab}}{}^{\mathbf{c}} &= \frac{1}{2}\hat{g}^{\mathbf{cd}}\left(\partial_{\mathbf{a}}\hat{g}_{\mathbf{bd}} + \partial_{\mathbf{b}}\hat{g}_{\mathbf{ad}} - \partial_{\mathbf{d}}\hat{g}_{\mathbf{ab}}\right) \\
&= \Gamma_{\mathbf{ab}}{}^{\mathbf{c}} = \frac{1}{2}\Omega^{-2}g^{\mathbf{cd}}\left(\partial_{\mathbf{a}}(\Omega^2 g_{\mathbf{bd}}) + \partial_{\mathbf{b}}(\Omega^2 g_{\mathbf{ad}}) - \partial_{\mathbf{d}}(\Omega^2 g_{\mathbf{ab}})\right) \\
&= \Gamma_{\mathbf{ab}}{}^{\mathbf{c}} + 2g^{\mathbf{a}}_{(\mathbf{b}}\nabla_{\mathbf{c})}\ln\Omega - g_{\mathbf{bc}}\nabla^{\mathbf{a}}\ln\Omega\,.
\end{aligned}$$

We denote by $C_{bc}{}^a$ the difference between the two Christoffel symbols ; note that this is a true tensor field (contrary to the Christoffel symbols) expressing the difference between the Levi-Civita connections ∇ and $\hat{\nabla}$ of the two metrics g and $\hat{g}$:

$$C_{\mathbf{bc}}{}^{\mathbf{a}} := \hat{\Gamma}_{\mathbf{ab}}{}^{\mathbf{c}} - \Gamma_{\mathbf{ab}}{}^{\mathbf{c}}\,,\; C_{bc}{}^a = 2g^a_{(b}\nabla_{c)}\ln\Omega - g_{bc}\nabla^a\ln\Omega\,,\; C_{bc}{}^a = C_{(bc)}{}^a\,. \tag{2.21}$$

This tensor can be used to express the difference between the Riemann tensors for g and $\hat{g}$: first we write

$$\begin{aligned}
\hat{\nabla}_a\hat{\nabla}_b\omega_c &= \nabla_a(\nabla_b\omega_c - C_{bc}{}^d\omega_d) \\
&\quad - C_{ab}{}^e(\nabla_e\omega_c - C_{ec}{}^d\omega_d) \\
&\quad - C_{ac}{}^e(\nabla_b\omega_e - C_{be}{}^d\omega^d)\,.
\end{aligned}$$

This gives

$$\begin{aligned}
-\hat{R}_{abc}{}^d\omega_d &= (\hat{\nabla}_a\hat{\nabla}_b - \hat{\nabla}_b\hat{\nabla}_a)\omega_c \\
&= -R_{abc}{}^d\omega_d - 2\nabla_{[a}C_{b]c}{}^d\omega_d \\
&\quad -0 \\
&\quad -2C_{c[a}{}^e\nabla_{b]}\omega_e + 2C_{c[a}{}^eC_{b]e}{}^d\omega_d\,,
\end{aligned}$$

whence

$$\hat{R}_{abc}{}^d - R_{abc}{}^d = 2(\nabla_{[a}C_{b]c}{}^d) - 2C_{c[a}{}^eC_{b]e}{}^d\,. \tag{2.22}$$

By taking the trace of (2.22), we can obtain the relation between $\mathrm{Scal}_{\hat{g}}$ and Scal_g. But we must be careful, taking the trace means using a metric to raise an index, $\hat{g}$ for $\hat{R}$ and g for R. The index d is already raised, so we just need to contract with $\hat{g}^b_d = g^b_d$, but we also need to contract with $\hat{g}^{ac} = \Omega^{-2}g^{ac}$. We get

$$\mathrm{Scal}_{\hat{g}} = \hat{R}_{ab}{}^{ab} = \Omega^{-2}(R_{ab}{}^{ab} + g^{ac}2(\nabla_{[a}C_{b]c}{}^b) - 2g^{ac}C_{c[a}{}^eC_{b]e}{}^b)$$

and after a long but straightforward calculation, provided we are careful and do not make mistakes, we find the following result[2]

Theorem 2.9. *Consider a spacetime $(\mathcal{M}, g)$ and a metric $\hat{g}$ in the conformal class of g with conformal factor Ω, i.e. $\hat{g} = \Omega^2 g$, then*

$$\mathrm{Scal}_{\hat{g}} = \Omega^{-2}\mathrm{Scal}_g + 6\Omega^{-3}\Box_g\Omega\,.$$

[2]A more detailed study of the modification of the different parts of the curvature under conformal rescalings is given in [26], with a different sign convention for Lorentzian metrics though, so some conversions are necessary, and in the formalism of Weyl spinors in [23].

This will allow us to establish that a simple modification of the wave equation satisfies a property called conformal invariance.

Definition 2.22 (Conformal invariance)**.** *The conformal invariance of a covariant equation means that there exists $s \in \mathbb{R}$ such that a field ϕ satisfies the equation for the metric g if and only if $\Omega^s \phi$ satisfies the equation for $\hat{g}$.*

The wave equation is not conformally invariant. However, a slight modification of this equation involving the scalar curvature is conformally invariant. We shall refer to it as the conformal wave equation :

$$\Box_g \phi + \frac{1}{6}\mathrm{Scal}_g \phi = 0\,. \tag{2.23}$$

More precisely, we have the following fundamental result which is a consequence of Theorem 2.9 :

Corollary 2.1. *We consider a spacetime $(\mathcal{M}, g)$ and a metric $\hat{g}$ in the conformal class of g with conformal factor Ω, i.e. $\hat{g} = \Omega^2 g$. Then we have the equality of operators acting on scalar fields on $\mathcal{M}$:*

$$\Box_g + \frac{1}{6}\mathrm{Scal}_g = \Omega^3 \left(\Box_{\hat{g}} + \frac{1}{6}\mathrm{Scal}_{\hat{g}} \right) \Omega^{-1}\,. \tag{2.24}$$

Proof. We express the right-hand side of (2.24) in terms of the factor Ω and the metric g :

$$\begin{aligned} \Omega^3 \left(\Box_{\hat{g}} + \frac{1}{6}\mathrm{Scal}_{\hat{g}} \right) \Omega^{-1} &= \Omega^3 \left(\Box_{\hat{g}} + \frac{1}{6}\Omega^{-2}\mathrm{Scal}_g + \Omega^{-3}\Box_g \Omega \right) \Omega^{-1} \\ &= \Omega^3 \Box_{\hat{g}} \Omega^{-1} + \frac{1}{6}\mathrm{Scal}_g + \Omega^{-1}(\Box_g \Omega)\,. \end{aligned}$$

We apply the first term of the right-hand side to a scalar field ϕ and develop the expression in a local coordinate basis

$$\begin{aligned} \Omega^3 \Box_{\hat{g}} \Omega^{-1}\phi &= \Omega^3 \frac{1}{\sqrt{|\hat{g}|}} \partial_{\mathbf{a}} \sqrt{|\hat{g}|}\, \hat{g}^{\mathbf{ab}} \partial_{\mathbf{b}} \Omega^{-1}\phi \\ &= \Omega^{-1} \frac{1}{\sqrt{|g|}} \partial_{\mathbf{a}} \Omega^2 \sqrt{|g|}\, g^{\mathbf{ab}} \partial_{\mathbf{b}} \Omega^{-1}\phi \\ &= 2(\partial_{\mathbf{a}}\Omega) g^{\mathbf{ab}} \partial_{\mathbf{b}} (\Omega^{-1}\phi) + \Omega \Box_g (\Omega^{-1}\phi) \\ &= 2\langle \nabla\Omega\,,\ \nabla(\Omega^{-1}\phi)\rangle_g + \Omega(\Box_g \Omega^{-1})\phi + 2\Omega \langle \nabla\Omega^{-1}\,,\ \nabla\phi\rangle_g + \Box_g \phi \\ &= \Box_g \phi + 2\Omega^{-1}\langle \nabla\Omega\,,\ \nabla\phi\rangle_g - 2\Omega^{-2}\langle \nabla\Omega\,,\ \nabla\Omega\rangle_g \phi - 2\Omega^{-1}\langle \nabla\Omega\,,\ \nabla\phi\rangle_g \\ &\quad -\Omega\phi\nabla_a(\Omega^{-2}\nabla^a\Omega) \\ &= \Box_g \phi - 2\Omega^{-2}\langle \nabla\Omega\,,\ \nabla\Omega\rangle_g \phi + 2\Omega^{-2}\langle \nabla\Omega\,,\ \nabla\Omega\rangle_g \phi - \Omega^{-1}(\Box_g \Omega)\phi \\ &= \Box_g \phi - \Omega^{-1}(\Box_g \Omega)\phi\,. \end{aligned}$$

Putting things together gives (2.24) and proves the theorem. □

This has the immediate consequence :

Corollary 2.2. *Let $\phi \in \mathcal{D}'(\mathcal{M})$, the following conditions are equivalent :*

1. *ϕ satisfies (2.23) in the sense of distributions on $\mathcal{M}$;*

2. $\hat{\phi} := \Omega^{-1}\phi$ *satisfies*

$$\Box_{\hat{g}}\hat{\phi} + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\hat{\phi} = 0$$

in the sense of distributions on $\mathcal{M}$.

2.4 Exercises

Exercise 2.1. *Obtain the expression (2.2) of the Minkowski metric in spherical coordinates starting from its expression (2.1) in Cartesian coordinates.*

Exercise 2.2. *Calculate the Christoffel symbols associated to the Minkowski metric for Cartesian coordinates and for spherical coordinates. Conclude that the Christoffel symbols are not a tensor field.*

Exercise 2.3. *Check that the expression* (2.11) *of the volume form in definition 2.13 is independent of the coordinate system. Prove the equivalence of* (2.11) *and* (2.12).

Exercise 2.4. *Prove proposition 2.10.*

Exercise 2.5. *Prove that the* 10 *vectors listed in the last section of this chapter are indeed Killing vector fields.*

Exercise 2.6. *Prove expression* (2.20) *of the D'Alembertian.*

Chapter 3

The wave equation on flat space-time : integral formulae, Cauchy problem

The scalar wave equation, in the flat case, is the hyperbolic evolution equation on $\mathbb{R}_t \times \mathbb{R}^n_x$

$$\Box \phi = 0\,, \text{ where } \Box = \partial_t^2 - \Delta_x \text{ is the D'Alembertian on } \mathbb{R} \times \mathbb{R}^n\,. \tag{3.1}$$

3.1 Integral formulae

There are integral formulae giving either the general solution of equation (3.1) or of the Cauchy problem

$$\partial_t^2 \phi - \Delta_x \phi = 0 \text{ on } \mathbb{R}_t \times \mathbb{R}^n_x\,,\ \phi(0,.) = f\,,\ \partial_t \phi(0,.) = g\,. \tag{3.2}$$

3.1.1 $n = 1$

The first well-known formula giving solutions to the wave equation is due to D'Alembert ; it provides the general solution of (3.1) for $n = 1$. It was given by D'Alembert in 1747 [6] in the following form :

$$\phi(t,x) = F(x+t) + G(x-t)\,. \tag{3.3}$$

The proof is a simple exercise using a change of variables.

Remark 3.1. *1. It is indeed an integral formula, it can be written as follows*

$$\phi(t,x) = \int_{\{-1,1\}} \Phi(u, x+ut) \mathrm{d}(\delta_{-1} + \delta_1)(u) = \int_{S^0} \Phi(u, x+ut) \mathrm{d}\sigma(u)\,,$$

where $\Phi(-1,.) = G$ and $\Phi(1,.) = F$.

2. There is an apparent limitation attached to this formula. The functions F and G need to be continuous for it to make sense. There is no guarantee it is indeed providing the fully general solution of the wave equation. But the formula can be reformulated as follows :

$$\phi = \tau_{-t} F + \tau_t G\,,$$

where τ_t is the translation with respect to t defined as follows for continuous functions

$$(\tau_t f)(x) = f(x-t)$$

and which naturally extends to distributions by duality and invariance of the Lebesgue measure under translations.

3. *There is another common way of expressing D'Alembert's formula when one is interested in solving the Cauchy problem (3.2) for $n = 1$, it is as follows :*

$$\phi(t,x) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\mathrm{d}s\,. \qquad (3.4)$$

To make sense of it, g must be locally integrable on $\mathbb{R}$, which is a fairly weak constraint. It is easy to go from (3.3) to (3.4). We proceed as follows :

$$\begin{aligned}
f(x) := \phi(0,x) &= F(x) + G(x)\,,\\
g(x) := \frac{\partial\phi}{\partial t}(0,x) &= F'(x) - G'(x)\,,\\
\phi(t,x) &= F(x+t) + G(x-t)\\
&= \frac{1}{2}[(F(x+t) + G(x+t)) + (F(x+t) - G(x+t))]\\
&\quad + \frac{1}{2}[(F(x-t) + G(x-t)) - (F(x+t) - G(x+t))]\\
&= \frac{1}{2}f(x+t) + F(0) - G(0) + \frac{1}{2}\int_0^{x+t} g(s)\mathrm{d}s\\
&\quad + \frac{1}{2}f(x-t) - (F(0) - G(0)) - \frac{1}{2}\int_0^{x-t} g(s)\mathrm{d}s\\
&= \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\mathrm{d}s\,.
\end{aligned}$$

Note also that (3.4) gives us the expression of F and G in terms of the Cauchy data :

$$F(x) = \frac{1}{2}\left(f(x) + \int_0^x g(s)\mathrm{d}s\right)\,,\ G(x) = \frac{1}{2}\left(f(x) - \int_0^x g(s)\mathrm{d}s\right)\,.$$

3.1.2 $n = 1$ with symmetry

Proposition 3.1. *The form (3.4) of D'Alembert's formula shows that some symmetries of the data are transferred to the solution :*

1. *if f and g are even, then the solution (3.4) will be even in x for all t ;*

2. *if f and g are odd, then the solution (3.4) will be odd in x for all t.*

Proof. The proof simply uses the fact that if g is odd (resp. even), then its primitive which vanishes at the origin is even (resp. odd). The rest is an obvious direct calculation. □

3.1.3 $n = 3$

In 3 space dimensions, the classic formula is the Kirchhoff formula (or D'Adhémar-Fresnel-Kirchhoff formula), dating from the late XIXth century. It is in the spirit of the second form of the D'Alembert formula and gives the solution (provided it is $\mathcal{C}^2$) of the Cauchy problem (3.2) for $n = 3$ in terms of integrals of the initial data :

$$\begin{aligned}\phi(t,x) &= \frac{1}{4\pi t^2}\int_{S(x,t)} (f(y) + (y-x).\nabla f(y) + tg(y))\,\mathrm{d}\sigma(y) \\ &= \frac{1}{4\pi}\int_{S^2} (f(x+t\omega) + t\omega.(\nabla f)(x+t\omega) + tg(x+t\omega))\,\mathrm{d}^2\omega\,.\end{aligned}$$

Proof of the formula. It is done by the method of spherical means. We write the proof for $t > 0$, it can be extended to $t < 0$ simply by a time reflexion (meaning changing t into $-t$ and g into $-g$). Let $\phi \in \mathcal{C}^2(\mathbb{R}_t \times \mathbb{R}^3_x)$ a solution of (3.1) for $n = 3$. We define the average $U(x,t,r)$ of $\phi(t,y)$ on the sphere $S^2(x,r)$ in $\mathbb{R}^3$:

$$U(x,t,r) = \frac{1}{4\pi r^2}\int_{S^2(x,r)} \phi(t,y)\mathrm{d}\sigma(y) = \frac{1}{4\pi}\int_{|\xi|=1} \phi(t,x+r\xi)\mathrm{d}\sigma(\xi)\,. \quad (3.5)$$

The fact that ϕ satisfies the wave equation implies that U satisfies a partial differential equation purely in the variables t and r. Let us start by evaluating the derivative of U with respect to r :

$$\begin{aligned}\frac{\partial U}{\partial r}(x,t,r) &= \frac{1}{4\pi}\int_{|\xi|=1} \xi^i \frac{\partial \phi}{\partial x^i}(t,x+r\xi)\mathrm{d}\sigma(\xi) \text{ (outgoing flux of the gradient)} \\ &= \frac{1}{4\pi}\int_{|\xi|<1} \mathrm{div}_\xi(\nabla\phi(t,x+r\xi))\mathrm{d}\xi \\ &= \frac{r}{4\pi}\int_{|\xi|<1} \Delta\phi(t,x+r\xi)\mathrm{d}\xi \\ &= \frac{r}{4\pi}\int_{|\xi|<1} \frac{\partial^2\phi}{\partial t^2}(t,x+r\xi)\mathrm{d}\xi \\ &= \frac{r}{4\pi}\frac{\partial^2}{\partial t^2}\int_{|\xi|<1} \phi(t,x+r\xi)\mathrm{d}\xi \qquad (3.6) \\ &= \frac{1}{4\pi r^2}\frac{\partial^2}{\partial t^2}\int_{|y-x|<r} \phi(t,y)\mathrm{d}y\,.\end{aligned}$$

Now we express the volume integral in terms of the spherical average function U :

$$\frac{1}{4\pi}\int_{|y-x|<r} \phi(t,y)\mathrm{d}y = \int_{]0,r[} \rho^2 U(x,t,\rho)\mathrm{d}\rho\,.$$

It follows that

$$r^2\frac{\partial U}{\partial r}(x,t,r) = \frac{\partial^2}{\partial t^2}\int_{]0,r[} \rho^2 U(x,t,\rho)\mathrm{d}\rho = \int_{]0,r[} \rho^2 \frac{\partial^2}{\partial t^2} U(x,t,\rho)\mathrm{d}\rho\,.$$

Taking the derivative with respect to r we get

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial t^2}\,.$$

Hence $\psi(x,t,r) = rU(x,t,r)$ satisfies

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial r^2} = 0$$

which is the wave equation in one space dimension. The function ψ, for a given x, is a priori only defined on $\mathbb{R}_t \times \mathbb{R}^+_x$ but the second expression of U in (3.5) is defined for any $r \in \mathbb{R}$, is even in r and is $\mathcal{C}^2$ in (x,t,r) for $r > 0$. Moreover, (3.6) shows that

$$\frac{\partial U}{\partial r} \to 0 \text{ as } r \to 0\,.$$

So ψ extends naturally for each x as a function that is $\mathcal{C}^2$ on $\mathbb{R}_t \times \mathbb{R}_r$ (in fact in $\mathcal{C}^2(\mathbb{R}_x \times \mathbb{R}_t \times \mathbb{R}_r)$) and odd in r, which consequently satisfies the wave equation on the whole (t,r)-plane for each x. We denote by $\tilde{\psi}$ the extension of ψ to $\mathbb{R}_x \times \mathbb{R}_t \times \mathbb{R}_r$ that is odd in r. Hence, putting

$$\tilde{f}(x,r) = \tilde{\psi}(x,0,r)\,,\ \tilde{g}(x,r) = \frac{\partial \tilde{\psi}}{\partial t}(x,0,r)\,,$$

using D'Alembert's formula we get

$$\tilde{\psi}(x,t,r) = \frac{1}{2}(\tilde{f}(x,r+t) + \tilde{f}(x,r-t)) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{g}(x,s)\mathrm{d}s\,.$$

Using the fact that $\tilde{f}$ and $\tilde{g}$ are odd in r, we deduce the following expressions for $\psi(x,t,r)$ in the domains $r \geq t \geq 0$ and $t \geq r \geq 0$ respectively :

$$\begin{aligned}
\psi(x,t,r) &= \frac{1}{2}(\tilde{f}(x,r+t) + \tilde{f}(x,r-t)) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{g}(x,s)\mathrm{d}s \quad \text{for } r \geq t > 0\,,\\
\psi(x,t,r) &= \frac{1}{2}(\tilde{f}(x,r+t) - \tilde{f}(x,t-r)) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{g}(x,s)\mathrm{d}s \quad \text{for } t > r \geq 0
\end{aligned}$$

Now we divide by r and take the limit as $r \to 0$. So only the form for $t > r > 0$ is useful and we obtain

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{\tilde{f}(x,r+t) - \tilde{f}(x,t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{g}(x,s)\mathrm{d}s \right)\\
&= \frac{\partial \tilde{\psi}}{\partial r}(x,0,r=t) + \frac{\partial \tilde{\psi}}{\partial t}(x,0,r=t)\\
&= t\frac{\partial U}{\partial r}(x,0,r=t) + U(x,0,r=t) + t\frac{\partial U}{\partial t}(x,0,r=t)
\end{aligned}$$

and using the second expression of U in (3.5) we obtain the formula. □

This formula has a remarkable consequence which is often referred to as the Huygens principle (or strong Huygens principle in contrast with some weak versions we shall encounter later in the course) and stating that for the wave equation in 3 space dimensions, the information travels exactly at speed 1 :

Theorem 3.1 (Huygens principle)**.** *For $n = 3$, if the data for the wave equation (3.1) at $t = 0$ are supported in the ball $B(0, R)$, then the associated solution ϕ satisfies*

$$\phi(t, x) = 0 \text{ for } |x| \leq |t| - R\,.$$

In 1903, Whittaker [27] obtained another integral formula which is comparable to the first form of D'Alembert's formula. It provides "general" solutions of the wave equation on $\mathbb{R}_t \times \mathbb{R}^3$ with no particular relation to the Cauchy problem :

$$\phi(t, x) = \int_{S^2} \Phi(x.\omega - t, \omega)\mathrm{d}\omega\,. \tag{3.7}$$

It is quite different from the Kirchhoff formula, the most striking aspect being the presence of only one arbitrary function on $\mathbb{R} \times S^2$ instead of two functions on $\mathbb{R}^3$. We shall see that its natural interpretation is in terms of scattering theory. Whittaker's proof was a direct calculation, similar to the one he used to obtain an integral formula characterizing harmonic functions on $\mathbb{R}^3$. We will not give his proof here. We shall see later a different and much less direct proof of this formula in the Lax-Phillips version of scattering theory.

3.1.4 $n = 2$

Deduce a formula from the $n = 3$ case. In even space dimensions, the propagation exhibits a very different behaviour from the odd dimensional cases, namely, the Huygens principle is valid only for space dimensions $n \geq 3$ and odd. We will only be interested in the odd space dimension cases and mostly in $n = 3$.

3.2 The Cauchy problem

The Cauchy problem (3.2) can be made sense of and solved in very general function spaces, provided we keep some sort of time regularity to allow us to give a meaning to the initial data conditions. Several methods can be used to solve (3.2). Some will be adapted to these very general function spaces, others will bring their own sets of function spaces, which, although less general, will provide ideal frameworks for developing scattering theories.

3.2.1 Spectral approach

In this section we work with $n \geq 3$ (see footnote 1 below). We write the wave equation in its Hamiltonian form, i.e. as a Schrödinger equation :

$$\partial_t U = iAU\,,\; U := \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix}\,,\; A = -i \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\,. \tag{3.8}$$

Theorem 3.2. *The operator A is self-adjoint on $\mathcal{H} = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, completion of $\mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{C}_0^\infty(\mathbb{R}^n)$ in the norm*

$$\|U\|^2_{\mathcal{H}} := \int_{\mathbb{R}^n} (|\nabla_x u_1|^2 + |u_2|^2)\mathrm{d}x\,.$$

Proof. First for $U \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have $AU \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{C}_0^\infty(\mathbb{R}^n) \subset \mathcal{H}$, so the domain of A contains $\mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{C}_0^\infty(\mathbb{R}^n)$ and is therefore dense in $\mathcal{H}$. Let us prove that A is symmetric. Let

$$U = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \; V = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{C}_0^\infty(\mathbb{R}^n) ,$$

$$\begin{aligned} \langle AU, V \rangle_{\mathcal{H}} &= -i \langle \begin{pmatrix} \phi_2 \\ \Delta\phi_1 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rangle_{\mathcal{H}} \\ &= -i \int_{\mathbb{R}^n} \left(\nabla\phi_2 . \nabla\overline{\psi_1} + \Delta\phi_1 \overline{\psi_2} \right) \mathrm{d}x \\ &= i \int_{\mathbb{R}^n} \left(\phi_2 . \Delta\overline{\psi_1} + \nabla\phi_1 . \nabla\overline{\psi_2} \right) \mathrm{d}x \\ &= \langle \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, -i \begin{pmatrix} \psi_2 \\ \Delta\psi_1 \end{pmatrix} \rangle_{\mathcal{H}} \\ &= \langle U, AV \rangle_{\mathcal{H}} . \end{aligned}$$

The symmetry on $D(A)$ follows by density. It remains to show that $D(A^*) \subset D(A)$. For $n \geq 3$, $\dot{H}^1(\mathbb{R}^n)$ is a space of distributions[1] so it is easy to understand A^* as a differential operator and to determine its domain. Let

$$U = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H} ,$$

then $U \in D(A^*)$ if and only if the map

$$V \in D(A) \mapsto \langle AV, U \rangle_{\mathcal{H}}$$

extends as a linear continuous map on $\mathcal{H}$. This map is a distribution which we can evaluate in terms of ϕ_1 and ϕ_2 in the usual manner : consider

$$V = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{C}_0^\infty(\mathbb{R}^n) \times \mathcal{C}_0^\infty(\mathbb{R}^n) ,$$

$$\begin{aligned} \langle AV, U \rangle_{\mathcal{H}} &= \langle -i \begin{pmatrix} \psi_2 \\ \Delta\psi_1 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rangle_{\mathcal{H}} \\ &= i \int_{\mathbb{R}^n} \left(\overline{\phi_2} . \Delta\psi_1 + \nabla\overline{\phi_1} . \nabla\psi_2 \right) \mathrm{d}x \\ &= i \langle \overline{\phi_2}, \Delta\psi_1 \rangle_{\mathcal{D}', \mathcal{C}_0^\infty} + i \langle \nabla\overline{\phi_1}, \nabla\psi_2 \rangle_{\mathcal{D}', \mathcal{C}_0^\infty} \\ &= -i \langle \nabla\overline{\phi_2}, \nabla\psi_1 \rangle_{\mathcal{D}', \mathcal{C}_0^\infty} - i \langle \Delta\overline{\phi_1}, \psi_2 \rangle_{\mathcal{D}', \mathcal{C}_0^\infty} . \end{aligned}$$

This extends as a continuous linear map on $\mathcal{H}$ if and only if

$$\nabla\overline{\phi_2} \text{ and } \Delta\overline{\phi_1} \text{ are in } L^2(\mathbb{R}^n) ,$$

[1] This is not the case for $n = 1$ and $n = 2$, see Soga 1983 [25], p. 732.

which is equivalent to $AU \in \mathcal{H}$, i.e. to $U \in D(A)$. Therefore $D(A^*) = D(A)$ and the proof is complete. □

Consequently, by Stone's theorem and remark 2.9, the Cauchy problem (3.2) is well-posed in $\mathcal{H}$ and all the successive domains of A in $\mathcal{H}$.

Remark 3.2. *One might feel that the purely spectral approach for the wave equation is somewhat unsatisfactory, the space $\mathcal{H}$ being a little awkward because of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$. This would be particularly true for $n = 1$ and $n = 2$ since $\dot{H}^1(\mathbb{R})$ is not even a distribution space (see footnote 1 above). But even for $n \geq 3$, the domains of A are not easy to understand because of the lack of L^2 control on the first component of U. However, such function spaces are natural for constructing a scattering theory for the wave equation on asymptotically flat backgrounds, whether one uses a spectral approach or a conformal approach (see the example of the Schwarzschild metric with the spectral approach [7, 8, 9] and the conformal approach [19]).*

3.2.2 Fourier transform

We look for a solution $\phi \in \mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}'(\mathbb{R}^n))$ (resp. $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}(\mathbb{R}^n))$) of (3.2). If ϕ is in such a distribution space, then ϕ satisfies (3.2) if and only if its Fourier transform in space $\hat{\phi}(t,\xi)$ satisfies

$$\partial_t^2 \hat{\phi} + |\xi|^2 \hat{\phi} = 0 \text{ on } \mathbb{R}_t \times \mathbb{R}^n_x\,,\ \hat{\phi}(0,.) = \hat{f}\,,\ \partial_t \hat{\phi}(0,.) = \hat{g}\,. \tag{3.9}$$

The problem (3.9) has a unique solution in $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}'(\mathbb{R}^n))$ (resp. $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}(\mathbb{R}^n))$) given by

$$\hat{\phi}(t,\xi) = \hat{f}(\xi)\cos(t|\xi|) + \hat{g}(\xi)\frac{\sin(t|\xi|)}{|\xi|}\,. \tag{3.10}$$

This is due to the fact that the functions $\cos(t|\xi|)$ and $\frac{\sin(t|\xi|)}{|\xi|}$ are smooth in t with values in smooth (in fact analytic) functions in ξ with moderate growth at infinity, and are therefore continuous multipliers of both spaces $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}'(\mathbb{R}^n))$ and $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}(\mathbb{R}^n))$. This proves the following theorem, the last statement in the theorem being a straightforward consequence of what we just established and the equation.

Theorem 3.3. *Given $f, g \in \mathcal{S}'(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$), the Cauchy problem (3.2) for the wave equation has a unique solution in $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}'(\mathbb{R}^n))$ (resp. $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{S}(\mathbb{R}^n))$) given by*

$$\phi(t,.) = \mathcal{F}_\xi^{-1}\left(\hat{f}(\xi)\cos(t|\xi|) + \hat{g}(\xi)\frac{\sin(t|\xi|)}{|\xi|}\right)\,.$$

This solution is in fact in $\mathcal{C}^\infty(\mathbb{R}_t\,;\ \mathcal{S}'(\mathbb{R}^n))$ (resp. $\mathcal{C}^\infty(\mathbb{R}_t\,;\ \mathcal{S}(\mathbb{R}^n))$)

Moreover it is immediate to check, using the characterization of Sobolev spaces via the Fourier transform, that the Cauchy problem is also well-posed in any Sobolev space :

Corollary 3.1. *Given $s \in \mathbb{R}$, $f \in H^s(\mathbb{R}^n)$, $g \in H^{s-1}(\mathbb{R}^n)$, the associated solution ϕ of the Cauchy problem (3.2) satisfies :*

$$\phi \in \mathcal{C}(\mathbb{R}_t\,;\ H^s(\mathbb{R}^n)) \cap \mathcal{C}^1(\mathbb{R}_t\,;\ H^{s-1}(\mathbb{R}^n))\,.$$

3.2.3 Fundamental solutions

G the backward fundamental solution ($G(x,y) = \delta((x'-y')^2)\theta(-x_0+y_0)$ essentially). Then write

$$\phi(x) = \int_{\mathbb{R}}^4 (\phi(y)\Box G(x,y) - G(x,y)\Box\phi(y))\,\mathrm{d}^4y = \int_{\mathbb{R}}^4 \nabla.\,(\phi(y)\nabla G(x,y) - G(x,y)\nabla\phi(y))\,\mathrm{d}^4y\,.$$

Integrate by parts on the future of some initial data hypersurface (null or spacelike) and since the fundamental solution has support limited in the future, we get a boundary term that is purely on the data hypersurface and formally reads

$$\phi(x) = \int_\Sigma (\phi(y)\nabla_n G(x,y) - G(x,y)\nabla_n\phi(y))\,\mathrm{d}^3y\,.$$

3.3 Exercises

Exercise 3.1. *Prove D'Alembert's formula (3.3).*

Exercise 3.2. *Prove that the Cauchy problem for the Klein-Gordon equation on $\mathbb{R}_t \times \mathbb{R}^n_x$*

$$\partial_t^2\phi - \Delta\phi + m^2\phi = 0$$

is well-posed in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Exercise 3.3. *Same exercise as the previous one with equation*

$$\partial_t^2\phi - \Delta\phi + m^2\phi + P(t,x)\phi = 0$$

where P is a continuous bounded function on $\mathbb{R}_t \times \mathbb{R}^n_x$.

Chapter 4

Energy estimates

4.1 The flat case

In this section, we present the methods of energy estimates for the wave equation on flat spacetime.

The Fourier transform approach provides the existence of solutions in spaces of very weak regularity (tempered distributions) and very strong regularity (the Schwartz class of smooth rapidly decreasing functions). The well-posedness in the Schwartz class together with energy estimates allow us to recover, for $s \in \mathbb{N}^*$, the results of corollary 3.1, with the important difference that this strategy carries over to general curved spacetimes. We describe energy estimates and their many applications in the next section ; the use of these techniques to solve the Cauchy problem in various Sobolev spaces will be presented there.

There are two essentially equivalent ways of understanding the principle of energy estimates. The first, which is familiar to most PDE analysts, is to multiply equation (3.1) by a well-chosen directional derivative of the solution ϕ and to integrate the result by parts on a domain of $\mathbb{R}^{n+1}$ with piecewise $\mathcal{C}^1$ boundary. The second is much more geometrical and comes from the physical/geometrical invariance properties of the equation ; it is the basis of so-called vector field methods or geometric energy estimates techniques.

4.1.1 Analytical approach : finite propagation speed

We explain this description of energy estimate on a particular example and use it to establish the finite propagation speed of the solution. Consider $\phi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n)$ solution of (3.1) : we have

$$0 = \partial_t \phi \left(\partial_t^2 \phi - \Delta_x \phi \right) .$$

We integrate this by parts on the domain

$$\Omega_{R,T} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \,;\; 0 \leq t \leq T \,,\; |x| \leq R + T - t\} \tag{4.1}$$

and denote the three pieces of the boundary of $\Omega_{R,T}$ by

$$\begin{aligned}\Sigma_T &= \{(t,x)\in\mathbb{R}\times\mathbb{R}^n\,;\ t=T\,,\ |x|\leq R\}\,, &(4.2)\\ \Sigma_0 &= \{(t,x)\in\mathbb{R}\times\mathbb{R}^n\,;\ t=0\,,\ |x|\leq R+T\}\,, &(4.3)\\ S &= \{(t,x)\in\mathbb{R}\times\mathbb{R}^n\,;\ 0\leq t\leq T\,,\ |x|=R+T-t\}\,. &(4.4)\end{aligned}$$

We obtain

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\Sigma_T}\left(|\partial_t\phi|^2+|\nabla\phi|^2\right)\mathrm{d}^3x\\ &\quad -\frac{1}{2}\int_{\Sigma_0}\left(|\partial_t\phi|^2+|\nabla\phi|^2\right)\mathrm{d}^3x\\ &\quad +\frac{1}{2\sqrt{2}}\int_S\left(|\partial_t\phi|^2+|\nabla\phi|^2+2\partial_t\phi\frac{x}{|x|}.\nabla\phi\right)\mathrm{d}\sigma\,.\end{aligned}$$

The last integral being non negative, we see that

$$\frac{1}{2}\int_{\Sigma_T}\left(|\partial_t\phi|^2+|\nabla\phi|^2\right)\mathrm{d}^3x\leq\frac{1}{2}\int_{\Sigma_0}\left(|\partial_t\phi|^2+|\nabla\phi|^2\right)\mathrm{d}^3x\,. \tag{4.5}$$

In particular, if the solution is zero for $|x|\leq R+T$ at $t=0$, this implies that it must also be zero at $t=T$ for $|x|\leq R$, i.e. the information propagates at most at speed 1.

Remark 4.1. *Note that this result is weaker than the Huygens principle which gives an exact propagation speed. But it is also more general : unlike the Huygens principle, this property will be valid for perturbations of the wave equation by first or zero order terms and it can also be extended, using the same method, to similar equations on curved backgrounds.*

Finite propagation speed and theorem (3.3) entail the following result

Theorem 4.1. *Given $f,g\in\mathcal{C}_0^\infty(\mathbb{R}^n)$, the Cauchy problem (3.2) for the wave equation has a unique solution in $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{C}_0^\infty(\mathbb{R}^n))$. This solution is in fact in $\mathcal{C}^\infty(\mathbb{R}_t\,;\ \mathcal{C}_0^\infty(\mathbb{R}^n))$.*

By duality, we deduce well-posedness with general distribution data :

Corollary 4.1. *Given $f,g\in\mathcal{D}'(\mathbb{R}^n)$, the Cauchy problem (3.2) for the wave equation has a unique solution in $\mathcal{C}^1(\mathbb{R}_t\,;\ \mathcal{D}'(\mathbb{R}^n))$. This solution is in fact in $\mathcal{C}^\infty(\mathbb{R}_t\,;\ \mathcal{D}'(\mathbb{R}^n))$.*

4.1.2 Geometrical approach

From now on, we consider only the case $n=3$ corresponding to the framework of special relativity. We denote by $(\mathbb{M},\eta)$ the 4-dimensional Minkowski spacetime.

The wave equation (3.1) has a conserved stress-energy tensor. It is the symmetric 2-tensor

$$T_{ab}=\partial_a\phi\partial_b\phi-\frac{1}{2}\langle\nabla\phi\,,\ \nabla\phi\rangle_\eta\eta_{ab}\,, \tag{4.6}$$

where η is the Minkowski metric

$$\eta=\mathrm{d}t^2-e_{\mathbb{R}^3}\,,\ e_{\mathbb{R}^3}\text{ being the euclidean metric on }\mathbb{R}^3\,,$$

and $\langle . \, , . \rangle$ is the inner product induced by η, i.e.

$$\langle \nabla \phi \, , \nabla \phi \rangle_\eta = \eta^{ab} \nabla_a \phi \nabla_b \phi = |\partial_t \phi|^2 - |\nabla_x \phi|^2 \, .$$

The stress-energy tensor (4.6) satisfies the following fundamental property.

Proposition 4.1.

$$\nabla^a T_{ab} = (\nabla_b \phi) \, \Box \phi \tag{4.7}$$

and therefore

$$\nabla^a T_{ab} = 0 \tag{4.8}$$

whenever ϕ satisfies the wave equation.

Proof. It is a direct calculation :

$$\begin{aligned}
\nabla^a T_{ab} &= \nabla^a \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} \nabla_c \phi \nabla^c \phi \eta_{ab} \right) \\
&= (\Box \phi) \nabla_b \phi + \nabla_a \phi \nabla^a \nabla_b \phi - (\nabla^a \nabla_c \phi) \, (\nabla^c \phi) \eta_{ab} \\
&= (\Box \phi) \nabla_b \phi + \nabla_c \phi \nabla_b \nabla^c \phi - (\nabla^a \nabla_c \phi) \, (\nabla^c \phi) \eta_{ab} \text{ no torsion} \\
&= (\Box \phi) \nabla_b \phi + (\nabla^c \phi)(\nabla^a \nabla_c \phi) \eta_{ab} - (\nabla^a \nabla_c \phi) \, (\nabla^c \phi) \eta_{ab} \\
&= (\Box \phi) \nabla_b \phi \, . \quad \Box
\end{aligned}$$

A “family of local observers” is described by a timelike vector field. From a stress energy tensor, an **energy current** can be inferred by contracting it with a timelike vector field. Note that it is also sometimes interesting to consider energy currents associated with spacelike or null vector fields, they do not correspond to a physical measurement of energy current by a realistic observer but they can give useful information nonetheless, like local dispersion of energy for instance.

The conservation law (4.8) is not directly usable because it does not readily provide a conserved current (i.e. a divergence-free vector field). However, the symmetries of flat spacetime will allow us to infer many conserved currents from T_{ab}.

Proposition 4.2. *Let K be a Killing vector field on $\mathbb{M}$, then the vector field*

$$J^a = K^b T_b^a$$

is divergence-free.

Proof. It is a direct consequence of the Killing equation and the symmetry of the stress-energy tensor :

$$\nabla^a J_a = K^b \nabla^a T_{ab} + T_{ab} \nabla^a K^b = K^b \nabla^a T_{ab} + T_{ab} \nabla^{(a} K^{b)}$$

and this is zero since K is Killing and by the conservation law (4.8). $\Box$

Minkowski spacetime has a 10-dimensional group of isometries : the Poincaré group. Its associated Lie algebra is the 10-dimensional vector space of all Killing vector fields of $\mathbb{M}$, a basis of which is made of :

- ∂_t, ∂_{x^1}, ∂_{x^2}, ∂_{x^3}, generating translations ;
- $x^1\partial_{x^2} - x^2\partial_{x^1}$, $x^2\partial_{x^3} - x^3\partial_{x^2}$, $x^3\partial_{x^1} - x^1\partial_{x^3}$, generating spatial rotations ;
- $t\partial_{x^1} + x^1\partial_t$, $t\partial_{x^2} + x^2\partial_t$, $t\partial_{x^3} + x^3\partial_t$, generating boosts.

This gives us 10 independent conserved currents.

An important property of the stress-energy tensor for the wave equation is that when we contract it with a future-oriented timelike vector field and calculate its flux across a spacelike hypersurface with future-oriented normal, we obtain a positive energy. This is the so-called "dominant energy condition".

Proposition 4.3 (Dominant energy condition)**.** *The stress-energy tensor T_{ab} satisfies the dominant energy condition : for every future-oriented causal vector field V, the vector field $T^a_bV^b$ is itself causal and future-pointing. Another equivalent way of stating the dominant energy condition is the following : for all future-oriented causal vector fields V, W, we have $T_{ab}V^aW^b \geq 0$.*

Proof. Let V be a future-oriented causal vector field, i.e.

$$V = v^0\partial_t + V'\,,\ v^0 \geq |V'|\,,$$

put

$$W^a = T^a_bV^b\,.$$

We have

$$\begin{aligned} W^0 &= \partial_t\phi\nabla_V\phi - \frac{1}{2}((\partial_t\phi)^2 - |\nabla_x\phi|^2)V^0 \\ &= \frac{1}{2}V^0((\partial_t\phi)^2 + |\nabla_x\phi|^2) + \partial_t\phi\nabla_{V'}\phi \\ &\geq \frac{1}{2}V^0((\partial_t\phi)^2 + |\nabla_x\phi|^2) - |\partial_t\phi|\,|V'|\,|\nabla_x\phi|^2 \\ &\geq \frac{1}{2}((\partial_t\phi)^2 + |\nabla_x\phi|^2)(V^0 - |V'|) \geq 0\,. \end{aligned}$$

So if W^a is causal, it is future-oriented. Let us check the causality :

$$\begin{aligned} g_{ab}W^aW^b &= g_{ab}(\nabla^a\phi\nabla_V\phi - \frac{1}{2}\langle\nabla\phi\,,\ \nabla\phi\rangle V^a)(\nabla^b\phi\nabla_V\phi - \frac{1}{2}\langle\nabla\phi\,,\ \nabla\phi\rangle V^b) \\ &= (\nabla_V\phi)^2\langle\nabla\phi\,,\ \nabla\phi\rangle - (\nabla_V\phi)^2\langle\nabla\phi\,,\ \nabla\phi\rangle + \frac{1}{4}\langle\nabla\phi\,,\ \nabla\phi\rangle^2\langle V\,,\ V\rangle \\ &= \frac{1}{4}\langle\nabla\phi\,,\ \nabla\phi\rangle^2\langle V\,,\ V\rangle \geq 0 \text{ since } V^a \text{ is causal.} \end{aligned}$$

This proves the proposition. □

This property allows to establish estimate (4.5) by a different approach using more geometrical ingredients. We consider the energy current

$$J^a = T^a_b(\partial_t)^b = T^a_0\,,$$

corresponding to the perception of a static observer (whose velocity 4-vector is given by ∂_t) ; recall that the current is conserved because ∂_t is Killing. We integrate the divergence of J over the domain $\Omega_{R,T}$ with boundary made of Σ_T, Σ_0 and S as defined in (4.1), (4.2), (4.3) and (4.4). Denoting by E_{Σ_T} and E_{Σ_0} the energy fluxes across Σ_T and Σ_0 oriented by ∂_t and by E_S the outgoing energy fluxes across S, we get

$$E_{\Sigma_T} + E_S - E_{\Sigma_0} = 0\,,$$

all fluxes being calculated using the expression in Theorem 2.4. For the first two fluxes, we take $l = n = \partial_t$:

$$E_{\Sigma_T} = \frac{1}{2}\int_{\Sigma_T} \left(|\partial_t\phi|^2 + |\nabla\phi|^2\right) \mathrm{d}^3x\,, \tag{4.9}$$

$$E_{\Sigma_0} = \frac{1}{2}\int_{\Sigma_0} \left(|\partial_t\phi|^2 + |\nabla\phi|^2\right) \mathrm{d}^3x\,. \tag{4.10}$$

As for E_S, taking

$$n = \frac{1}{\sqrt{2}}(\partial_t - \partial_r)\,,\ l = \frac{1}{\sqrt{2}}(\partial_t + \partial_r)\,,$$

we have

$$E_S = \int_S T_{ab}(\partial_t)^a N^b L \lrcorner \mathrm{dVol} \geq 0$$

by the dominant energy condition. This gives (4.5).

4.2 Energy estimates on a general spacetime

The multiplyer technique can be used in a general curved framework just as in the flat case using a local coordinate system. We present here the geometrical method involving a stress-energy tensor and a choice of observer (or vector field in general), for the wave equation

$$\Box_g \phi = 0\,, \tag{4.11}$$

on a spacetime $(\mathcal{M}, g)$. Equation (4.11) has a conserved stress-energy tensor

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}\langle\nabla\phi\,,\ \nabla\phi\rangle_g g_{ab}\,, \tag{4.12}$$

satisfying

$$\nabla^a T_{ab} = (\nabla_b\phi)\,\Box_g\phi\,,$$

the proof being identical to the flat case.

Once again, we have a conservation law that cannot be used directly and we must contract T_{ab} with a vector field V^a (usually timelike but not always) in order to get an energy current

$$J^a = K^b T_b^a\,.$$

In a general situation, we have no Killing vector field and we get the following expression for the divergence of the energy current :

$$\nabla^a J_a = \nabla_V \phi \,\Box_g \phi + T_{ab} \nabla^{(a} V^{b)} \,,$$

which, for ϕ solution of (4.11), simplifies to

$$\nabla^a J_a = T_{ab} \nabla^{(a} V^{b)} \,, \tag{4.13}$$

Remark 4.2. *Note that T_{ab} satisfies the dominant energy condition, the proof being identical to the flat case using an orthonormal basis at each point.*

Now consider S a closed hypersurface whose interior we denote Ω, S being oriented by the outgoing normal. We have the following equality from the divergence theorem (Theorem 2.4) :

$$E_S = \int_\Omega T_{ab} \nabla^{(a} V^{b)} \mathrm{dVol} \,.$$

If V^a is causal and future-oriented, we know that on parts of S where the outgoing normal is also causal and future-oriented, the flux is non-negative.

4.3 Exercises

Exercise 4.1. *Prove equality (4.7).*

Exercise 4.2. *Show that on any space-time $(\mathcal{M}, g)$, for any scalar field ϕ and for any $m \in \mathbb{R}$, the tensor*

$$T_{ab} := \partial_a \phi \partial_b \phi - \frac{1}{2} \langle \nabla \phi \,,\ \nabla \phi \rangle_g g_{ab} + \frac{1}{2} m^2 \phi^2$$

satisfies the dominant energy condition.

Exercise 4.3. *Prove equality (4.9).*

Chapter 5

Scattering theory

The principle of scattering theory (at least of time-dependent scattering theory) is to analyse the asymptotic behaviour of solutions to a field equation along its characteristics by finding a simplification of the equation in the asymptotic region considered and proving that the field approaches solutions to the simplified system in this region. A complete scattering theory will also say that the field is completely described by the simplified solutions it approaches asymptotically.

For the wave equation on $\mathbb{R}_t \times \mathbb{R}_x$, the scattering theory is immediate. There are two asymptotic regions, also called scattering channels : $r \to -\infty$ and $r \to +\infty$. Consider a generic solution of equation (3.1) for $n = 1$ such that both its incoming and outgoing parts have finite energy, i.e.

$$\phi(t, x) = F(x+t) + G(x-t) \text{ with } F, G \in H^1(\mathbb{R}) .$$

We recall the important property of $H^1(\mathbb{R})$:

Proposition 5.1. *Let $f \in H^1(\mathbb{R})$ then f is continuous on $\mathbb{R}$ and tends to zero at infinity.*

Proof. For any $f \in \mathcal{C}_0^\infty(\mathbb{R})$, we have

$$(f(x))^2 = \int_{-\infty}^{x} 2f(t)f'(t)\mathrm{d}t \leq \int_{-\infty}^{x} ((f'(t))^2 + (f(t))^2)\mathrm{d}t \leq \|f\|_{H^1}^2 .$$

Whence

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})} . \tag{5.1}$$

This implies, by a standard density argument (exercise 5.1), that $H^1(\mathbb{R}) \hookrightarrow \mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. As a consequence of this, the equality

$$(f(x))^2 = \int_{-\infty}^{x} 2f(t)f'(t)\mathrm{d}t ,$$

which is valid for any $f \in \mathcal{C}_0^\infty(\mathbb{R})$, also extends to elements of $H^1(\mathbb{R})$. Indeed, for $f \in H^1(\mathbb{R})$, consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty(\mathbb{R})$ which converges towards f in $H^1(\mathbb{R})$. We have for each $n \in \mathbb{N}$,

$$(f_n(x))^2 = \int_{-\infty}^{x} 2f_n(t)f_n'(t)\mathrm{d}t .$$

Now $f_n \to f$ and $f_n' \to f'$ in $L^2(\mathbb{R})$, which entails $f_n f_n' \to f f'$ in $L^1(\mathbb{R})$ and gives the convergence of the integral in the right-hand side. But also $f_n \to f$ in $\mathcal{C}^0(\mathbb{R})$ which implies the convergence of the left-hand side. We conclude by remarking that for $f \in H^1(\mathbb{R})$, $ff' \in L^1(\mathbb{R})$ and therefore

$$\lim_{x \to -\infty} \int_{-\infty}^{x} 2f(t)f'(t)\mathrm{d}t = 0\,.$$

The limit at $+\infty$ can be treated similarly. □

If we follow ϕ along an incoming null geodesic $t = -x + C$, we get

$$\phi(-x + C, x) = F(C) + G(2x - C) \to F(C) \text{ as } x \to -\infty\,.$$

Similarly, along an outgoing null geodesic $t = x + C$,

$$\phi(x + C, x) = F(2x + C) + G(-C) \to G(-C) \text{ as } x \to +\infty\,.$$

We see that along the incoming null lines ϕ approaches $F(x + t)$ which is a solution of the simplified equation

$$(\partial_t - \partial_x)\upsilon = 0\,,$$

and along the incoming null lines ϕ approaches $G(x - t)$ which is a solution of the simplified equation

$$(\partial_t + \partial_x)\upsilon = 0\,.$$

Moreover the solution ϕ is entirely characterized by the solutions of the simplified equations it approaches in the two scattering channels.

We shall start with a similar construction for the wave equation on 4-dimensional Minkowski spacetime using the classic method of Cook which relies on some sort of Huyghens principle. Then, we shall present an alternative approach that is based on spectral theory and makes contact with very geometrical structures : the Lax-Phillips theory [14]. Its essential ingredient is a translation representer of the evolution. We first explain the construction of the translation representation on the simple example of a differential system, then describe Lax and Phillips's treatment of the wave equation on $\mathbb{M}$ and its relation to the Whittaker formula.

5.1 A classic scattering construction : Cook's method

We shall see here the usual ingredients of scattering theory for the first time : spaces of incoming and outgoing data, comparison dynamics, identifying operator, wave operators and scattering operator.

We start by expressing the wave equation on $\mathbb{M}$ in spherical coordinates

$$\partial_t^2 \phi - \partial_r^2 \phi - \frac{2}{r}\partial_r \phi - \frac{1}{r^2}\Delta_{S^2}\phi = 0\,. \tag{5.2}$$

In scattering theory, there is a crucial difference between short-range perturbations which fall-off like $r^{-\alpha}$ with $\alpha > 1$ and long-range perturbations which fall off like $r^{-\alpha}$ with $\alpha \leq 1$. This is of course a question of integrability of these quantities at infinity and the remarkable thing is that the space dimension is irrelevant, it is always a matter of integrability in 1-dimension ; this will become clearer when we see Cook's method. Short-range perturbations can be treated naturally provided we have some weak version of the Huygens principle, but long-range perturbations require an in depth modification of the construction which reveals the profound change they induce in the asymptotic behaviour. So it is crucial to understand, when long-range terms are present, whether they are genuine or artificial. Here the term $\frac{2}{r}\partial_r\phi$ is artificially long-range since it can be eliminated by a simple rescaling of the unknown function. Putting

$$\psi = r\phi \,, \tag{5.3}$$

we get that $\phi \in \mathcal{D}'(\mathbb{R}^4)$ satisfies the wave equation on $\mathbb{M}$ (equivalently (5.2)) if and only if ψ is a solution of the simplified equation

$$\partial_t^2\psi - \partial_r^2\psi - \frac{1}{r^2}\Delta_{S^2}\psi = 0 \tag{5.4}$$

We can explain this a little more systematically. Recall that the operator A (here expressed in spherical coordinates)

$$A = -i\begin{pmatrix} 0 & 1 \\ \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2} & 0 \end{pmatrix}$$

is self-adjoint on $\mathcal{H} = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and putting

$$h = -(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2})\,, \text{ i.e. } h = -\Delta_{\mathbb{R}^3}\,,$$

the $\mathcal{H}$ inner product is given by

$$\langle \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \rangle_{\mathcal{H}} = \langle h\phi_1, \zeta_1\rangle_{L^2(\mathbb{R}^3\,,\ r^2\mathrm{d}r\mathrm{d}\omega)} + \langle \phi_2, \zeta_2\rangle_{L^2(\mathbb{R}^3\,,\ r^2\mathrm{d}r\mathrm{d}\omega)}\,.$$

We now consider the unitary operator

$$\mathcal{R} \,:\; L^2(\mathbb{R}^3\,,\ r^2\mathrm{d}r\mathrm{d}\omega) \to L^2(\mathbb{R}^3\,,\ \mathrm{d}r\mathrm{d}\omega)\,,\ \mathcal{R}\phi = r\phi\,.$$

Then by conjugation by $\mathcal{R}$, we have

$$\mathcal{R}h\mathcal{R}^* = -\partial_r^2 - \frac{1}{r^2}\Delta_{S^2}\,,$$

the operator $\mathcal{R}^*$ being simply the multiplication by $1/r$ from $L^2(\mathbb{R}^3\,,\ \mathrm{d}r\mathrm{d}\omega)$ to $L^2(\mathbb{R}^3\,,\ r^2\mathrm{d}r\mathrm{d}\omega)$, and

$$\mathcal{R}A\mathcal{R}^* = -i\begin{pmatrix} 0 & 1 \\ \partial_r^2 + \frac{1}{r^2}\Delta_{S^2} & 0 \end{pmatrix} =: B\,,$$

where $\mathcal{R}$ and $\mathcal{R}^*$ are understood as acting on each component. It follows that B is self-adjoint on H, completion of $L^2(\mathbb{R}^3\,,\ \mathrm{d}r\mathrm{d}\omega)^2$ in the norm

$$\begin{aligned} \langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rangle_H &= \langle (-\partial_r^2 - \frac{1}{r^2}\Delta_{S^2})\phi_1, \xi_1\rangle_{L^2(\mathbb{R}^3\,,\ \mathrm{d}r\mathrm{d}\omega)} + \langle \psi_2, \xi_2\rangle_{L^2(\mathbb{R}^3\,,\ \mathrm{d}r\mathrm{d}\omega)}\,, \\ \text{i.e. } \| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \|_H^2 &= \int_{\mathbb{R}_r^+ \times S^2} (|\partial_r\phi_1|^2 + \frac{1}{r^2}|\nabla_{S^2}\phi_1|^2 + |\phi_2|^2)\,\mathrm{d}r\mathrm{d}\omega\,. \end{aligned}$$

Let us now give the steps of the scattering construction.

- **Comparison dynamics.** The asymptotic region is $r \to +\infty$. In this region, the equation (5.4) simplifies to
$$\partial_t^2 \upsilon - \partial_r^2 \upsilon = 0 \,. \tag{5.5}$$
The Hamiltonian forms of equations (5.4) and (5.5) are given by
$$\partial_t U = iBU \,,\ \partial_t V = iB_0 V \,,$$
$$U := \begin{pmatrix} \psi \\ \partial_t \psi \end{pmatrix} ,\ B = -i \begin{pmatrix} 0 & 1 \\ \partial_r^2 + \frac{1}{r^2}\Delta_{S^2} & 0 \end{pmatrix} ,$$
$$V := \begin{pmatrix} \upsilon \\ \partial_t \upsilon \end{pmatrix} ,\ B_0 = -i \begin{pmatrix} 0 & 1 \\ \partial_r^2 & 0 \end{pmatrix} .$$
The operator B is self-adjoint with dense domain on H and B_0 is self-adjoint with dense domain on $H_0 = (\dot{H}^1(\mathbb{R}\,;\ L^2(S^2)) \times L^2(\mathbb{R} \times S^2))$. Note that the range of the variable r in H is $\mathbb{R}^+$ whereas it is the whole real axis for H_0. The propagators e^{itB} and e^{itB_0} are strongly continuous 1-parameter groups of unitary operators on H and H_0 respectively.

- **Free outgoing and incoming data.** Since (5.5) is the wave equation on $\mathbb{R}_t \times \mathbb{R}_r$, we know that every solution is a sum of an incoming and an outgoing progressive wave in the (t, r) variables. Each of these two types of solutions is characterized by a special space of initial data. We define the spaces of incoming and outgoing data
$$H_0^\pm := \{ V = {}^{\mathrm{t}}(\upsilon_1, \upsilon_2) \in H_0 \,,\ \upsilon_2 = \mp \partial_r \upsilon_1 \} \,.$$
We have $H_0 = H_0^+ \oplus H_0^-$ and for any $V \in H_0^\pm$, $(e^{itB_0}V)(r) = V(r \pm t)$.

- **Inverse wave operators.** The situation we are studying here is very special in that the full dynamics satisfies a strong Huyghens principle. This means that the scattering theory is essentially trivial (apart from purely formal difficulties related to function spaces) ; in particular, the construction of inverse wave operators is immediate. The principle of construction of wave operators is simple : we start with some initial data for one dynamics in some dense subspace of the corresponding function space, evolve it for a time t, then evolve it backwards for a time t with the other dynamics and take the limit as $t \to \infty$. For the inverse wave operators, we start with the full dynamics and then apply the time reversed simplified dynamics. Since the function spaces on which the two dynamics act are different, we need an "identifying operator" between the two spaces. Consider a cut-off function
$$\chi \in \mathcal{C}^\infty(\mathbb{R}^+) \,,\ \chi(0) = 0 \,,\ \chi \equiv 1 \text{ on } [1, +\infty[$$
and define the bounded operator
$$\mathcal{J} \ :\ H \to H_0 \,,\ \mathcal{J}U = \begin{cases} \chi U \text{ on } \mathbb{R}^+ \,, \\ 0 \text{ on } \mathbb{R}^- \,. \end{cases} \tag{5.6}$$
Theorem 5.1. *The inverse wave operators*
$$\tilde{W}^\pm = s - \lim_{t \to \pm\infty} e^{-itB_0} \mathcal{J} e^{itB} \tag{5.7}$$
are well defined for smooth compactly supported data and extend as partial isometries from H *to* H_0.

Proof. We prove the theorem for $\tilde{W}^+$, the proof is similar for $\tilde{W}^-$. We consider smooth compactly supported data $V \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^2$. By the strong Huyghens principle, taking $R > 0$ such that $\mathrm{supp}(V) \subset B(0,R)$, for $t > R$ we have $e^{-itB}V \equiv 0$ for $0 \leq r \leq R - t$ and therefore in particular $\chi e^{-itB}V = e^{-itB}V$ for $t > R+1$. The principle of Cook's method is a very simple observation : the existence of the limit

$$\lim_{t\to+\infty} e^{-itB_0}\mathcal{J}e^{itB}V\,, \tag{5.8}$$

is equivalent to the property

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-itB_0}\mathcal{J}e^{itB}V \in L^1(\mathbb{R}^+\,;\ H_0)\,. \tag{5.9}$$

Let us prove (5.9) :

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{-itB_0}\mathcal{J}e^{itB}V &= -ie^{-itB_0}(B_0\mathcal{J} - \mathcal{J}B)e^{itB}V \\ &= -ie^{-itB_0}\mathcal{J}\begin{pmatrix} 0 & 0 \\ \frac{1}{r^2}\Delta_{S^2} & 0\end{pmatrix}e^{itB}V \text{ for } t > R+1\,.\end{aligned}$$

We calculate the norm of this quantity in H_0 for $t > R+1$, denoting $\psi(t,r,\omega)$ the first component of $e^{itB}V$:

$$\begin{aligned}\| -ie^{-itB_0}\mathcal{J}\begin{pmatrix} 0 & 0 \\ \frac{1}{r^2}\Delta_{S^2} & 0\end{pmatrix}e^{itB}V\|^2_{H_0} &= \|\mathcal{J}\begin{pmatrix} 0 & 0 \\ \frac{1}{r^2}\Delta_{S^2} & 0\end{pmatrix}e^{itB}V\|^2_{H_0} \\ &= \int_{\mathbb{R}^+\times S^2} |\frac{1}{r^2}\Delta_{S^2}\psi(t,r,\omega)|^2\mathrm{d}r\mathrm{d}\omega \\ &= \int_{[t-R,t+R]\times S^2} |\frac{1}{r^2}\Delta_{S^2}\psi(t,r,\omega)|^2\,\mathrm{d}r\mathrm{d}\omega \\ &\leq \frac{1}{(t-R)^4}\|\Delta_{S^2}\psi(t)\|^2_{L^2([t-R,t+R]\times S^2\,;\ \mathrm{d}r\mathrm{d}\omega)}\,.\end{aligned}$$

The last expression can be estimated by a norm in H of an angular derivative of $e^{itB}V$ which in turn can be estimated by the same quantity at $t = 0$. To show this, we use first a Poincaré estimate :

Lemma 5.1. *Let $f \in H^1(\mathbb{R})$ supported in $[R_1, R_2]$, then*

$$\|f\|_{L^2(\mathbb{R})} \leq (R_2 - R_1)\|f'\|_{L^2(\mathbb{R})}\,.$$

Proof. We prove the result in the case where f is smooth, using the fundamental theorem of calculus

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R})} &= \int_{R_1}^{R_2} |f(t)|^2\mathrm{d}t \\ &= \int_{R_1}^{R_2} |\int_{R_1}^{t} f'(x)\mathrm{d}x|^2\mathrm{d}t \\ &\leq \int_{R_1}^{R_2}(R_2 - R_1)\int_{R_1}^{R_2} |f'(x)|^2\mathrm{d}x\mathrm{d}t \leq (R_2 - R_1)^2\|f'\|^2_{L^2(\mathbb{R})}\,.\end{aligned}$$

This gives the result in the smooth case and the inequality extends by density to functions in H^1. ☐

It follows that

$$\| - ie^{-itB_0}\mathcal{J}\left(\begin{array}{cc} 0 & 0 \\ \frac{1}{r^2}\Delta_{S^2} & 0 \end{array}\right)e^{itB}V\|^2_{H_0} \quad \leq \quad \frac{4R^2}{(t-R)^4}\|\partial_r\Delta_{S^2}\psi(t)\|^2_{L^2([t-R,t+R]\times S^2\,;\,\mathrm{d}r\mathrm{d}\omega)}\,.$$

This is controlled by

$$\frac{4R^2}{(t-R)^4}\|\Delta_{S^2}e^{-itB}V\|^2_H$$

which is equal to

$$\frac{4R^2}{(t-R)^4}\|e^{-itB}\Delta_{S^2}V\|^2_H$$

since Δ_{S^2} commutes with the equation (5.4) and therefore with e^{-itB}. It follows that (5.9) is true and the limit (5.8) exists for all smooth and compactly supported V. Since the operator $\mathcal{J}$ is bounded, this suffices to define the operator $\tilde{W}^+$ on H by density, but it is not clear then that it is defined as the strong limit (5.7).

Remark 5.1. *We even know that $\mathcal{J}$ has norm 1, hence for each t, $e^{itB_0}\mathcal{J}e^{-itB}$ is bounded from H to H_0 and has norm 1, spherically symmetric data supported sufficiently far away from the unit ball realizing the sup.*

Let us prove that the limit (5.8) exists for all $V \in H$. Let $V \in H$, consider $\{V_n\}_{n\in\mathbb{N}}$ a sequence in $(\mathcal{C}_0^\infty(\mathbb{R}^3))^2$ converging towards V in H. For $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that for all $n \geq n_0$, $\|V - V_n\|_H < \varepsilon/4$. Then for all $t \in \mathbb{R}$, thanks to remark 5.1, we have

$$\|e^{-itB_0}\mathcal{J}e^{itB}(V - V_{n_0})\|_{H_0} < \varepsilon/4\,. \tag{5.10}$$

Now we use the fact that the limit

$$\lim_{t\to\pm\infty} e^{-itB_0}\mathcal{J}e^{itB}V_{n_0}$$

exists : we take $a > 0$ large enough so that for all $t_1 > t_0 > a$,

$$\|e^{-it_1B_0}\mathcal{J}e^{it_1B}V_{n_0} - e^{-it_0B_0}\mathcal{J}e^{it_0B}V_{n_0}\|_{H_0} < \varepsilon/2\,.$$

This gives that for all $t_1 > t_0 > a$,

$$\begin{array}{rcl} \|e^{-it_1B_0}\mathcal{J}e^{it_1B}V - e^{-it_0B_0}\mathcal{J}e^{it_0B}V\|_{H_0} & \leq & \|e^{-it_1B_0}\mathcal{J}e^{it_1B}(V - V_{n_0})\|_{H_0} \\ & & +\|e^{-it_0B_0}\mathcal{J}e^{it_0B}(V - V_{n_0})\|_{H_0} \\ & & +\|e^{-it_1B_0}\mathcal{J}e^{it_1B}V_{n_0} - e^{-it_0B_0}\mathcal{J}e^{it_0B}V_{n_0}\|_{H_0} \\ & < & \varepsilon\,. \end{array}$$

This proves the existence of the limit

$$\lim_{t\to\pm\infty} e^{-itB_0}\mathcal{J}e^{itB}V\,.$$

The operator $\tilde{W}^+$ is then well defined, by the strong limit (5.7), as a linear operator from H to H_0. As a strong limit of operators whose norms are all equal to 1, it is bounded and satisfies $\|\tilde{W}^+\|_{\mathcal{L}(H,H_0)} \leq 1$. The proof is similar for $\tilde{W}^-$.

Let us now show that $\tilde{W}^\pm$ preserve the norm. It suffices to prove this for $V \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^2$ as it will extend by density to all $V \in H$. Let $V \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^2$ and $R > 0$ such that $\mathrm{supp}V \subset B(0,R)$. Denoting by ψ the first component of $e^{itB}V$, we have

$$\begin{aligned} \|e^{-itB_0}\mathcal{J}e^{itB}V\|_{H_0}^2 &= \|\mathcal{J}e^{itB}V\|_{H_0}^2 \\ &= \int_{\mathbb{R}^+\times S^2} (|\partial_r\psi(t,r,\omega)|^2 + |\partial_t\psi(t,r,\omega)|^2)\mathrm{d}r\mathrm{d}\omega \text{ for } t > R+1\,, \\ &= \|e^{itB}V\|_H^2 - \int_{\mathbb{R}^+\times S^2} \frac{1}{r^2}|\nabla_{S^2}\psi(t,r,\omega)|^2\,\mathrm{d}r\mathrm{d}\omega \\ &= \|e^{itB}V\|_H^2 - \int_{[t-R,t+R]\times S^2} \frac{1}{r^2}|\nabla_{S^2}\psi(t,r,\omega)|^2\,\mathrm{d}r\mathrm{d}\omega\,. \end{aligned}$$

We show that the last intergal tends to zero using again a Poincaré inequality :

$$\begin{aligned} &\int_{[t-R,t+R]\times S^2} \frac{1}{r^2}|\nabla_{S^2}\psi(t,r,\omega)|^2\,\mathrm{d}r\mathrm{d}\omega \\ &\leq \frac{1}{2(t-R)^2}\int_{[t-R,t+R]\times S^2} \frac{1}{r^2}|\Delta_{S^2}\psi(t,r,\omega)|^2 + |\psi(t,r,\omega)|^2\,\mathrm{d}r\mathrm{d}\omega \\ &\leq \frac{4R^2}{2(t-R)^2}\int_{[t-R,t+R]\times S^2} \frac{1}{r^2}|\partial_r\Delta_{S^2}\psi(t,r,\omega)|^2 + |\partial_r\psi(t,r,\omega)|^2\,\mathrm{d}r\mathrm{d}\omega \\ &\leq \frac{4R^2}{2(t-R)^2}(\|e^{itB}\Delta_{S^2}V\|_H^2 + \|e^{itB}V\|_H^2) \\ &\leq \frac{4R^2}{2(t-R)^2}(\|\Delta_{S^2}V\|_H^2 + \|V\|_H^2) \to 0 \text{ as } t \to +\infty\,. \end{aligned}$$

It follows that

$$\|\tilde{W}^+V\|_{H_0} = \|V\|_H\,.$$

The proof is similar for $\tilde{W}^-$. □

This establishes an important property of the solutions of (5.4) :

Corollary 5.1. *For any solution ψ of (5.4), there exist $\upsilon^\pm$ solutions of (5.5) such that*

$$\lim_{t\to\pm\infty}\left\|\mathcal{J}\left(\begin{array}{c}\psi(t)\\ \partial_t\psi(t)\end{array}\right) - \left(\begin{array}{c}\upsilon^\pm(t)\\ \partial_t\upsilon^\pm(t)\end{array}\right)\right\|_{H_0} = 0\,.$$

Moreover, $\upsilon^\pm$ are respectively an outgoing and an incoming solution of (5.5), i.e.

$$\left(\begin{array}{c}\upsilon^\pm|_{t=0}\\ \partial_t\upsilon^\pm|_{t=0}\end{array}\right) \in H_0^\pm\,.$$

In other words, $F_0^\pm := \mathrm{Ran}(\tilde{W}^\pm) \subset H_0^\pm$; F_0^+ (resp. F_0^-) is a closed subspace of H_0^+ (resp. H_0^-) and $\tilde{W}^+$ (resp. $\tilde{W}^-$) is an isomorphism between H and F_0^+ (resp. F_0^-).

Proof. The first part is a direct consequence of the theorem. Let $V \in H$, put $Y^\pm := \tilde{W}^\pm V$, we have

$$Y^\pm = \lim_{t\to\pm\infty} e^{-itB_0}\mathcal{J}e^{itB}V\,,$$

whence

$$\lim_{t\to\pm\infty} \|e^{-itB_0}\mathcal{J}e^{itB}V - Y^\pm\|_{H_0} = 0$$

and since e^{-itB_0} is a unitary operator on H_0,

$$\|e^{-itB_0}\mathcal{J}e^{itB}V - Y^\pm\|_{H_0} = \|\mathcal{J}e^{itB}V - e^{itB_0}Y^\pm\|_{H_0}\,.$$

Now the fact that $\tilde{W}^\pm$ are partial isometries implies that they have closed range and that they are isomorphisms from H onto their range. For $V \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^2$, $e^{itB}V$ vanishes for $r < t - R$, whence if $\mathcal{J}e^{itB}V$ approaches a solution $e^{itB_0}Y^+$ of (5.5) as $t \to +\infty$, we must have $Y^+ \in H_0^+$. Similarly in the past. □

- **Direct wave operators.** We can also define direct wave operators that to data for the simplified dynamics associate data for the full equation. In our case, all the work has been done for the inverse wave operators so the direct wave operators may appear as a redundant feature. In most scattering constructions however, they are the natural first step of the construction of the scattering theory and the existence of the inverse wave operators is the main difficulty : this is because in most cases the complete dynamics does not satisfy a strong Huygens principle and the simplified and complete dynamics usually operate on functions spaces that are more directly comparable than our spaces H and H_0 (the norm in H_0 completely loses control over the angular derivatives).

Definition 5.1. *We define the direct wave operators as the inverse of the inverse wave operators, i.e. their adjoints :*

$$W^\pm := (W^\pm)^{-1} = (W^\pm)^* \;:\; F_0^\pm \to H\,,$$
$$W^\pm = s - \lim_{t\to\pm\infty} e^{-itB}\mathcal{J}^* e^{itB_0} \textit{ for data in } \mathcal{C}_0^\infty\,.$$

Proposition 5.2. *We have the intertwining relations*

$$W^\pm B_0 = BW^\pm\,,\; B_0\tilde{W}^\pm = \tilde{W}^\pm B\,.$$

Proofs. The definition requires some proof : we need to check that $(W^\pm)^{-1} = (W^\pm)^*$ and is indeed given by the limit above. We write the proof for W^+, it is similar for W^-. First, W^+ is an isometry from H to F_0^+ equipped with the H_0 norm. Hence for $V \in H$,

$$\langle V\,,\; V\rangle_H = \langle W^+V\,,\; W^+V\rangle_{H_0} = \langle (W^+)^*W^+V\,,\; V\rangle_H\,.$$

So by the polarization identity, $(W^+)^*$ is a left inverse of W^+ (in fact this property is equivalent to W^+ being a partial isometry). Since W^+ is an isomorphism from H to F_0^+, it has only one left-inverse which is its inverse. Now since W^+ is defined as a strong limit of operators, its adjoint is given by the strong limit of the adjoints and the definition is therefore valid. It remains to establish the intertwining relations. This is the object of exercise 5.3. □

- **Scattering operator.** It is the operator that to the past scattering data associates the future scattering data and thus summarizes the full evolution of the field :

$$S = W^+ \tilde{W}^- .$$

It is an isometry from F_0^- to F_0^+.

5.2 The Lax-Phillips approach

5.2.1 Finite dimensional case : translation representation

Consider the equation for a time-dependent vector in $\mathbb{C}^n$:

$$\partial_t V(t) = iAV(t)$$

where A is an $n \times n$ hermitian matrix A with n distinct eigenvalues σ_1, ..., σ_n. Let $\{e_1\,,\ \dots\,,\ e_n\}$ be an orthonormal basis of eigenvectors of A. A vector $V \in \mathbb{R}^n$ can be described as the function $\tilde{V}$ from $\mathbb{R}$ to itself that is zero everywhere except for

$$\tilde{V}(\sigma_i) := \langle V, e_i \rangle .$$

The vector AV is then simply represented as the function $\sigma \tilde{V}(\sigma)$, i.e. the action of A is represented as the multiplication by the spectral parameter σ. Similarly, the unitary group e^{itA} is described as the multiplication by $e^{it\sigma}$. This is a **spectral representation** of the matrix A and its associated unitary group.

A Fourier transform in σ then gives naturally a **translation representation** of the group :

$$\mathcal{F}_\sigma(\widetilde{e^{itA}V})(r) = \mathcal{F}_\sigma(e^{it\sigma}\tilde{V})(r) = \hat{\tilde{V}}(r-t) .$$

Remark 5.2. *Of course for it all to make sense, the Fourier transform must be understood on $\mathcal{S}'(\mathbb{R})$ or on a discrete L^2 space over the spectrum of A.*

5.2.2 The wave equation : spectral representation

Consider the wave equation on Minkowski spacetime in its Hamiltonian form (3.8). Recall that the operator A is self-adjoint on $\mathcal{H} = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, completion of $\mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$ in the norm

$$\|U\|^2 := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla_x u_1|^2 + |u_2|^2) \mathrm{d}^3 x ,$$

the factor $\frac{1}{2}$ being there for later convenience. Look at the eigenvalues of A, i.e. $\sigma \in \mathbb{R}$ such that $AU = \sigma U$:

$$\left\{ \begin{array}{rcl} u_2 &=& i\sigma u_1 , \\ \Delta u_1 &=& i\sigma u_2 , \\ &=& -\sigma^2 u_1 . \end{array} \right. \tag{5.11}$$

Lemma 5.2. *The system (5.11) has no solution in $\mathcal{H}$, i.e. the point-spectrum of A is empty.*

Proof. Taking the Fourier transform of (5.11) gives

$$(\sigma^2 - |\xi|^2)\hat{u}_1 = 0\,.$$

Hence, $\mathrm{supp}(\hat{u}_1) \subset \{|\xi| = |\sigma|\}$. The same is therefore true of the support of $\widehat{\nabla u_1}$, i.e. the support of $\widehat{\nabla u_1}$ is negligeable and since $\widehat{\nabla u_1}$ is an element of $L^2(\mathbb{R}^3)$, it follows that $\widehat{\nabla u_1} = 0$, which in turn implies that u_1 is constant. Now using the fact that $u_1 \in \dot{H}^1(\mathbb{R}^3)$, which means not only that $\nabla u_1 \in L^2(\mathbb{R}^3)$ but also that there exists a sequence of smooth compactly supported functions whose gradient converges to that of u_1 in $L^2(\mathbb{R}^3)$, we get that $u_1 = 0$. We also have $u_2 = 0$ by the first equation. □

However (5.11) has solutions in $\mathcal{S}'(\mathbb{R}^3)$: we see that for each $\sigma \in \mathbb{R}$, we have a whole 2-sphere of solutions which are the plane waves

$$e_{\sigma,\omega}(x) = \begin{pmatrix} e^{-i\sigma x.\omega} \\ i\sigma e^{-i\sigma x.\omega} \end{pmatrix}, \; \omega \in S^2\,.$$

At this point it is not yet clear that we have enough solutions to generate them all. Let us hope so for the moment and proceed exactly as in the finite dimensional case : consider $U \in \mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}
\tilde{U}(\sigma,\omega) &:= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\langle U, e_{\sigma,\omega}\rangle_{\mathcal{H}} \\
&= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}(\nabla u_1 \overline{\nabla e^{-i\sigma x.\omega}} + u_2 \overline{i\sigma e^{-i\sigma x.\omega}})\mathrm{d}^3x \\
&= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}(u_1(\overline{-\Delta e^{-i\sigma x.\omega}}) + u_2 \overline{i\sigma e^{-i\sigma x.\omega}})\mathrm{d}^3x \\
&= \frac{1}{2}\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}(\sigma^2 u_1 - i\sigma u_2)e^{i\sigma x.\omega}\mathrm{d}^3x \\
&= \frac{1}{2}(\sigma^2\hat{u}_1(-\sigma\omega) - i\sigma\hat{u}_2(-\sigma\omega))\,.
\end{aligned}$$

We now have a result which proves that we have found enough tempered distribution solutions of (5.11) to generate them all.

Proposition 5.3. *Although the intermediate calculations do not, the final formula extends to $\mathcal{H}$ and the map that to U associates $\tilde{U}$ extends as an isometry from $\mathcal{H}$ onto $L^2(\mathbb{R}_\sigma \times S^2_\omega)$.*

Proof. We put for $f \in \mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$,

$$\tilde{f}(\sigma,\omega) := \sigma^2\hat{f}_1(-\sigma\omega) - i\sigma\hat{f}_2(-\sigma\omega)\,.$$

Let us show that $f \mapsto \tilde{f}$ is a linear continuous map from $\mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$ to $L^2(\mathbb{R}_\sigma \times S^2_\omega)$ for the norms $\|.\|_{\mathcal{H}}$ and $\|.\|_{L^2(\mathbb{R}_\sigma \times S^2_\omega)}$. For $f \in \mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$, we have

$$|\tilde{f}(\sigma,\omega)|^2 = \sigma^4|\hat{f}_1(-\sigma\omega)|^2 + \sigma^2|\hat{f}_2(-\sigma\omega)|^2 + i\sigma^3\hat{f}_1(-\sigma\omega)\bar{\hat{f}}_2(-\sigma\omega) - i\sigma^3\bar{\hat{f}}_1(-\sigma\omega)\hat{f}_2(-\sigma\omega)\,.$$

When changing the signs of σ and ω, the last two terms change sign and therefore their integral over $\mathbb{R}_\sigma \times S^2_\omega$ vanishes. Whence,

$$\|\tilde{f}\|^2_{L^2(\mathbb{R}_\sigma \times S^2_\omega)} = \frac{1}{4}\int_{\mathbb{R}\times S^2} \left(\sigma^2|\hat{f}_1(-\sigma\omega)|^2 + |\hat{f}_2(-\sigma\omega)|^2\right)\sigma^2 \mathrm{d}\sigma \mathrm{d}^2\omega = \|f\|^2_{\mathcal{H}}\,.$$

It follows that $f \mapsto \tilde{f}$ extends as a linear continuous map from $\mathcal{H}$ to $L^2(\mathbb{R}\times S^2)$ and that map is one-to-one and has closed range. We therefore only need to prove that its range is dense in order to prove that it is an isometry. Let $F \in \mathcal{C}^\infty(\mathbb{R}\times S^2)$, let us find $f \in \mathcal{H}$ such that $F = \tilde{f}$. We must have

$$\begin{aligned} F(\sigma,\omega) &= \frac{1}{2}(\sigma^2\hat{f}_1(-\sigma\omega) - i\sigma\hat{f}_2(-\sigma\omega))\,,\\ F(-\sigma,-\omega) &= \frac{1}{2}(\sigma^2\hat{f}_1(-\sigma\omega) + i\sigma\hat{f}_2(-\sigma\omega)) \end{aligned}$$

and therefore

$$\begin{aligned} \hat{f}_1(-\sigma\omega) &= \frac{F(\sigma,\omega)+F(-\sigma,-\omega)}{\sigma^2} \in \mathcal{C}^\infty_0(\mathbb{R}^3)\,,\\ \hat{f}_2(-\sigma\omega) &= i\frac{F(\sigma,\omega)-F(-\sigma,-\omega)}{\sigma} \in \mathcal{C}^\infty_0(\mathbb{R}^3)\,. \end{aligned}$$

This concludes the proof. □

This provides a spectral representation of A and its propagator :

$$\widetilde{AU} = \sigma\tilde{U}\,,\ \widetilde{e^{itA}U} = e^{it\sigma}\tilde{U}\,.$$

5.2.3 The wave equation : translation representation

Just as in the finite dimensional case, we take the Fourier transform in σ. Denote by

$$\mathcal{R}U(r,\omega) := \mathcal{F}_\sigma(\tilde{U}(.,\omega))(r)\,.$$

Then

$$\mathcal{R}(e^{itA}U)(r,\omega) = (\mathcal{R}U)(r-t,\omega)\,. \tag{5.12}$$

This representation is of course also an isometry from $\mathcal{H}$ onto $L^2(\mathbb{R}\times S^2)$.

5.2.4 Link with the Radon transform and asymptotic profiles

Definition 5.2. *Let $f \in \mathcal{C}^\infty_0(\mathbb{R}^3)$, we define its Radon transform as the function of $s\in\mathbb{R}$ and $\omega \in S^2$:*

$$Rf(s,\omega) = \int_{x.\omega=s} f(x)\mathrm{d}^2\sigma(x)\,,$$

i.e. $Rf(s,\omega)$ is the average of f on the plane with normal ω containing the point $s\omega$.

Proposition 5.4. *The Radon transform has the following properties :*

1. *If* $\text{supp} f \subset B(0,r)$ *then* $\text{supp}(Rf) \subset [-r,r] \times S^2$ *;*
2. $Rf \in \mathcal{C}^\infty(\mathbb{R} \times S^2)$ *;*
3. $Rf(s,\omega) = Rf(-s,-\omega)$ *;*
4. $R(\partial_{x^k} f)(s,\omega) = \omega_k \partial_s (Rf)(s,\omega)$, *whence* $R\Delta f = \partial_s^2 (Rf)$ *;*
5. *if* $g \in \mathcal{C}_0^\infty(\mathbb{R} \times S^2)$, *then*

$$\langle Rf, g \rangle_{L^2(\mathbb{R}\times S^2)} = \langle f, R^* g \rangle_{L^2(\mathbb{R}^3)} \, ;$$

R^* *is the formal adjoint of* R *given by*

$$R^* \phi(x) = \int_{S^2} \phi(x.\omega, \omega) \mathrm{d}^2\omega \, .$$

The definition and proposition above allow us to express the translation representation in a simple manner in terms of the Radon transform as well as to find an explicit formula for its inverse.

Theorem 5.2. *The translation representation has the following simple expression in terms of the Radon transform :*

$$\mathcal{R}U = \frac{1}{4\pi}(-\partial_s^2 R u_1 + \partial_s R u_2)(s,\omega) \, .$$

Moreover, for $k \in \mathcal{C}_0^\infty(\mathbb{R} \times S^2)$, *the map*

$$\mathcal{I} \, : \, \mathcal{C}_0^\infty(\mathbb{R}\times S^2) \to \mathcal{C}^\infty(\mathbb{R}^3) \times \mathcal{C}^\infty(\mathbb{R}^3) \, ,$$

$$(\mathcal{I}k)(x) = \left(\frac{1}{2\pi} R^* k, -\frac{1}{2\pi} \mathbb{R}^* \partial_s k \right) \, , \tag{5.13}$$

extends as an isometry from $L^2(\mathbb{R} \times S^2)$ *onto* $\mathcal{H}$ *which is the inverse of* $\mathcal{R}$, *i.e.*

$$\mathcal{R}\mathcal{I} = \mathrm{Id}_{L^2(\mathbb{R}\times S^2)} \, , \; \mathcal{I}\mathcal{R} = \mathrm{Id}_{\mathcal{H}} \, .$$

Remark 5.3. *Given a solution* ϕ *of the wave equation, formula* (5.13) *gives in particular* $\phi(0,x)$ *in terms of the translation representer* k *of* ϕ *:*

$$\phi(0,x) = \frac{1}{2\pi} \int_{S^2} k(x.\omega, \omega) \mathrm{d}^2\omega \, ,$$

and using the property (5.12), *we get*

$$\phi(t,x) = \frac{1}{2\pi} \int_{S^2} k(x.\omega - t, \omega) \mathrm{d}^2\omega \, ,$$

which is exactly Whittaker's formula (3.7).

This can be used to establish the **asymptotic profile** property.

Theorem 5.3 (Asymptotic profiles). *Assuming that the data ϕ_0, ϕ_1 are smooth and compactly supported*[1]*, denoting*

$$k(s,\omega) = \mathcal{R}U(s,\omega)\,,$$

we have

$$k(s,\omega) = -\lim_{r\to+\infty} r\partial_t\phi(r,(r+s)\omega)\,. \tag{5.14}$$

Proof. Since k is compactly supported, there exists $R>0$ such that $\operatorname{supp} k \subset [-R,R]\times S^2$. We have

$$\phi(t,x) = \frac{1}{2\pi}\int_{S^2} k(x.\zeta - t,\zeta)\mathrm{d}^2\zeta$$

and since k is $\mathcal{C}^1$ and compactly supported, we can differentiate under the integral

$$\partial_t\phi(t,x) = -\frac{1}{2\pi}\int_{S^2} \partial_s k(x.\zeta - t,\zeta)\mathrm{d}^2\zeta\,.$$

In particular, we have

$$\begin{aligned}\partial_t\phi(t,(t+s)\omega) &= -\frac{1}{2\pi}\int_{S^2}(\partial_s k)((t+s)\omega.\zeta - t,\zeta)\mathrm{d}^2\zeta \\ &= -\frac{1}{2\pi}\int_{S^2}(\partial_s k)((t+s)(\omega.\zeta - 1)+s,\zeta)\mathrm{d}^2\zeta\,.\end{aligned}$$

We now split the 2-sphere into a neighbourhood of the direction ω that becomes small as t becomes large,

$$V_{Rst} = \left\{\zeta\in S^2\,;\ 1-\omega.\zeta \le \frac{R+|s|}{|t+s|}\right\}\,,$$

and its complement :

$$\begin{aligned}\partial_t\phi(t,(t+s)\omega) &= -\frac{1}{2\pi}\int_{V_{Rst}}(\partial_s k)((t+s)(\omega.\zeta - 1)+s,\zeta)\mathrm{d}^2\zeta \\ &\quad -\frac{1}{2\pi}\int_{V_{Rst}^C}(\partial_s k)((t+s)(\omega.\zeta - 1)+s,\zeta)\mathrm{d}^2\zeta\,.\end{aligned}$$

When we are on the complement of V_{Rst},

$$\frac{R+|s|}{|t+s|} < 1-\omega.\zeta = |1-\omega.\zeta|\,,$$

so

$$R < |t+s||1-\omega.\zeta| - |s| \le |(t+s)(-1+\omega.\zeta)+s|\,,$$

and we see that

$$((t+s)\omega.\zeta - t,\zeta)\notin \operatorname{supp} k\,.$$

[1]In fact we merely need to assume that the data ϕ_0 and ϕ_1 are such that the corresponding asymptotic profile is $\mathcal{C}^1$ and compactly supported. Note that if the data are compactly supported, then so are the asymptotic profiles, but the converse is not true. The reason for this will appear very clearly in the conformal picture of scattering theory.

This implies that the second integral is zero. Whence

$$\partial_t\phi(t,(t+s)\omega) = -\frac{1}{2\pi}\int_{V_{Rst}} (\partial_s k)((t+s)(\omega.\zeta-1)+s,\zeta)\mathrm{d}^2\zeta\,.$$

We have localized the problem around the direction ω. We now add and substract a term where the dependence in ζ is frozen

$$\begin{aligned}\partial_t\phi(t,(t+s)\omega) = &-\frac{1}{2\pi}\int_{V_{Rst}} \left[(\partial_s k)((t+s)(\omega.\zeta-1)+s,\zeta) - (\partial_s k)((t+s)(\omega.\zeta-1)+s,\omega)\right]\mathrm{d}^2\zeta \\ &-\frac{1}{2\pi}\int_{V_{Rst}} (\partial_s k)((t+s)(\omega.\zeta-1)+s,\omega)\mathrm{d}^2\zeta\,.\end{aligned}$$

Using the fact that the area of V_{Rst} is bounded by a constant times $1/t$ for t large enough, the first term can be estimated by a constant times

$$\frac{1}{t}\mathrm{sup}_{1-\omega.\zeta\leq\frac{R+|s|}{|t+s|}}|(\partial_s k)((t+s)(\omega.\zeta-1)+s,\zeta) - (\partial_s k)((t+s)(\omega.\zeta-1)+s,\omega)|$$

which, since k is $\mathcal{C}^1$, is of the form $\varepsilon(t)/t$, where $\varepsilon(t)$ tends to zero as $t\to+\infty$. We now do an explicit calculation for the second term which we denote I. We use spherical coordinates (θ,φ) based on the direction ω :

$$\mathrm{d}^2\zeta = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi = \mathrm{d}\rho\,\mathrm{d}\varphi \text{ putting } \rho=\omega.\zeta=\cos\theta\,.$$

We have

$$\begin{aligned} I &= -\frac{1}{2\pi}\int_0^{2\pi}\int_{0\leq 1-\rho\leq\frac{R+|s|}{|t+s|}} (\partial_s k)((t+s)(\omega.\zeta-1)+s,\omega)\mathrm{d}\rho\mathrm{d}\varphi \\ &\quad \textit{we put } \tau = |t+s|(1-\rho) \\ &= -\frac{1}{2\pi}2\pi\frac{1}{|t+s|}\int_{0\leq\tau\leq R+|s|}\partial_s k(-\mathrm{sgn}(t+s)\tau+s,\omega)\mathrm{d}\tau \\ &= -\frac{1}{|t+s|}\int_{0\leq\tau\leq R+|s|} -\frac{1}{\mathrm{sgn}(t+s)}\partial_\tau\left[k(-\mathrm{sgn}(t+s)\tau+s,\omega)\right]\mathrm{d}\tau \\ &= \frac{1}{t+s}\left(k(-\mathrm{sgn}(t+s)(R+|s|)+s,\omega)-k(s,\omega)\right)\,. \end{aligned}$$

Since

$$|-\mathrm{sgn}(t+s)(R+|s|)+s|\geq R\,,$$

then

$$k(-\mathrm{sgn}(t+s)(R+|s|)+s,\omega)=0\,.$$

It follows

$$\partial_t\phi(t,(t+s)\omega) = \frac{1}{t}\varepsilon(t) - \frac{1}{t+s}k(s,\omega)$$

which proves the theorem. □

5.3 Conformal shortcuts

In this section, we shall see that using a geometrical method called conformal compactification, due to Roger Penrose (for a complete description, see [22] and [23] Vol. 2), we can recover on flat spacetime all the structure of scattering theory as well as precise decay properties. Before we describe the conformal compactification of Minkowski spacetime, we present a classic example of conformal compactification, well-known from undergraduate geometry courses but not always presented from the point of view of metric rescaling : the stereographic projection.

5.3.1 A classic example of conformal compactification

Consider the stereographic projection from the North pole of the unit 2-sphere to its equatorial plane. The formula relating in the points on the sphere in spherical coordinates (θ, φ) to those on the plane in polar coordinates (r, ψ) are

$$\psi = \phi\,,\;\; \theta = 2\arctan(1/r)\,.$$

Let us write the enclidean metric on the 2-sphere in terms of the variables r and ψ :

$$\begin{aligned} e_{S^2} &= \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \\ &= \left(2\frac{-1}{r^2}\frac{1}{1+\frac{1}{r^2}} \right)^2 \mathrm{d}r^2 + \sin^2\theta \mathrm{d}\psi^2 \\ &= \frac{4}{(1+r^2)^2}\mathrm{d}r^2 + \frac{4r^2}{(1+r^2)^2}\mathrm{d}\psi^2\,,\;\text{using the identity } \sin t = \frac{2\tan(t/2)}{1+\tan^2(t/2)}\,, \\ &= \frac{4}{(1+r^2)^2}\mathrm{d}r^2 + r^2\mathrm{d}\psi^2 \\ &= \frac{4}{(1+r^2)^2} e_{\mathbb{R}^2}\,, \end{aligned}$$

where $e_{\mathbb{R}^2}$ is the euclidean metric on $\mathbb{R}^2$. So we see that by multiplying the euclidean metric on $\mathbb{R}^2$ by Ω^2, where

$$\Omega = \frac{2}{1+r^2}\,,$$

we turn it into the euclidean metric on S^2. The thus rescaled metric is defined only away from the North pole, but it can be extended analytically to the whole 2-sphere. This is the conformal compactification of $\mathbb{R}^2$, which is the "metric" version of the usual Alexandroff compactification. It is called conformal because, since the metric is merely multiplied by a positive function, the angles, as measured using the metric, are unchanged.

Can we perform a compactification of a spacetime by rescaling its metric, just as we did with the euclidean metric on $\mathbb{R}^2$?

5.3.2 Conformal compactification of Minkowski spacetime

The contents of this section, and much more, can be found in [22]. The Minkowski metric in spherical coordinates is expressed as

$$\eta = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\omega^2\,,\;\; \mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\,.$$

We choose the advanced and retarded coordinates

$$u = t - r\,,\ v = t + r\,. \tag{5.15}$$

The metric η in terms of these new coordinates takes the form

$$\eta = \mathrm{d}u\mathrm{d}v - \frac{(v-u)^2}{4}\mathrm{d}\omega^2\,.$$

We now introduce new null coordinates that allow us to describe the whole of Minkowski space as a bounded domain :

$$p = \arctan u\,,\ q = \arctan v\,. \tag{5.16}$$

We obtain

$$\eta = (1+u^2)(1+v^2)\mathrm{d}p\,\mathrm{d}q - \frac{(v-u)^2}{4}\mathrm{d}\omega^2\,.$$

Finally coming back to time and space coordinates as follows,

$$\begin{aligned} \tau &= p + q = \arctan(t-r) + \arctan(t+r)\,, \\ \zeta &= q - p = \arctan(t+r) - \arctan(t-r)\,, \end{aligned} \tag{5.17}$$

we get

$$\eta = \frac{(1+u^2)(1+v^2)}{4}\left(\mathrm{d}\tau^2 - \mathrm{d}\zeta^2\right) - \frac{(v-u)^2}{4}\mathrm{d}\omega^2\,.$$

Choosing the conformal factor

$$\Omega^2 = \frac{4}{(1+u^2)(1+v^2)} = \frac{4}{(1+\tan^2 p)(1+\tan^2 q)} = (2\cos p\cos q)^2\ , \tag{5.18}$$

we obtain the rescaled metric

$$\begin{aligned} \mathfrak{e} \ &:= \ \Omega^2\eta = \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \frac{(v-u)^2}{(1+u^2)(1+v^2)}\mathrm{d}\omega^2 \\ &= \ \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \left((\tan q - \tan p)\cos p\cos q\right)^2\mathrm{d}\omega^2 \\ &= \ \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \left(\sin q\cos p - \sin p\cos q\right)^2\mathrm{d}\omega^2 \\ &= \ \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \left(\sin(q-p)\right)^2\mathrm{d}\omega^2 \\ &= \ \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \left(\sin\zeta\right)^2\mathrm{d}\omega^2 \\ &= \ \mathrm{d}\tau^2 - \sigma^2_{S^3}\,, \end{aligned}$$

where $\sigma^2_{S^3}$ is the euclidian metric on the 3-sphere. Minkowski space is now described as the diamond

$$\mathbb{M} = \{|\tau| + \zeta \le \pi\,,\ \zeta \ge 0\,,\ \omega \in S^2\}\,.$$

The metric $\mathfrak{e}$ is the Einstein metric, it extends analytically to the whole Einstein cylinder $\mathfrak{E} = \mathbb{R}_\tau \times S^3_{\zeta,\theta,\varphi}$. The full conformal boundary of Minkowski space can be defined in this framework. It is described as

$$\partial\mathbb{M} = \{|\tau| + \zeta = \pi\,,\ \zeta \ge 0\,, \omega \in S^2\}\,.$$

Several parts can be distinguished.

- Future and past null infinities :

$$\mathscr{I}^+ = \left\{ (\tau\,,\ \zeta\,,\ \omega)\,;\ \tau+\zeta=\pi\,,\ \zeta\in]0,\pi[\,,\ \omega\in S^2 \right\}\,,$$
$$\mathscr{I}^- = \left\{ (\tau\,,\ \zeta\,,\ \omega)\,;\ \zeta-\tau=\pi\,,\ \zeta\in]0,\pi[\,,\ \omega\in S^2 \right\}\,.$$

Proposition 5.5. *The hypersurfaces $\mathscr{I}^\pm$ are smooth null hypersurfaces for $\mathfrak{e}$ (hence the terminology "null infinities"). Their null generators are respectively the vector fields*

$$\partial_\tau - \partial_\zeta \text{ for } \mathscr{I}^+ \text{ and } \partial_\tau + \partial_\zeta \text{ for } \mathscr{I}^-\,.$$

Proof. They are clearly smooth hypersurfaces since $\mathfrak{e}$ is analytic up to $\mathscr{I}^\pm$ and does not degenerate there : its determinent

$$\det(\mathfrak{e}) = -\sin^4\zeta\sin^2\theta$$

does not vanish on $\mathscr{I}^\pm$ (except for the usual coordinate singularity unavoidable when working with spherical coordinates). Now the vector fields $\partial_\tau-\partial_\zeta$ and $\partial_\tau+\partial_\zeta$ are null and tangent respectively to $\mathscr{I}^+$ and $\mathscr{I}^-$. They are orthogonal to the two other generators of $\mathscr{I}^\pm$: ∂_θ and ∂_φ. They are therefore normal to $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively. This proves the proposition. □

- Future and past timelike infinities :

$$i^\pm = \left\{ (\tau=\pm\pi\,,\ \zeta=0\,,\ \omega)\,;\ \omega\in S^2 \right\}\,.$$

They are smooth points for $\mathfrak{e}$ (2-spheres whose area is zero because they correspond to $\zeta=0$).

- Spacelike infinity :

$$i^0 = \left\{ (\tau=0\,,\ \zeta=\pi\,,\ \omega)\,;\ \omega\in S^2 \right\}\,.$$

It is also a smooth point for $\mathfrak{e}$.

The scalar curvature of $\mathfrak{e}$ can be calculated easily :

$$\frac{1}{6}\mathrm{Scal}_{\mathfrak{e}} = \Omega^{-3}\Box_\eta\Omega = 1\,. \tag{5.19}$$

5.3.3 Consequences of conformal invariance

In the case of Minkowski spacetime, we see that $\phi\in\mathcal{D}'(\mathbb{R}^4)$ satisfies (3.1) if and only if $\hat\phi := \Omega^{-1}\phi$ (Ω defined by (5.18)) satisfies

$$\Box_{\mathfrak{e}}\hat\phi + \hat\phi = 0\,, \tag{5.20}$$

where

$$\Box_{\mathfrak{e}} = \partial_\tau^2 - \Delta_{S^3}\,.$$

On the Einstein cylinder, the Cauchy problem for (5.20) can be solved in any Sobolev space on S^3. In particular, for data $\hat\phi|_{t=0}\,,\ \partial_\tau\hat\phi|_{t=0}\in\mathcal{C}^\infty(S^3)$, the associated solution of (5.20) is smooth on the whole Einstein cylinder.

Hence,

Theorem 5.4. *Assume that the data ϕ_0, ϕ_1 for the Cauchy problem for (3.1) are such that the corresponding data for the rescaled field*

$$\hat{\phi}_0 = \frac{1+r^2}{2}\phi_0\,,\ \hat{\phi}_1 = \frac{(1+r^2)^2}{4}\phi_1\,,$$

extend as smooth functions on S^3. Then the rescaled solution $\hat{\phi} = \Omega^{-1}\phi$ extends as a smooth function on $\overline{\mathbb{M}}$.

5.3.4 Local decay

From theorem 5.4, we see that for suitable data, we can infer the rate of fall-off of the solution in all directions (timelike and null) :

Proposition 5.6. *Under the hypotheses of theorem 5.4, the solution ϕ of (3.1) associated to the data ϕ_0, ϕ_1 at $t = 0$ satisfies the following properties.*

1. **Decay along null directions.** *There exist smooth functions $\hat{\phi}^{\pm} \in \mathcal{C}^\infty(\mathbb{R}\times S^2)$ such that*

$$\lim_{r\to+\infty} r\phi(t=r+u,r,\omega) = \frac{1}{\sqrt{1+u^2}}\,\hat{\phi}^+(u,\omega)\,,$$
$$\lim_{r\to+\infty} r\phi(t=-r+v,r,\omega) = \frac{1}{\sqrt{1+v^2}}\,\hat{\phi}^-(v,\omega)\,.$$

   The functions $\hat{\phi}^{\pm}$ are simply the traces of $\hat{\phi}$ on $\mathscr{I}^{\pm}$; the two functions in the right hand side of the limits above are referred to as the future and past asymptotic profiles of ϕ.

2. **Decay along timelike directions.** *There exist two constants $C^{\pm}$ such that*

$$\lim_{t\to\pm\infty} t^2\phi(t,r,\omega) = 2C^{\pm}\,.$$

   These constants are simply $C^{\pm} = \hat{\phi}(i^{\pm})$ (recall that $i^{\pm}$ are points on the Einstein cylinder, not 2-spheres).

In other words, the physical solution ϕ decays like $1/r$ along radial null geodesics and like $1/t^2$ along the integral lines of ∂_t.

Proof. It is the object of exercise 5.5. □

Proposition 5.6 is valid for solutions ϕ of the wave equation on Minkowski spacetime such that $\hat{\phi} = \Omega^{-1}\phi$ extends as a smooth function on $\mathfrak{E}$. Implicit in this hypothesis are some requirements on the fall-off of initial data for ϕ.

Proposition 5.7. *The smoothness of $\hat{\phi}_0$ and $\hat{\phi}_1$ on S^3 entails that there exist two constants C_0, C_1 such that*

$$\lim_{r\to+\infty} r^2\phi(0,r,\omega) = 2C_0\,,$$
$$\lim_{r\to+\infty} r^4\partial_t\phi(0,r,\omega) = 4C_1\,.$$

The constants C_0 and C_1 are the respective values of $\hat{\phi}_0$ and $\hat{\phi}_1$ at i^0 (which, like $i^{\pm}$ is a point on $\mathfrak{E}$).

Proof. The first limit is a straightforward consequence of the regularity of $\hat{\phi}$ on $\mathfrak{E}$ and its relation to ϕ. As for the second limit, we have

$$\partial_t \phi = (\partial_t \Omega)\, \hat{\phi} + \Omega \frac{\partial \tau}{\partial t} \partial_\tau \hat{\phi} + \Omega \frac{\partial \zeta}{\partial t} \partial_\zeta \hat{\phi}$$

and

$$\frac{\partial \Omega}{\partial t}|_{t=0} = 0\,,\ \frac{\partial \tau}{\partial t}|_{t=0} = \frac{2}{1+r^2}\,,\ \frac{\partial \zeta}{\partial t}|_{t=0} = 0\,,$$

which, noting that $\tau = 0 \ \Leftrightarrow \ t = 0$, gives

$$\partial_t \phi_{|t=0} = \frac{4}{(1+r^2)^2} \partial_\tau \hat{\phi}_{|\tau=0} = \frac{4}{(1+r^2)^2} \hat{\phi}_1\,.$$

This proves the proposition. □

5.3.5 A first glance at peeling properties

What happens to the fall-off rate along null geodesics if instead of ϕ we consider $(\partial_t + \partial_r)\phi$ and $(\partial_t - \partial_r)\phi$? We proceed similalry, working with coordinates (u, v, ω) and (τ, ζ, ω) :

$$(\partial_t + \partial_r)\phi = \partial_v \phi = (\partial_v \Omega)\hat{\phi} + \Omega \frac{\partial \tau}{\partial v} \partial_\tau \hat{\phi} + \Omega \frac{\partial \zeta}{\partial v} \partial_\zeta \hat{\phi}\,.$$

We have

$$\Omega = \frac{2}{\sqrt{(1+u^2)(1+v^2)}}\,,\ \partial_v \Omega = \frac{-2v}{(1+u^2)^{1/2}(1+v^2)^{3/2}}\,,$$
$$\frac{\partial \tau}{\partial v} = \frac{1}{1+v^2}\,,\ \frac{\partial \zeta}{\partial v} = \frac{1}{1+v^2}\,.$$

So along an outgoing null geodesic, where we have $v \simeq 2r$, $\partial_v \phi$ falls-off like $1/r^2$.

As for $\partial_u \phi$:

$$(\partial_t - \partial_r)\phi = \partial_u \phi = (\partial_u \Omega)\hat{\phi} + \Omega \frac{\partial \tau}{\partial u} \partial_\tau \hat{\phi} + \Omega \frac{\partial \zeta}{\partial u} \partial_\zeta \hat{\phi}\,.$$

We have

$$\partial_u \Omega = \frac{-2u}{(1+u^2)^{3/2}(1+v^2)^{1/2}}\,,$$
$$\frac{\partial \tau}{\partial v} = \frac{1}{1+u^2}\,,\ \frac{\partial \zeta}{\partial u} = -\frac{1}{1+u^2}\,.$$

So along an outgoing null geodesic, $\partial_u \phi \simeq 1/r$.

This is one aspect of the peeling. Another way of describing it is that $\hat{\phi}$ has a Taylor expansion in v near $\mathscr{I}^+$ at any order, this gives an asymptotic expansion for ϕ at any order along outgoing null geodesics.

5.3.6 Scattering

The ingredients of scattering theory in the conformal picture

We have seen in the Lax-Phillips approach to scattering theory, more particularly in equation (5.14), that the scattering data can be understood as the limit along outgoing radial null geodesics of the time derivative of the solution multiplied by r ; these are the traces on $\mathscr{I}^\pm$ of $\widehat{\partial_t \phi}$. On the compactified picture, it seems much more natural to consider the traces $\hat{\phi}^\pm$ of ϕ. They are another description of the asymptotic behaviour of the solution, referred to as **Friedlander's radiation fields**, and we shall construct our conformal version of scattering theory using these scattering data instead of the Lax-Phillips asymptotic profiles. We shall see however that the knowledge of one is equivalent to that of the other.

Energy estimates

We consider the stress energy tensor for equation (5.20)

$$T_{ab} = T_{(ab)} = \partial_a \psi \partial_b \psi - \frac{1}{2} \mathfrak{e}_{ab} \mathfrak{e}^{cd} \partial_c \psi \partial_d \psi + \frac{1}{2} \psi^2 \mathfrak{e}_{ab} \tag{5.21}$$

and contract it with the Killing vector field ∂_τ. This yields the conservation law

$$\nabla^a \left(K^b T_{ab} \right) = 0 \,. \tag{5.22}$$

The energy 3-form $K^a T_{ab} \mathrm{d}^3 x^b = K^a T_a^b \partial_b \lrcorner \, \mathrm{dVol}^4$ has the expression

$$K^a T_{ab} \mathrm{d}^3 x^b = \psi_\tau \nabla \psi \lrcorner \, \mathrm{dVol}^4 + \frac{1}{2} \left(-\psi_\tau^2 + |\nabla_{S^3} \psi|^2 + \psi^2 \right) \partial_\tau \lrcorner \, \mathrm{dVol}^4 \,. \tag{5.23}$$

Integrating (5.23) on an oriented hypersurface S defines the energy flux across this surface, denoted $\mathcal{E}_S(\psi)$. For instance, denoting $X_\tau = \{\tau\} \times S^3$ the level hypersurfaces of the function τ

$$\mathcal{E}_{X_\tau}(\psi) = \frac{1}{2} \int_{X_\tau} \left(\psi_\tau^2 + |\nabla_{S^3} \psi|^2 + \psi^2 \right) \mathrm{d}\mu_{S^3} \,,$$

and parametrizing $\mathscr{I}^+$ as $\tau = \pi - \zeta$,

$$\begin{aligned} \mathcal{E}_{\mathscr{I}^+}(\psi) &= \frac{1}{\sqrt{2}} \int_{\mathscr{I}^+} \left(-2\psi_\tau \psi_\zeta + \psi_\tau^2 + |\nabla_{S^3} \psi|^2 + \psi^2 \right) \mathrm{d}\mu_{S^3} \\ &= \frac{1}{\sqrt{2}} \int_{\mathscr{I}^+} \left(|\psi_\tau - \psi_\zeta|^2 + \frac{1}{\sin^2 \zeta} |\nabla_{S^2} \psi|^2 + \psi^2 \right) \mathrm{d}\mu_{S^3} \,. \end{aligned}$$

This is a natural H^1 norm of ψ on $\mathscr{I}^+$, involving only the tangential derivatives of ψ along $\mathscr{I}^+$.

Now consider a smooth solution ψ of (5.20). The conservation law (5.22) tells us that (5.23) is closed, hence, integrating it on the closed hypersurface made of the union of X_0 and $\mathscr{I}^+$, we obtain

$$\mathcal{E}_{\mathscr{I}^+}(\psi) = \mathcal{E}_{X_0}(\psi) \tag{5.24}$$

and since ∂_τ is a Killing vector, for any $k \in \mathbb{N}$, $\partial_\tau^k \psi$ satisfies equation (5.20), whence

$$\mathcal{E}_{\mathscr{I}^+}(\partial_\tau^k \psi) = \mathcal{E}_{X_0}(\partial_\tau^k \psi)\,.$$

Using equation (5.20), for $k = 2p$, $p \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{E}_{X_0}(\partial_\tau^k \psi) &= \|\partial_\tau^{2p}\psi\|^2_{H^1(X_0)} + \|\partial_\tau^{2p+1}\psi\|^2_{L^2(X_0)} \\ &= \|(1-\Delta_{S^3})^p\psi\|^2_{H^1(X_0)} + \|(1-\Delta_{S^3})^p\partial_\tau\psi\|^2_{L^2(X_0)} \\ &\simeq \|\psi\|^2_{H^{2p+1}(X_0)} + \|\partial_\tau\psi\|^2_{H^{2p}(X_0)}\,, \end{aligned} \tag{5.25}$$

and for $k = 2p+1$, $p \in \mathbb{N}$,

$$\begin{aligned} \mathcal{E}_{X_0}(\partial_\tau^k \psi) &= \|\partial_\tau^{2p+1}\psi\|^2_{H^1(X_0)} + \|\partial_\tau^{2p+2}\psi\|^2_{L^2(X_0)} \\ &= \|(1-\Delta_{S^3})^p\partial_\tau\psi\|^2_{H^1(X_0)} + \|(1-\Delta_{S^3})^{p+1}\psi\|^2_{L^2(X_0)} \\ &\simeq \|\psi\|^2_{H^{2p+2}(X_0)} + \|\partial_\tau\psi\|^2_{H^{2p+1}(X_0)}\,. \end{aligned} \tag{5.26}$$

Hence, we have for each $k \in \mathbb{N}$:

$$\|\psi\|^2_{H^{k+1}(X_0)} + \|\partial_\tau\psi\|^2_{H^k(X_0)} \simeq \mathcal{E}_{X_0}(\partial_\tau^k\psi) = \mathcal{E}_{\mathscr{I}^+}(\partial_\tau^k\psi) \simeq \|\partial_\tau^k\psi\|^2_{H^1(\mathscr{I}^+)}$$

and using the fact that the H^k norm controls all the lower Sobolev norms, this gives us the apparently stronger equivalence

$$\|\psi\|^2_{H^{k+1}(X_0)} + \|\partial_\tau\psi\|^2_{H^k(X_0)} \simeq \sum_{p=0}^{k} \|\partial_\tau^p\psi\|^2_{H^1(\mathscr{I}^+)}\,. \tag{5.27}$$

Interpretation as a complete scattering theory

The energy equality (5.24) entails the following result.

Proposition 5.8. *The trace operators*

$$T^\pm\,,\ (\hat\phi_0\,,\ \hat\phi_1) \longmapsto \hat\phi|_{\mathscr{I}^\pm}\,,$$

are well defined from $(\mathcal{C}^\infty(X_0))^2$ to $\mathcal{C}^\infty(\mathscr{I}\pm)$. They extend uniquely as bounded operators (still denoted $T^\pm$) from $H^1(X_0)\times L^2(X_0)$ to $H^1(\mathscr{I}^\pm)$, that are one-to-one and with closed range.

Moreover the trace operators are in fact surjective onto $H^1(\mathscr{I}^+)$. This is a consequence of a general theorem that has been used for some time by different people and under different forms. A clear form of this theorem for the wave equation can be found in a short paper by L. Hörmander from 1990 [12], it is given under a slightly different form in this course (see theorem 7.3) : the well-posedness of the Goursat problem (which is established in all its generality in section 7.1). Hence :

Theorem 5.5. *The trace operators $T^\pm$ are isometries from $H^1(X_0)\times L^2(X_0)$ onto $H^1(\mathscr{I}^\pm)$.*

This is all we need to construct a scattering operator. The trace operators play the role of inverse wave operators, they associate asymptotic profiles (also referred to as scattering data), in a sense fairly close to that in the Lax-Phillips theory, to solutions of the full equation. The scattering operator summarizes the complete evolution of the field by associating to the past asymptotic profile the future asymptotic profile. The definition is therefore natural.

Definition 5.3 (Scattering operator)**.** *The scattering operator in this conformal scattering theory is given by*

$$S := T^+(T^-)^{-1}\,.$$

Recovering the structures of analytic scattering theories

This scattering theory, although it has been constructed in a geometrical way, retains some important analytic properties of the scattering constructions performed using the Lax-Phillips approach or Cook's method.

The wave operators can easily be interpreted as providing the comparison between two dynamics, the simplified dynamics being given by the flow of radial null geodesics. Even the identifying operator has a role to play : the simplified dynamics will associate to the asymptotic profile a function that is not well-defined on each level hypersurface of t, because of the rays focalizing inside the spacetime. This is solved easily using a smooth cut-off inside a fixed compact in space (in the physical spacetime) and allows to interpret the inverse trace operators as direct wave operators defined exactly as in the section on Cook's method.

Provided we use a different compactification, namely the partial compactification used in the case of the Schwarzschild metric in section 6.1.5, we see that the radiation fields are another type of translation representer for the solution. Indeed, in this compactification, the vector field ∂_t remains Killing and extends smoothly to null infinity as the null generator of $\mathscr{I}$. Taking as data the solution at time t instead of 0 means pulling to the full solution by a time $-t$ along the flow of ∂_t. This modifies the radiation fields (considered in the variables used for this compactification) by a translation of $-t$ along $\mathscr{I}$ (more precisely by pulling it of $-t$ along the flow of the null generator of $\mathscr{I}$).

The translation representer is a feature associated with a timelike Killing vector that extends to $\mathscr{I}$. It will be present if we construct a (conformal) scattering theory on Schwarzschild's spacetime. On time-dependent geometries however, we will lose this property. The interpretation of the trace operators as wave operators defined by comparing with a simplified dynamics will remain though, and the simplified dynamics will still be given as the flow of a congruence of null geodesics near $\mathscr{I}$.

5.4 Exercises

Exercise 5.1. *The inequality* (5.1) *being valid for all smooth compactly supported functions, prove that it extends to all elements of* $H^1(\mathbb{R})$.

Exercise 5.2. *Show that the identifying operator* $\mathcal{J}$ *defined in* (5.6) *is bounded and calculate its norm.*

Exercise 5.3. *The object of this exercise is to prove the intertwining relations given in proposition 5.2.*

1. *Show that for any $t \in \mathbb{R}$, we have*

$$e^{itB_0} W^{\pm} = W^{\pm} e^{itB} .$$

2. *Infer from this the intertwining relations.*

Exercise 5.4. *We study the Cauchy problem with data set at $\tau = 0$ for equation (5.20) on the Einstein cylinder.*

1. *Prove that the Cauchy problem is well-posed on $H^1(S^3) \times L^2(S^3)$.*
2. *Prove that it is well-posed on $H^{2k+1}(S^3) \times H^{2k}(S^3)$ for all $k \in \mathbb{N}$.*
3. *Prove that it is well-posed on $H^{k+1}(S^3) \times H^k(S^3)$ for all $k \in \mathbb{N}$ (this question can be treated independently of the previous one).*

Exercise 5.5. *Prove proposition 5.6.*

Chapter 6

The Schwarzschild metric, asymptotically simple spacetimes

The Schwarzschild metric is an exact solution of the Einstein vacuum equations. It was discovered by Karl Schwarzschild in 1917 and is the first non trivial solution to appear historically. What people found worrying at the time was the fact that the metric was singular not only at the origin but worse, on a sphere of positive radius. The solution was quickly dismissed as physically irrelevent because of this singularity. Eddington was the first to realize that the sphere was not a singularity of the metric but merely a coordinate singularity. He found a coordinate system which allowed him to give the correct interpretation of the physical meaning of the sphere. Finkelstein subsequently rediscovered this coordinate system in 1958, hence the name of Eddington-Finkelstein coordinates. After Oppenheimer and Snyder proposed a model for the collapse of a star where it appeared that the phenomenon could go well beyond white dwarfs and create a singularity, people suddenly remembered Schwarzschild's solution and the study of what John Wheeler would call black holes a few years later really started. Kruskal and Szekeres completed the picture and built the maximal analytic extension on the Schwarzschild metric.

The Schwarzschild spacetime is a reference model for all asymptotically flat universes containing energy/matter. The metric describing any such universe, when restricted to the leaves of a foliation by asymptotically flat spacelike hypersurfaces, is generically a short-range perturbation (i.e. a perturbation in $1/r^2$, r being for example the geodesic distance to a given point on the slice) at infinity of the Schwarzschild metric.

Asymptotically simple spacetimes are an attempt, due to Roger Penrose, at defining generic cosmological models of asymptotically flat spacetimes. A special class of asymptotically simple spacetimes, which will be of particular interest to us, coincindes with Schwarzschild's spacetime in a neighbourhood of infinity.

6.1 The Schwarzschild metric

The Schwarzschild metric is expressed (in a coordinate system (t, r, ω) referred to as Schwarzschild coordinates), on $\mathbb{R}_t \times]0, +\infty[_r \times S^2_\omega$ as

$$g = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2\,,\ \mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}^2\varphi\,,\ F(r) = 1 - \frac{2M}{r}\,, \tag{6.1}$$

where m is the mass of the black hole and $\mathrm{d}\omega^2$ is the euclidian metric on the 2-sphere. Expressed in the form (6.1), this metric appears to have two singularities corresponding to $r = 2M$ and $r = 0$. The sphere $\{r = 2M\}$, referred to as the event horizon, is merely a coordinate singularity, the metric can be extended analytically through it, while the origin $\{r = 0\}$ which is a true curvature singularity. The horizon separates the space-time in two domains :

- the exterior of the black hole $\{r > 2M\}$ is a static domain where $\partial/\partial t$ is timelike and $\partial/\partial r$ spacelike ;
- the interior of the black hole $\{r < 2M\}$, is a dynamic region where $\partial/\partial t$ is spacelike, $\partial/\partial r$ timelike, so r should be thought of as a time variable inside the black hole, it is therefore oriented ; the usual understanding of a black hole says that things can fall into it but not come out of it ; this would correspond to the inertial frames in the interior being dragged towards the singularity at $\{r = 0\}$, i.e. $-\partial/\partial r$ being future oriented, but one may just as well consider the reverse time orientation which would correspond to a white hole ; nothing at this point indicates that one orientation is preferable to the other.

The two domains are globally hyperbolic. The surfaces

$$\{t\} \times]2M, +\infty[\times S^2_{\theta,\varphi}$$

are Cauchy hypersurfaces for the exterior and

$$\mathbb{R}_t \times \{r\} \times S^2_{\theta,\varphi}$$

are Cauchy hypersurfaces for the interior.

The shape of the lightcones outside and inside the black-hole is well described by the position of the null vectors

$$V^{\pm} := \frac{\partial}{\partial t} + F(r)\frac{\partial}{\partial r}\,.$$

The vectors V^+ and V^- get closer to each other as one approaches the horizon from the inside or the outside. The situation is however very different on either side of the horizon : outside the black hole, the light cones get narrower as one approaches the horizon, whereas inside they get wider (see figure 6.1). Schwarzschild's spacetime is asymptotically flat. This can be seen in the fact that as $r \to +\infty$, the metric g approaches the Minkowski metric in spherical coordinates. It is also apparent in the property that the curvature tends to zero as $r \to +\infty$ (see next paragraph). Note that asymptotically flat means asymptotically flat in space, certainly not in time, we have a curved spacetime that is static, therefore the curvature does not die out as time tends to infinity.

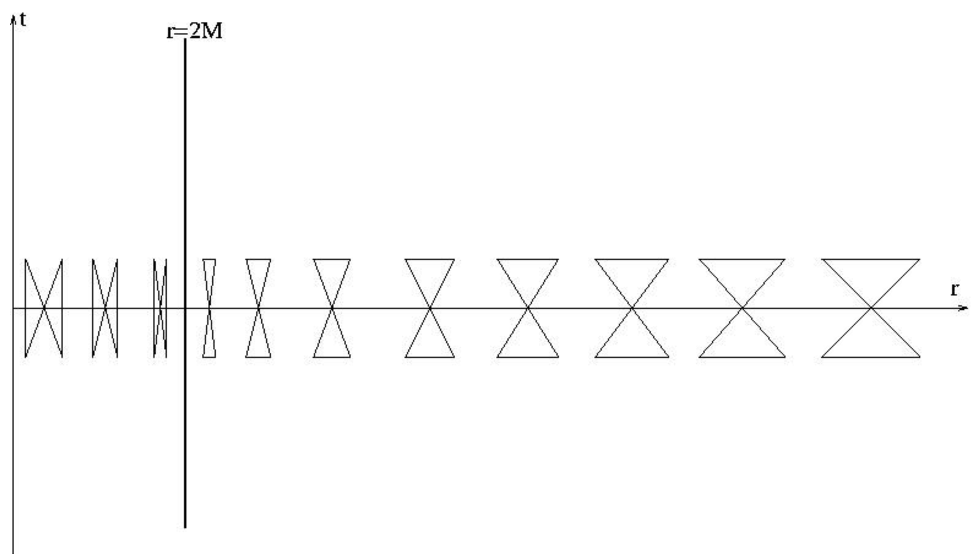

Figure 6.1: Profile of the light cones outside and inside the black-hole in the (t, r)-plane. The vectors $V^{\pm} = \partial_t \pm F\partial_r$ correspond to the upper parts of the cones.

6.1.1 Connection and curvature

In the Schwarzschild coordinates (t, r, θ, φ), the non zero Christoffel symbols of the Levi-Civita connection are

$$\Gamma^0_{01} = \frac{M}{r(r-2M)}\,,\ \Gamma^1_{00} = \frac{M(r-2M)}{r^3}\,,\ \Gamma^1_{11} = -\frac{M}{r(r-2M)}\,,$$
$$\Gamma^1_{22} = -(r-2M)\,,\ \Gamma^1_{33} = -(r-2M)\sin^2\theta\,,$$
$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}\,,\ \Gamma^2_{33} = -\sin\theta\cos\theta\,,\ \Gamma^3_{23} = \cot\theta\,,$$

and the non-zero components of the Riemann tensor

$$R_{0101} = -\frac{M(r-2M)}{r^2}\sin^2\theta\,,\ R_{0202} = \frac{2M}{r^3}\,,\ R_{0303} = -\frac{M(r-2M)}{r^2}\,,$$
$$R_{1212} = \frac{M}{r-2M}\sin^2\theta\,,\ R_{1313} = -2Mr\sin^2\theta\,,$$
$$R_{2323} = \frac{M}{r-2M}\,.$$

If, instead of the Schwarzschild coordinate basis, we evaluate the components of the Riemann tensor with respect to an orthonormal basis with vectors proportional to the coordinate basis vectors, namely (adopting Chandrasekhar's notations for frame indices between brackets)

$$e_{(0)}{}^a\partial_a = \frac{1}{\sqrt{F}}\frac{\partial}{\partial t}\,,\ e_{(1)}{}^a\partial_a = \sqrt{F}\frac{\partial}{\partial r}\,,\ e_{(2)}{}^a\partial_a = \frac{1}{r}\frac{\partial}{\partial\theta}\,,\ e_{(3)}{}^a\partial_a = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\,,$$

we find

$$R_{1010} = -R_{3232} = \frac{2M}{r^3}\,,\ R_{3131} = R_{1212} = R_{3030} = -R_{2020} = \frac{M}{r^3}\,,$$

and we see that the curvature, expressed in this frame, blows up at $\{r = 0\}$ but not at the horizon.

Remark 6.1. *Of course, if we express the components of the Riemann tensor with respect to the Schwarzschild coordinate basis, its components will be singular at $r = 0$ but also at $r = 2M$, as can readily be seen from the expression of the metric. This does not mean anything since the basis is not orthonormal. Orthonormality however is not enough to guarantee that the explosion of the coefficients of the curvature tensor corresponds to a real explosion of the curvature and not a singularity on the basis ; a basis could be singular by having an angular momentum that becomes infinite locally.*

A more usual way of seeing whether the curvature is singular is to calculate the curvature scalar which is an intrinsic quantity and is defined as follows :

$$g^{ae}g^{bf}g^{ci}g^{dj}R_{abcd}R_{efij} = R_{abcd}R^{abcd}\,.$$

It is easily calculated in the orthonormal frame above using the symmetries of the Riemann tensor :

$$R_{abcd}R^{abcd} = 32\frac{M^2}{r^6}\,.$$

The Ricci tensor of the Schwarzschild metric is zero, Schwarzschild's spacetime is a solution of the Einstein vacuum equations. Of course, the scalar curvature vanishes also.

6.1.2 Symmetries, Killing vectors

Schwarzschild's spacetime has a four-dimensional space of global Killing vector fields, generated by

$$\partial_t\,,\ \ \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi\,,\ \ \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi\,,\ \ \partial_\varphi\,,$$

which are the timelike (outside the black hole) Killing vector field ∂_t already mentionned above and the three generators of the rotation group. In other words, the symmetry group of Schwarzschild's spacetime is $\mathbb{R} \times SO(3)$.

6.1.3 The exterior of the black hole

We first consider the Schwarzschild geometry from the point of view of an observer static with respect to infinity. Such observers only see the exterior of the black hole and their perception of space-time is described by the time function t of the Schwarzschild coordinates outside the black hole. To their eyes, light rays falling into the black hole slow down infinitely as they approach the horizon and never cross it. One way of seeing this is to calculate the radial null geodesics.

Indeed, the fastest way of falling into the black hole, since the spacetime is spherically symmetric (i.e. in particular without rotation), is to go towards it radially and at the speed of light. Let us first evaluate the radial null directions. A radial vector at a given point (t, r, θ, φ) is of the form

$$V = \alpha\partial_t + \beta\partial_r\,.$$

For it to be null, α and β must satisfy

$$\frac{\beta}{\alpha} = F$$

since

$$g(V,V) = \alpha^2 F - \beta^2 F^{-1} .$$

So the two future oriented[1] radial null directions at a given point outside the black hole are those of the vectors

$$V^{\pm} = \partial_t \pm F\partial_r .$$

The apparent radial speed of these vectors for an observer static at infinity and measured using the variable r is $\pm F(r)$, it is ± 1 at infinity and slows down continuously to zero as one considers points closer and closer to the black hole horizon. Moreover, their integral curves are geodesics :

Proposition 6.1. *The radial null vectors $V^{\pm}$ satisfy*

$$\nabla_{V^+} V^+ = \frac{2M}{r^2} V^+ \, , \; \nabla_{V^-} V^- = -\frac{2M}{r^2} V^- \, .$$

Proof. Let us check this property for V^+. Dropping the "+" superscript for simplicity, using the values of the Christoffel symbols given above, we have

$$\begin{aligned} \nabla_V V^{\mathbf{a}} \partial_{\mathbf{a}} &= V^{\mathbf{b}} \nabla_{\mathbf{b}} V^{\mathbf{a}} \partial_{\mathbf{a}} \\ &= V^0 \nabla_0 V^{\mathbf{a}} \partial_{\mathbf{a}} + V^1 \nabla_1 V^{\mathbf{a}} \partial_{\mathbf{a}} \\ &= \partial_t (V^{\mathbf{a}}) \partial_{\mathbf{a}} + \Gamma^{\mathbf{a}}_{0\mathbf{b}} V^{\mathbf{b}} \partial_{\mathbf{a}} + F\partial_r (V^{\mathbf{a}}) \partial_{\mathbf{a}} + F\Gamma^{\mathbf{a}}_{1\mathbf{b}} V^{\mathbf{b}} \partial_{\mathbf{a}} \\ &= 0 + \Gamma^0_{01} V^1 \partial_t + \Gamma^1_{00} V^0 \partial_r + F\partial_r (V^1) \partial_r + F\Gamma^0_{10} V^0 \partial_t + F\Gamma^1_{11} V^1 \partial_r \\ &= \frac{MF^{-1}}{r^2} F\partial_t + \frac{MF}{r^2} \partial_r + F\frac{2M}{r^2} \partial_r + F\frac{MF^{-1}}{r^2} \partial_t - F\frac{MF^{-1}}{r^2} F\partial_r \\ &= \frac{2M}{r^2} V \, . \end{aligned}$$

The calculation is absolutely similar for V^- and left as an exercice. □

We note that the t,r-speed of radial light rays slows down as they approach the horizon. The question is whether this slowing down is strong enough to make t non-integrable along their worldlines. The answer is clearly yes since

$$\int_{2M}^{R} \frac{\mathrm{d}r}{F(r)} = \int_{2M}^{R} \frac{r\mathrm{d}r}{r-2M} = +\infty \text{ for any } R > 2M \, .$$

This can be done in a more explicit way by introducing the Regge-Wheeler variable

$$r_* = r + 2M\mathrm{Log}(r-2M) \tag{6.2}$$

which varies from $-\infty$ to $+\infty$ as r varies from $2M$ to $+\infty$. It satisfies

$$\frac{dr_*}{dr} = F^{-1}$$

and the metric g takes the form

$$g = F\left(\mathrm{d}t^2 - \mathrm{d}r_*^2\right) - r^2 \mathrm{d}\omega^2 .$$

[1]Future-oriented provided we choose outside the black hole the time orientation given by ∂_t.

The radial null vectors take the expression

$$V^{\pm} = \partial_t \pm \partial_{r_*}$$

and their integral lines parametrized by r_* are the straight lines

$$\gamma^{\pm}_{C,\omega_0}(r_*) = \{(t, r_*, \omega)\,;\ \omega = \omega_0\,,\ t = \pm r_* + C\}\ ,\ C \in \mathbb{R}\,,\ \omega_0 \in S^2\,.$$

The horizon $\{r = 2M\}$ (corresponding to $r_* \to -\infty$) is reached in infinite time t. A remarkable consequence of this property is that if we choose for a covariant field equation (Dirac, Maxwell, or the wave equation for instance) some initial data at time $t = 0$ whose support is contained in $\{r \geq 2M + \varepsilon\}$, $\varepsilon > 0$, then the support of the solution will only reach the horizon when t becomes infinite.

The intuitive description of a black hole tells us that the more we approach the horizon from the exterior, the harder it becomes to escape the attraction, until at the horizon, even a photon cannot escape anymore. But it is easier and easier to go towards the black hole. In terms of light-cones, this seems to indicate a picture where the lightcones are tilted towards the horizon and become tangent to the horizon as we reach it. When representing the lightcones in the Schwarzschild coordinates however, this does not appear to be correct after all. How do we solve this canondron? We will see that the intuitive picture has some degree of realism when we build the maximal analytic extension of the Schwarzschild spacetime, which gives the correct picture of the horizon.

An important consequence of this remark is that the interior of the black hole and the exterior should not be considered as co-existing simultaneously for the time t, in other words, a $t =$ constant slice for $r \in]0, +\infty[$ has no physical meaning whatsoever. Such hypersurfaces will be represented and put in their proper perspective once we have constructed the maximal extension of Schwarzschild's spacetime.

The spacelike geometry of the exterior of the black hole

The exterior of the black hole is globally hyperbolic. We consider the foliation by Cauchy hypersurfaces induced by the time function t, i.e. the slices are

$$\Sigma_t = \{t\} \times]2M, +\infty[_r \times S^2_\omega,\ t \in \mathbb{R},$$

with the induced Riemannian metric

$$h = F^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\omega^2. \tag{6.3}$$

The 3+1 decomposition of the geometry is given by (calling $\mathcal{M}$ the exterior of the black hole) :

$$\mathcal{M} = \mathbb{R}_t \times \Sigma\,,\ \Sigma =]2M, +\infty[_r \times S^2_\omega\,,\ g = F\mathrm{d}t^2 - h = \frac{N^2}{2}\mathrm{d}t^2 - h \tag{6.4}$$

with the lapse function $N = \sqrt{2}F^{1/2}$. The exterior of the black hole is static : $\frac{\partial}{\partial t}$ is a Killing vector field (since g does not depend on t), is timelike outside the black hole and is everywhere orthogonal to the Cauchy hypersurfaces Σ_t. The time orientation is chosen by deciding that $\frac{\partial}{\partial t}$ is future pointing and the normalized vector field T^a is then

$$T^a \partial_a = \sqrt{2}\, F^{-1/2} \frac{\partial}{\partial t} = \frac{2}{N} \frac{\partial}{\partial t} \,.$$

We consider a generic spacelike slice (Σ, h). The metric h appears singular at $r = 2M$. This is merely due to the choice of coordinates ; introducing as the new radial variable $u(r)$ the h-distance to the horizon, we show that (Σ, h) is a smooth manifold and that the horizon $H = \{2M\}_r \times S^2_{\theta,\varphi}$ is a smooth boundary.

Given $p = (r, \omega) \in \Sigma$, the h-distance from p to the horizon is given by

$$u(r) = \int_{[2M,r]} F^{-1/2}(s) \mathrm{d}s = \int_{[2M,r]} \frac{\sqrt{s}}{\sqrt{s - 2M}} \mathrm{d}s \,. \tag{6.5}$$

This distance is finite and H thus appears as the boundary of (Σ, h). Since

$$\frac{\mathrm{d}u}{\mathrm{d}r} = F^{-1/2},$$

the metric h can be written as

$$h = \mathrm{d}u^2 + r^2 \mathrm{d}\omega^2 \tag{6.6}$$

and

$$\Sigma =]0, +\infty[_u \times S^2_\omega \,.$$

The function $u(r)$ is continuous and strictly increasing from $[2M, +\infty[$ onto $[0, +\infty[$, it is $\mathcal{C}^\infty$ on $]2M, +\infty[$ but it is not differentiable at $2M$. However, the inverse function satisfies

Lemma 6.1. *The function $u \longmapsto r(u)$ is $\mathcal{C}^\infty$ on $[0, +\infty[$ and all its derivatives are uniformly bounded on $[0, +\infty[$. In particular, the first derivative $\frac{\mathrm{d}r}{\mathrm{d}u} = F^{1/2}$ (and therefore also the lapse function) is uniformly bounded as well as all its derivatives on $[0, +\infty[$.*

Proof of lemma 6.1 : the first and second derivatives $F^{1/2}$ and M/r^2 are continuous on $[0, +\infty[_u$ whence r is $\mathcal{C}^2$ on $[0, +\infty[_u$. If r is $\mathcal{C}^k$ on $[0, +\infty[_u$, then so is the second derivative and the lemma is thus proved by induction. □

This entails that h is smooth on $\bar{\Sigma} = [0, +\infty[_u \times S^2_\omega$; $(\bar{\Sigma}, h)$ is a smooth manifold with boundary. Moreover

Theorem 6.1. *The metric h is uniformly equivalent to the euclician metric on the exterior of the unit ball in $\mathbb{R}^3$*

$$\mathrm{d}u^2 + (1+u)^2 \mathrm{d}\omega^2 \,.$$

Proof. We see that

$$\frac{1+u}{r} \to \frac{1}{2M} \text{ as } r \to 2M \,,$$

$$\frac{1+u}{r} \to 1 \text{ as } r \to +\infty \text{ since } F(r) \to 1$$

and moreover $(1+u)/r$ is continuous on $[2M, +\infty[_r$, hence, there exists $C > 0$ such that

$$C < \frac{1+u}{r} < \frac{1}{C} \text{ for } 2M \leq r < +\infty \,.$$

This proves the theorem. □

Bending of light-rays : the photon sphere

We consider an extreme example of bending of light rays by gravity in the schwarzschild geometry : the photon sphere, which is a sphere of trapped geodesics around the black hole. Let us consider in the equatorial plane a null vector that is purely rotational, i.e. of the form $V = a\partial_t + b\partial_\varphi$, for example, we can take

$$V = r\partial_t + \sqrt{1 - \frac{2M}{r}}\partial_\varphi \,.$$

The integral curves of this vector field are circles in the equator (helices if we consider the time as well as space variables) whose tangent vectors are null. What is the acceleration of such curves? This is the following simple calculation :

$$\begin{aligned}
\nabla_V V = V^a \nabla_a V^b \partial_b &= \left(V^{\mathbf{a}} \nabla_{\mathbf{a}} V^{\mathbf{b}} + \Gamma^{\mathbf{b}}_{\mathbf{ac}} V^{\mathbf{c}}\right) \partial_{\mathbf{b}} \\
&= \left(V^0 \partial_t V^{\mathbf{b}} + V^3 \partial_\varphi V^{\mathbf{b}} + V^0 \Gamma^{\mathbf{b}}_{0\mathbf{c}} V^{\mathbf{c}} + V^3 \Gamma^{\mathbf{b}}_{3\mathbf{c}} V^{\mathbf{c}}\right) \partial_{\mathbf{b}} \\
&= \left(V^0 \Gamma^{\mathbf{b}}_{0\mathbf{c}} V^{\mathbf{c}} + V^3 \Gamma^{\mathbf{b}}_{3\mathbf{c}} V^{\mathbf{c}}\right) \partial_{\mathbf{b}} \\
&= r\left(\Gamma^0_{01} V^1 \partial_t + \Gamma^1_{00} V^0 \partial_r\right) \\
&\quad + \sqrt{1 - \frac{2M}{r}} \left(\Gamma^1_{33} V^3 \partial_r + \Gamma^3_{31} V^1 \partial_\varphi + \Gamma^3_{32} V^2 \partial_\varphi\right) \\
&= r\Gamma^1_{00} V^0 \partial_r + \sqrt{1 - \frac{2M}{r}} \Gamma^1_{33} V^3 \partial_r \\
&= \left(r^2 \frac{M}{r^3}(r - 2M) + \left(1 - \frac{2M}{r}\right)(-r)\left(1 - \frac{2M}{r}\right)\right) \partial_r \\
&= \left(1 - \frac{2M}{r}\right)(3M - r)\partial_r \,.
\end{aligned}$$

As could be expected, the acceleration is purely radial. It points towards the black hole if $r > 3M$, away from the black hole if $r < 3M$ and it is zero if $r = 3M$. This means that the integral curves of V for $r = 3M$ are geodesics : there are some "photon trajectories" orbiting the black hole at $r = 3M$. This is a very strong effect of light bending which requires a black hole or a very dense body of radius lower than three times its mass.

6.1.4 Maximal extension

After having adopted, in the previous section, the point of view of an observer static with respect to infinity, and thus limited our study to the exterior of the black hole foliated using Schwarzschild's time coordinate, we describe here briefly the global geometry of Schwarzschild's space-time. We define the Eddington-Finkelstein and the Kruskal-Szekeres coordinates inside and outside the black hole. These will allow us to show that the horizon is not a singularity of the metric. The maximal analytic extension of Schwarzschild's space-time will then appear naturally. Most of the material of this section is standard, it can be found under various forms in [3], [11] and [17] for example.

Eddington-Finkelstein coordinates

There are two types of Eddington-Finkelstein coordinates respectively referred to as advanced and retarded, or, more to the point, incoming and outgoing. They are based on the incoming (resp. outgoing) radial null geodesics.

The incoming Eddington-Finkelstein coordinates are

$$v = t + r_*, r, \theta, \varphi \,,$$

where $r_* = r + 2M \log(r - 2M)$ is the Regge-Wheeler coordinate. The Schwarzschild metric, in these coordinates, reads

$$g = \left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - r^2 \mathrm{d}\omega^2 \,. \tag{6.7}$$

This is fine outside the black hole but not inside where the expression of r_* is no longer valid. If we define r_* inside the black hole as

$$r_* = r + 2M \log(2M - r) \,, \tag{6.8}$$

r_* varies from $-\infty$ to $2M \log(2M)$ as r varies from $2M$ to 0. We keep the definition $v = t + r_*$ inside the black hole and we obtain the same expression (6.7) of the metric g. This is analytic on $\mathbb{R}_v \times]0, +\infty[_r \times S^2_\omega$ and does not degenerate anywhere (apart from the usual problem due to spherical coordinates) as we can see from the determinent of g :

$$\det g = -r^4 \sin^2 \theta \,.$$

The whole of Schwarzschild's spacetime is represented by the incoming Eddington-Finkelstein coordinates and we can wonder how to interpret the spacetime, and more particularly the horizon, physically.

A v =constant curve is a curve

$$(t = -r_* + v_0, r_*, \omega = \omega_0) \,,$$

with v_0 and ω_0 fixed ; i.e. this is an integral curve of the vector field $V^- = \partial_t - \partial_{r_*}$, in other words, a null geodesic. Outside the black hole, this is clearly the incoming radial null geodesic γ_{v_0,ω_0}. If we parametrize this curve by r, then it is an analytic curve in all positive values of r, in particular we see that the incoming null geodesic γ_{v_0,ω_0} outside the black hole extends analytically inside the black hole as the same $v = v_0$. As we follow the geodesic from infinity inwards, we move towards the future and r decreases (with r_* decreasing from $+\infty$ to $-\infty$ as r decreases from $+\infty$ to $2M$), the geodesic then crosses the horizon $\{r = 2M\}$ and keeps going towards the singularity at the origin (r_* increasing from $-\infty$ to $2M \log(2M)$ as r decreases from $2M$ to 0). The interior of the black hole is thus understood as lying in the future of the exterior. The correct time orientation of the interior of the black hole, consistent with that given by ∂_t outside the black hole, would appear to be given by $-\partial_r$.

The horizon is seen as the hypersurface $\mathbb{R}_v \times \{2M\}_r \times S^2_\omega$ and separates the exterior from the interior. Moreover, the horizon appears as a null hypersurface. Indeed, the metric does not degenerate there, but its restriction to the horizon is the 2-metric

$$-(2M)^2 \mathrm{d}\omega^2 \,,$$

whereas the horizon is a 3-surface. This means that one of the tangent vectors to the horizon is null. At each point of the hypersurface $\{r = 2M\}$, the space of tangent vectors is spanned by ∂_v, ∂_θ and ∂_φ. The "squared norm" of ∂_v for the metric g is given by

$$g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \left(1 - \frac{2M}{r}\right) .$$

So ∂_v is null for $r = 2M$. The correct picture of Schwarzschild's spacetime in incoming Eddington-Finkelstein coordinates is given by (FIGURE IncomEF) and we see that once inside the black hole, we cannot come back out of it.

We now perform a similar construction based on the outgoing Eddington-Finkelstein coordinates :

$$u = t - r_*, r, \theta, \varphi ,$$

and the Schwarzschild metric in these coordinates takes the expression

$$g = \left(1 - \frac{2M}{r}\right) \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - r^2\mathrm{d}\omega^2 . \tag{6.9}$$

Similarly to the incoming case, this is analytic on $\mathbb{R}_u \times]0, +\infty[_r \times S^2_\omega$ and does not degenerate anywhere. The whole of Schwarzschild's spacetime is again represented, but the physical picture is different. Following an outgoing radial null geodesic (a u =constant line) towards the future, we emerge from the singularity at $r = 0$, cross the interior of the "black hole", the horizon, emerge from the "black hole" and go towards infinity. The black hole does not appear to be so black in this case since light rays emerge from it. The horizon is again a null hypersurface but this time it cannot be crossed from the exterior to the interior. This is a very different description of Schwarzschild's spacetime corresponding not to a black hole, but to a white hole (see figure OutgoEF). The time orientation of the interior consistent with the one given by ∂_t outside the black hole would now seem to correspond to ∂_r.

What we have constructed using the incoming and the outgoing Eddington-Finkelstein coordinates are similar objects but with the opposite time orientation. We shall see in the next section that the two descriptions are both present in the most complete picture of Schwarzschild's spacetime : the maximal analytic extension of it, also know as the Kruskal manifold.

Kruskal-Szekeres coordinates

Outside the black hole, Kruskal Szekeres coordinates (T, X, ω), ω denoting the angular variables of the Schwarzschild coordinate system, are defined by

$$T = \frac{1}{2} e^{\frac{r_*}{4M}} \left(e^{\frac{t}{4M}} - e^{-\frac{t}{4M}}\right) , \; X = \frac{1}{2} e^{\frac{r_*}{4M}} \left(e^{\frac{t}{4M}} + e^{-\frac{t}{4M}}\right) , \tag{6.10}$$

where r_* is the Regge-Wheeler variable outside the black hole given by (6.2)

$$r_* = r + 2M\mathrm{Log}(r - 2M) .$$

This coordinate system maps the exterior of the black hole $\mathbb{R}_t \times]2M, +\infty[_r \times S^2_\omega$ onto the quadrant $\{X > |T|\}$ of $\mathbb{R}_T \times \mathbb{R}_X \times S^2_\omega$. The horizon now appears as the hypersurface $\{(T, X, \omega) ;\ T =$

$X > 0\,,\ \omega \in S^2\}$. The outgoing (resp. incoming) radial null geodesics, represented in (t, r_*, ω) coordinates as the straight lines $\{(t, r_* = t + s, \omega)\,;\ t \in \mathbb{R}\}$ (resp. $\{(t, r_* = -t + s, \omega)\,;\ t \in \mathbb{R}\}$) for fixed $s \in \mathbb{R}$ and $\omega \in S^2$, are described in Kruskal-Szekeres coordinates as the straight lines $\{(T, X = T + S, \omega)\}$ (resp. $\{(T, X = -T + S, \omega)\}$) for fixed S and ω.

Inside the black hole, the definition is very similar. We consider the Regge-Wheeler coordinate adapted to this domain (given by (6.8))

$$r_* = r + 2M\mathrm{Log}|r - 2M| = r + 2M\mathrm{Log}(2M - r)\,,$$

the expression of the variables T and X in terms of t and r_* is then given by

$$T = \frac{1}{2}e^{\frac{r_*}{4M}}\left(e^{-\frac{t}{4M}} + e^{\frac{t}{4M}}\right)\,,\ X = \frac{1}{2}e^{\frac{r_*}{4M}}\left(e^{-\frac{t}{4M}} - e^{\frac{t}{4M}}\right)\,. \tag{6.11}$$

The interior of the black hole $\mathbb{R}_t \times]0, 2M[_r \times S^2_\omega$ is mapped onto the domain $\{(T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2\,;\ |X| < T < \sqrt{X^2 + 2M}\}$ and the singularity at $r = 0$ is represented as the product of S^2_ω with the hyperbola in the (T, X)-plane : $\{(T, X)\,;\ T^2 - X^2 = 2M\,,\ T > 0\}$.

The expression of the metric in Kruskal-Szekeres coordinates is the same inside and outside the black hole

$$g = \frac{16M^2}{X^2 - T^2}\left(1 - \frac{2M}{r}\right)\left(\mathrm{d}T^2 - \mathrm{d}X^2\right) - r^2\mathrm{d}\omega^2\,.$$

This can be simplified using the fact that

$$X^2 - T^2 = (r - 2M)e^{\frac{r}{2M}} \tag{6.12}$$

and we obtain

$$g = \frac{16M^2}{r}e^{-\frac{r}{2M}}\left(\mathrm{d}T^2 - \mathrm{d}X^2\right) - r^2\mathrm{d}\omega^2 \tag{6.13}$$

where r is determined implicitely in terms of T and X by (6.12). The function $(r - 2M)e^{\frac{r}{2M}}$ is analytic in r and strictly increasing from $]0, +\infty[$ onto $]-2M, +\infty[$. It follows that r is an analytic function of $X^2 - T^2$, and therefore of (T, X), on $-2M < X^2 - T^2 < +\infty$. An immediate consequence is the analyticity of the metric g on the whole Schwarzschild manifold, described in (T, X, ω) coordinates as $\{(T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2\,;\ T + X > 0\,,\ T < \sqrt{X^2 + 2M}\}$ (the singularity at $r = 0$ is not considered as a subset of the Schwarzschild manifold).

This construction is another way of showing that the metric g is not singular at the horizon of the black hole ; the expression (6.13) of g and the description of the horizon in (T, X, ω) coordinates reveal it to be a smooth null hypersurface of Schwarzschild's space-time. This can be seen as an alternative to the construction we performed earlier with the incoming Eddington-Finkelstein coordinates. This has an advantage over the previous construction however, it can now be extended into the "maximal Schwarzschild spacetime".

Maximal Schwarzschild space-time

As we have seen above, the metric (6.13) can be extended analytically on the region

$$\mathcal{M}^{\mathcal{K}} = \left\{(T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2_\omega\,;\ X^2 - T^2 > -2M\right\}\,.$$

We obtain a new space-time $(\mathcal{M}^{\mathcal{K}}, g)$ called the Kruskal extension, or maximal analytic extension, of Schwarzschild's space-time. It contains four blocks separated by a bifurcate horizon $\{|T| = |X|\}$ (see figure 6.2) :

$$\begin{aligned}
\mathrm{I} &:= \left\{(T,X,\omega)\,,\ X > |T|\,,\ \omega \in S^2\right\}\,,\\
\mathrm{II} &:= \left\{(T,X,\omega)\,,\ |X| < T < \sqrt{2M + X^2}\,,\ \omega \in S^2\right\}\,,\\
\mathrm{III} &:= \left\{(T,X,\omega)\,,\ X < -|T|\,,\ \omega \in S^2\right\}\,,\\
\mathrm{IV} &:= \left\{(T,X,\omega)\,,\ -|X| > T > -\sqrt{2M + X^2}\,,\ \omega \in S^2\right\}\,.
\end{aligned}$$

Blocks I and III are exteriors (corresponding to $r > 2M$) and the blocks II and IV are interiors (corresponding to $0 < r < 2M$). The realization of the Schwarzschild manifold that we constructed using the incoming (resp. outgoing) Eddington-Finkelstein coordinates is the union of blocks I and II (resp. I and IV) with the part of the horizon between them.

The union of blocks III and IV with the part of the horizon between them is also a realization of the Schwarzschild manifold ; it is isometric to the union of blocks I and II with the adequate part of the horizon with the time orientation reversed. More explicitely, blocks III and IV are the image of the Schwarzschild space-time, described in Schwarzschild coordinates, by the transformations (6.10) and (6.11) with the signs of T and X reversed.

The space-time $(\mathcal{M}^{\mathcal{K}}, g)$ is best pictured by a Penrose diagram, which can be constructed by defining the new coordinates (which are not smooth and only of practical use to get a picture of the general structure of $\mathcal{M}^{\mathcal{K}}$, not for any calculation) :

$$\alpha = \arctan\left(\frac{T+X}{\sqrt{2M}}\right) - \arctan\left(\frac{T-X}{\sqrt{2M}}\right)\,,$$

$$\beta = \arctan\left(\frac{T+X}{\sqrt{2M}}\right) + \arctan\left(\frac{T-X}{\sqrt{2M}}\right)\,.$$

This diagram will make more sense very soon after we have constructed the complete boundary (except for a few "points") ; this is done in the next section. Note that $(\mathcal{M}^{\mathcal{K}}, g)$ is globally hyperbolic, the hypersurface $\{\tau = 0\}$ is a Cauchy hypersurface.

6.1.5 Conformal compactification

Schwarzschild's spacetime contains mass. This is apparent in the asymptotic behaviour of the metric : some terms are proportional to the mass M of the black hole and fall off in $1/r$ at infinity. These terms prevent the construction of a complete regular compactification similar to what can be done with Minkowski spacetime. A partial compactification however is possible and yields in the limit $M \to 0$ a partial compactification of Minkowski spacetime where only $\mathscr{I}^{\pm}$ are defined but neither $i^{\pm}$ nor i^0. This compactification is performued using the variables $u = t - r_*$ and $v = t + r_*$. The lines of constant (u, ω), resp. (v, ω), are outgoing, resp. incoming, radial null geodesics. They are referred to as the principal null geodesics because their tangent vectors are double roots of the Weyl tensor (see [23] for more details).

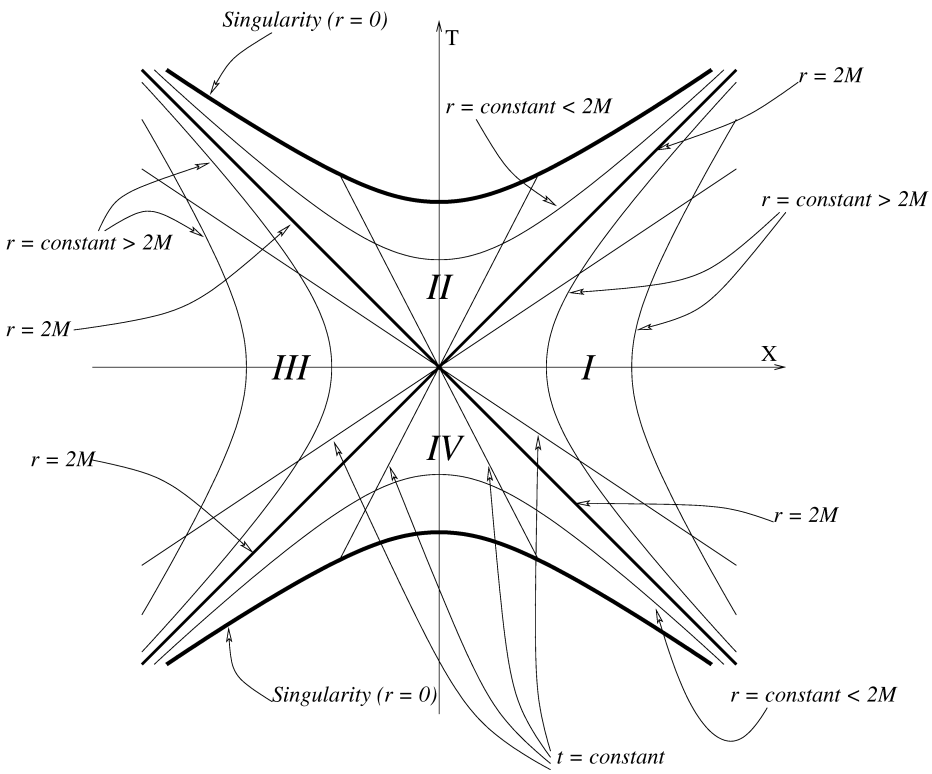

Figure 6.2: The maximal analytic extension of Schwarzschild's space-time in Kruskal-Szekeres coordinates : domains I and III correspond to $r > 2M$, domain II represents the interior of the black hole and domain IV the interior of the white hole.

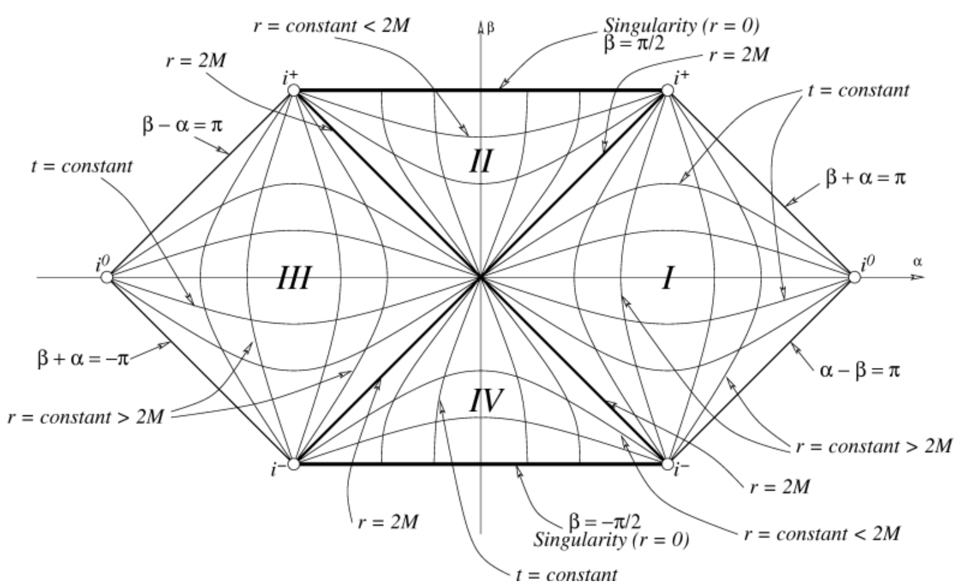

Figure 6.3: The Penrose diagram of maximal Schwarzschild space-time

In terms of variables $u = t - r_*$, $R = 1/r$, θ and φ, the Schwarzschild metric g takes the form

$$g = (1 - 2MR)\mathrm{d}u^2 - \frac{2}{R^2}\mathrm{d}u\mathrm{d}R - \frac{1}{R^2}\mathrm{d}\omega^2 .$$

Rescaling the metric with the conformal factor $\Omega = R = 1/r$, we obtain

$$\hat{g} = R^2 g = R^2(1 - 2MR)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}R - \mathrm{d}\omega^2 ,$$

which extends as an analytic metric on the domain $\mathbb{R}_u \times \left[0, \frac{1}{2M}\right[_R \times S^2_{\theta,\varphi}$. Similarly to the Minkowski case, we can add a boundary to the exterior of the black hole : the hypersurface $\mathbb{R}_u \times \{0\}_R \times S^2_{\theta,\varphi}$. A point $(u_0, 0, \theta_0, \varphi_0)$ on the boundary is reached along the outgoing radial null geodesic

$$\begin{aligned}\gamma_{u_0,\theta_0,\varphi_0}(r) &= (t = r + 2M\mathrm{Log}(r - 2M) + u_0\,,\ r\,,\ \theta = \theta_0\,,\ \varphi = \varphi_0) \\ &= \left(u = u_0\,,\ R = \frac{1}{r}\,,\ \theta = \theta_0\,,\ \varphi = \varphi_0\right)\end{aligned}$$

as $r \to +\infty$ and there is a one to one correspondence between the points on the boundary and the outgoing radial null geodesics. The hypersurface therefore represents future null infinity, $\mathscr{I}^+$, for the Schwarzschild metric.

Using theorem 2.9 and the fact that the scalar curvature of the Schwarzschild metric is zero, we can calculate the scalar curvature of the rescaled metric $\hat{g} = R^2 g$ and we find

$$\frac{1}{6}\mathrm{Scal}_{\hat{g}} = 2MR .$$

6.2 Asymptotically simple spacetimes

Asymptotically simple spacetimes were introduced by Roger Penrose (see for example [22]) as generic models of asymptotically flat spacetimes. Here, the notion of asymptotic flatness is to be understood in a stronger sense as in the case of Schwarzschild's spacetime ; the curvature falls off to zero at infinity in all directions : timelike, spacelike or null. There definition is formulated in terms of condormal compactification and the "degree of flatness at infinity" is expressed in terms of the regularity of the conformally rescaled metric at the conformal boundary. We shall not worry about the quantitative fall-off of the curvature at infinity here and we therefore only consider asymptotically simple spacetimes for which the metric is $\mathcal{C}^\infty$ at the conformal boundary.

Definition 6.1. *A spacetime $(\mathcal{M}, g)$ is said to be asymptotically simple if $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^4$ and there exist a positive function Ω on $\mathcal{M}$ and a smooth spacetime with boundary $(\bar{\mathcal{M}}, \hat{g})$, such that :*

1. *$\hat{g} = \Omega^2 g$ on $\mathcal{M}$, Ω vanishes at the boundary of $\bar{\mathcal{M}}$ but $\mathrm{d}\Omega$ is nowhere zero there ;*

2. *$\mathcal{M}$ is the interior of $\bar{\mathcal{M}}$, the boundary of $\bar{\mathcal{M}}$ is the union of two points $i^\pm$, the past light-cone of i^+ (denoted $\mathscr{I}^+$) and the future light-cone of i^- (denoted $\mathscr{I}^-$) ;*

3. *every inextendible null geodesic acquires a future end-point on $\mathscr{I}^+$ and a past end-point on $\mathscr{I}^-$.*

The place where $\mathscr{I}^+$ and $\mathscr{I}^-$ meet, i.e. spacelike infinity i^0, is a in general a singularity of the conformal structure, it is not part of the "compactified spacetime" $\bar{\mathcal{M}}$ which is therefore not compact.

Such spacetimes were initially considered by many people as rather empty idealizations of "real" asymptotically flat spacetimes because it was not known whether the Einstein vacuum equations admitted any asymptotically simple solutions. Recently, the works of Chrusciel-Delay, Corvino-Schoen and Klainerman-Nicolò [4, 5, 13] established the existence of a large class of such Einstein spacetimes. They are generically time-dependent, which prevents the use of standard analytic methods for the construction of a scattering theory.

As already mentionned, the Schwarzschild metric is a model for asymptotically flat spacetimes containing energy/matter in the following sense : at first order, any such spacetime differs at infinity from the flat one by a Schwarzschild-type contribution that falls-off like $1/r$, r being, say, the spacelike geodesic distance from a given timelike curve for a choice of spacelike asymptotically flat slicing. The spacetimes constructed by Klainerman and Nicolò in [13] are asymptotic to the Schwarzschild metric on each slice in this sense. Such structures are delicate to manipulate analytically because all conservation laws fail and are replaced by "approximate conservation laws" that, essentially, work in the same manner, but require some care and painful calculations. The spacetimes of Corvino/Schoen-Chrusciel/Delay are simplified versions of the Klainerman-Nicolò universes in that they are exactly diffeomorphic to the Schwarzschild metric in a neighbourhood of spacelike infinity (in the simplest case, other versions are diffeomorphic to rotating metrics near i^0). This means that near spacelike infinity at least, we have the luxury of the symmetries of the Schwarzschild metric which grants us access to exact conservation laws. This makes the conformal scattering construction more clearcut and avoids cumbersome estimates.

In the next chapter, we shall work with asymptotically simple spacetimes that are diffeomorphic to the Schwarzschild spacetime in a neighbourhood of i^0.

6.3 Exercises

Exercise 6.1. Domain of influence

Let us consider for $2M < r_1 < r_2 < +\infty$, $-\infty < t_1 < t_2 < +\infty$, $0 < r_0 < 2M$, the domains defined in Schwarzschild coordinates by :

$$D_1 := \{(t, r, \theta, \varphi)\,,\ r_1 < r < r_2\,,\ t = 0\}\,,$$
$$D_2 := \{(t, r, \theta, \varphi)\,,\ r = r_0\,,\ t_1 < t < t_2\}\,.$$

1. *Determine the domain of influence of D_1 in the exterior of the back hole.*

2. *Determine the domain of influence of D_1 in the maximal extension of Schwarzschild's spacetime.*

3. *Determine the domain of influence of D_2 in the interior of the black hole.*

Exercise 6.2. Global hyperbolicity

1. *Find a Cauchy hypersurface in the exterior of the black hole.*
2. *Find a Cauchy hypersurface in the interior of the black hole.*
3. *Find a Cauchy hypersurface in the maximal extension of Schwarzschild's spacetime.*

Exercise 6.3. Free fall into the black hole

1. *Find all radial geodesics outside a Schwarzschild black hole.*
2. *Give a graphic interpretation of the fact that an object in free fall directly towards the black hole appears to a distant observer as becoming ever flatter as it approaches the horizon.*

Chapter 7

The wave equation on asymptotically simple spacetimes

7.1 Lars Hörmander's solution of the generalized Cauchy problem

As we have seen already in the simple example of Minkowski spacetime, a crucial ingredient of a conformal scattering theory is the resolution of the characteristic Cauchy problem, or Goursat problem, with data set on null infinity. Several methods are available that provide solutions to a Goursat problem for a wide class of hyperbolic equations. A classic approach is to find an integral formula for the solution using a Green function, i.e. a two-point function $G(p, q)$ such that, when the operator is applied to G in the variable p, it gives the Dirac distribution at the point q. Another no less classic approach uses energy estimates : it has been formulated very neatly by Lars Hörmander in a short paper in 1990 [12]. This section presents an equivalent form of his results under a more geometrical form.

The geometrical framework chosen by Hörmander is as follows. Let $\tilde{X}$ be a smooth globally hyperbolic $(n+1)$-dimensional spacetime, $n \geq 1$, that is spatially compact[1]. Then we have a smooth time function t on $\tilde{X}$ whose level hypersurfaces are Cauchy hypersurfaces and are diffeomorphic to a fixed n-dimensional manifold X (without boundary). Using a global timelike vector field, we can therefore realize $\tilde{X}$ as $\mathbb{R} \times X$. With this identification, the level-hypersurfaces of t are simply $X_t = \{t\} \times X$. We do this identification using the gradient of t. Since it is orthogonal to the level hypersurfaces of t, we can perform an orthogonal decomposition of the metric g into parts along ∇t and along the level hypersurfaces of t. We have

$$g = N^2 \mathrm{d}t^2 - h \tag{7.1}$$

where h is a smooth time-dependent Riemannian metric on X and N is the lapse-function, defined by

$$N = \frac{1}{g(\nabla t, \nabla t)} .$$

[1]Meaning that each closed spacelike hypersurface is compact.

On $\tilde{X}$, we consider a perturbed wave equation of the form

$$\Box_g u + u + L_1 u = 0 \tag{7.2}$$

where L_1 is a general first order differential operator

$$L_1 = b^a \nabla_a + c \tag{7.3}$$

where the vector field b^a, and scalar field c are assumed to be $\mathcal{C}^\infty$ on $\tilde{X}$. The hypersurface on which the initial data are specified can be a spacelike Cauchy hypersurface for a standard Cauchy problem, a light cone for a characteristic Cauchy problem (Goursat problem), or anything in between. It is defined as follows

$$\Sigma = \{(\varphi(x), x) ;\ x \in X\} \ ,\ \varphi \ :\ X \longrightarrow \mathbb{R} \, , \tag{7.4}$$

where φ is simply assumed to be Lipschitz on X, to allow for singularities such as the vertex of a light cone, and "weakly spacelike", by which we mean

$$g^{ab}(\varphi(x), x) \nabla_a (t - \varphi(x)) \nabla_b (t - \varphi(x)) \geq 0 \text{ almost everywhere on } X \, . \tag{7.5}$$

Condition (7.5) has a meaning, since Lipschitz functions are differentiable almost everywhere, and it simply says that Σ is allowed to be locally spacelike or null but not timelike, i.e. its normal vector field is required to be causal where it is defined. If Σ is uniformly spacelike, we are studying a standard Cauchy problem, with the slight difference that the hypersurface on which we set the data is not very regular. If Σ is (almost) everywhere null, we are looking at a Goursat problem. Hörmander's approach is to study both problems and anything in between at the same time by allowing the hypersurface Σ to be locally spacelike or null.

The well-posedness of the Cauchy problem (set on X_0 for example) for equation (7.2) is classic in many function spaces. We give an idea of the arguments involved to get well-posedness for finite energy data. They are similar to one of the approaches we used in the flat case. The key result is the following theorem, which we admit ; it is a consequence of an even more general result due to Leray [15].

Theorem 7.1. *On a globally hyperbolic spacetime $(\mathcal{M}, g)$, consider a Cauchy hypersurface Σ and a timelike, future-oriented vector field τ. For data $\phi_0, \phi_1 \in \mathcal{C}_0^\infty(\Sigma)$, the equation*

$$\Box_g \phi + L\phi = 0 \, ,$$

where L is a smooth first order differential operator, has a unique smooth solution on $\mathcal{M}$ such that

$$\phi|_\Sigma = \phi_0 \text{ and } \nabla_\tau \phi|_\Sigma = \phi_1 \, .$$

Then, all we need, for solving both the Cauchy problem and the Goursat problem, is energy estimates. To obtain these, we need to construct an energy current. We consider the stress-energy tensor for the Klein-Gordon equation

$$\Box_g \phi + \phi = 0$$

given by

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}\langle\nabla\phi\,,\ \nabla\phi\rangle_g g_{ab} + \frac{1}{2}\phi^2 g_{ab}\,.$$

It satisfies, for ϕ a solution of (7.2),

$$\nabla^a T_{ab} = (\Box_g\phi + \phi)\nabla_b\phi = -L_1\phi\nabla_b\phi\,. \tag{7.6}$$

We define the energy current associated with the vector field $\tau = \nabla t/(g(\nabla t, \nabla t))^{1/2}$ (which is the unit vector field along the direction of ∇t)

$$J^a := T^a_b\tau^b\,.$$

This current is of course not conserved but it satisfies an "approximate conservation law"

$$\nabla^a J_a = T_{ab}\nabla^{(a}\tau^{b)} - L_1\phi\nabla_\tau\phi\,. \tag{7.7}$$

The following energy estimate is straightforward, its proof is the object of Exercise 7.2 : for any $T>0$, there exists a constant $C>0$ such that,

$$E_{X_t}(\phi) \leq E_{X_s}(\phi)e^{C|t-s|}\ \forall t,s\in[-T,T]\,, \tag{7.8}$$

where

$$E_{X_t}(\phi) = \int_{X_t} *J_a \mathrm{d}x^a\,.$$

This and Theorem 7.1 imply the well-posedness of the Cauchy problem for finite energy data. This can be stated a little more precisely. Note that the regularity of the metric and the time function imply that the norms

$$\|.\|_{H^1(X_t)} = \sqrt{E_{X_t}(.)}$$

are equivalent for any two values of t, this equivalence being locally uniform in time. The same is true for the $L^2(X_t)$ norms induced on each X_t by the metric g. We define as the H^1 and L^2 norms on X the $H^1(X_0)$ and $L^2(X_0)$ norms, the spaces $H^1(X)$ and $L^2(X)$ being the completions of $\mathcal{C}^\infty_0(X)$ in these norms (which are the same as the completions in any other $H^1(X_t)$ and $L^2(X_t)$ norms). Then we have

Theorem 7.2. *For any* $(\phi_0\,,\ \phi_1)\in H^1(X)\times L^2(X)$*, there exists a unique solution of* (7.2)

$$\phi\in\mathcal{C}(\mathbb{R}_t\,;\ H^1(X))\cap\mathcal{C}^1(\mathbb{R}_t\,;\ L^2(X))$$

such that

$$\phi|_{X_0} = \phi_0 \text{ and } \nabla_\tau\phi|_{X_0} = \phi_1\,.$$

Proof. See Exercise 7.3. □

We denote by $\mathcal{E}$ the space of finite energy solutions of (7.2), i.e. the set of solutions of (7.2) in $\mathcal{C}(\mathbb{R}_t\,;\ H^1(X))\cap\mathcal{C}^1(\mathbb{R}_t\,;\ L^2(X))$. Note that this space can be canonically identified with $H^1(X)\times L^2(X)$ by taking the initial data for each solution. Using the energy on Σ of a solution ϕ, we also define a function space on Σ. It will be our natural space of data for the Cauchy problem on Σ. Let ϕ be a solution of (7.2) in $\mathcal{E}$, recall from proposition **??** that the energy flux of ϕ across Σ is given by

$$E_\Sigma(\phi) = \int_\Sigma *J_a \mathrm{d}x^a = \int_\Sigma \tau^a T_{ab} \nu^b (L \lrcorner \mathrm{dVol}) \, ,$$

where ν is a future-oriented normal vector field to Σ and L a future-oriented transverse vector field to Σ such that $g(L,\nu)=1$. Note that the measure $L\lrcorner\mathrm{dVol}$ will be uniformly equivalent to the measure μ_Σ lifted from the measure induced by g on X_0 via the parametrization (7.4) of Σ. In our case, we have an explicit choice of ν given by

$$\nu = \nabla(t - \varphi(x)) \, .$$

At points where $\nabla\varphi = 0$, we have $\nu = \nabla t$ and we can take $L = N^2 \nabla t$. We obtain the usual energy density on the X_t slices. At points where $\nabla\varphi \neq 0$, things will be different. If the vector ν remains spacelike, the energy density will still be equivalent to that on the slices X_t, it ν becomes null however, the equivalence will be lost. Let us see this in more detail. At a point where $\nabla\varphi \neq 0$, we choose an orthonormal frame as follows :

$$e_0 = \tau \, , \; e_1 = (\nabla\varphi)/(g(\nabla\varphi,\nabla\varphi))^{1/2} \, , \; e_2 \, , \; e_3 \, ,$$

and we put

$$l = \frac{1}{\sqrt{2}}(e_0 + e_1) \, , \; n = \frac{1}{\sqrt{2}}(e_0 - e_1) \, .$$

The vectors l and n are null and ν is null if and only if $g(\nabla t, \nabla t) + g(\nabla\varphi, \varphi) = 0$ since ∇t and $\nabla\varphi$ are g-orthogonal. We can decompose the gradient of ϕ along the basis $\{l, n, e_2, e_3\}$

$$\nabla\phi = (\nabla_n\phi) l + (\nabla_l)\phi n - (\nabla_{e_2}\phi) e_2 - (\nabla_{e_3}\phi) e_3$$

and we have

$$g(\nabla\phi,\nabla\phi) = 2(\nabla_n\phi)(\nabla_l\phi) + (\nabla_{e_2}\phi)^2 - (\nabla_{e_3}\phi)^2 \, .$$

We then use this to express $T_{ab}\tau^a\nu^b$:

$$\begin{aligned} T_{ab}\tau^a\nu^b &= \frac{1}{2} T_{ab}(l^a+n^a)(g(\nabla t,\nabla t)^{1/2}(l^b+n^b) - |g(\nabla\varphi,\nabla\varphi)|^{1/2}(l^b-n^b)) \\ &= \frac{1}{2}(g(\nabla t,\nabla t)^{1/2} - |g(\nabla\varphi,\nabla\varphi)|^{1/2}) T_{ab}(l^a+n^a) l^b \\ &\quad + \frac{1}{2}(g(\nabla t,\nabla t)^{1/2} + |g(\nabla\varphi,\nabla\varphi)|^{1/2}) T_{ab}(l^a+n^a) n^b \, . \end{aligned}$$

We develop each term as follows :

$$\begin{aligned} T_{ab}(l^a+n^a)l^b &= \nabla_n\phi\nabla_l\phi - \frac{1}{2}\langle\nabla\phi,\nabla\phi\rangle_g + \frac{1}{2}\phi^2 + (\nabla_l\phi)^2 \\ &= \frac{1}{2}((\nabla_l\phi)^2 + (\nabla_{e_2}\phi)^2 + (\nabla_{e_3}\phi)^2 + \phi^2) \, ; \\ T_{ab}(l^a+n^a)l^b &= \frac{1}{2}((\nabla_n\phi)^2 + (\nabla_{e_2}\phi)^2 + (\nabla_{e_3}\phi)^2 + \phi^2) \, . \end{aligned}$$

So we see that the energy on σ has the following form

$$\begin{aligned} E_\Sigma(\phi) &= \int_\Sigma \Big(\frac{1}{4}(g(\nabla t, \nabla t)^{1/2} - |g(\nabla\varphi, \nabla\varphi)|^{1/2})(\nabla_l \phi)^2 \\ &\quad + \frac{1}{4}(g(\nabla t, \nabla t)^{1/2} + |g(\nabla\varphi, \nabla\varphi)|^{1/2})(\nabla_n \phi)^2 \\ &\quad + \frac{1}{2} g(\nabla t, \nabla t)^{1/2}((\nabla_{e_2}\phi)^2 + (\nabla_{e_3}\phi)^2 + \phi^2)\Big) L \lrcorner \mathrm{dVol}\,. \end{aligned} \tag{7.9}$$

At points where ν is null, we have $g(\nabla t, \nabla t)^{1/2} - |g(\nabla\varphi, \nabla\varphi)|^{1/2} = 0$; we lose the information of the derivative along l, i.e. the derivative transverse to Σ, and we keep only information on derivatives along n, e_2 and e_3, i.e. tangent to Σ.

Considering ϕ and $\nabla_\tau \phi$ as independent functions, the energy (7.9) induces a semi-norm on the pairs $(\phi\,,\ \nabla_\tau\phi)$ that are restrictions to Σ of smooth functions on $\tilde{X}$. We denote this semi-norm by $\|(\phi\,,\ \nabla_\tau\phi)\|_{\mathcal{E}_\Sigma}$. It only fails to be a norm at points where Σ is null, by losing control over $\nabla_\tau\phi$. We denote by $\mathcal{E}_\Sigma$ the completion of restrictions to Σ of pairs of smooth functions on $\tilde{X}$ in the semi-norm $\|(.\,,\ .)\|_{\mathcal{E}_\Sigma}$. If Σ is null (almost) everywhere, $\mathcal{E}_\Sigma$ is simply a natural H^1 space on Σ, it really is a space of real valued functions, not pairs of them. In the general case, we introduce the density measure on Σ

$$\mathrm{d}\nu^0_{\Sigma} = \Big(g^{ab}(\varphi(x), x)\nabla_a(t - \varphi(x))\nabla_b(t - \varphi(x))\Big)\, \mathrm{d}\nu_\Sigma\,,$$

which is uniformly equivalent to

$$(g(\nabla t, \nabla t)^{1/2} - |g(\nabla\varphi, \nabla\varphi)|^{1/2})\mathrm{d}\nu_\Sigma\,,$$

is positive where Σ is spacelike and vanishes where Σ is null. The space $\mathcal{E}_\Sigma$ can be understood as follows

$$\mathcal{E}_\Sigma \simeq H^1(\Sigma) \oplus L^2(\Sigma; \mathrm{d}\nu^0_\Sigma)\,.$$

and the associated L^2 space $L^2(\Sigma; \mathrm{d}\nu^0_\Sigma)$.

The main result of [12] is the following :

Theorem 7.3. (Hörmander, 1990) *The map*

$$\begin{array}{cccc} \mathbb{T}_\Sigma & : & \mathcal{E} & \longrightarrow & \mathcal{E}_\Sigma \\ & & \phi & \longmapsto & \left(\phi_{|\Sigma}\ ,\ \nabla_\tau\phi_{|\Sigma}\right)\,, \end{array} \tag{7.10}$$

which is well defined for smooth solutions, extends as an isomorphism. In particular, there exists a constant $C > 0$ such that, for any $\phi \in \mathcal{E}$, we have

$$\|\mathbb{T}_\Sigma\phi\|^2_{\mathcal{E}_\Sigma} \leq C(\|\phi|_{t=0}\|^2_{H^1(X_0)} + \|\nabla_\tau\phi|_{t=0}\|^2_{L^2(X_0)})$$

and

$$\|\phi|_{t=0}\|^2_{H^1(X_0)} + \|\nabla_\tau\phi|_{t=0}\|^2_{L^2(X_0)} \leq C\,\|\mathbb{T}_\Sigma\phi\|^2_{\mathcal{E}_\Sigma}\,,$$

or equivalently (we might have to take a larger $C > 0$)

$$E_\Sigma(\phi) \leq C E_{X_0)}(\phi) \tag{7.11}$$

and

$$E_{X_0)}(\phi) \leq C E_\Sigma(\phi)\,. \tag{7.12}$$

Proof. The structure of the proof is divided in two main steps. First we establish energy estimates both ways for smooth solutions of the equation (7.2), these are exactly estimates (7.11) and (7.12). This will entail that the trace operator $\mathbb{T}_{\Sigma}$ extends as a bounded linear operator from $\mathcal{E}$ into $H^1(\Sigma) \oplus L^2(\Sigma; \mathrm{d}\nu^0_{\Sigma})$, that is one-to-one and with closed range. Then we shall construct solutions to the Cauchy problem on Σ for "smooth" data. This will entail the density of the range of $\mathbb{T}_{\Sigma}$ in $H^1(\Sigma) \oplus L^2(\Sigma; \mathrm{d}\nu^0_{\Sigma})$ and prove the theorem.

Step 1 : energy estimates.

Step 2 : solutions for smooth data.

7.2 Friedlander's radiation fields

They are the generalization to curved spacetimes of the scattering data $\hat{\phi}^{\pm}$ in the Minkwoski case. Friedlander introduced and used them on static spacetimes with a strong enough decay at infinity to ensure a smooth $\mathscr{I}$. Such a class of spacetimes, although it was the natural thing to consider as a non flat framework in which to extend the Lax-Phillips theory using conformal geometry, is unphysical in that it contains no solution of the Einstein cacuum equations apart from Minkowski spacetime. The notion of radiation fields however survives in all asymptotically simple spacetimes, and even in the much larger class of spacetimes admitting a smooth $\mathscr{I}$, even though the most analytically explicit aspects of the Lax-Phillips scattering theory cannot be recovered in such general cases (in particular the translation representer).

7.3 The conformal scattering construction

7.4 Exercises

Exercise 7.1. *This exercise is centered on the weakly spacelike condition (7.5) for the hypersurface Σ in Hörmander's approach to the Cauchy problem. We consider the hypersurface Σ defined by (7.4) and we assume that the function φ is smooth.*

1. *Prove that Σ is everywhere spacelike if and only if*

$$g^{ab}(\varphi(x),x)\nabla_a(t-\varphi(x))\nabla_b(t-\varphi(x)) > 0 \;\forall x \in X\,.$$

2. *Prove that Σ is null at a point $(\phi(x),x)$ if and only if*

$$g^{ab}(\varphi(x),x)\nabla_a(t-\varphi(x))\nabla_b(t-\varphi(x)) = 0\,.$$

Exercise 7.2. *Prove estimate (7.8).*

Exercise 7.3. *Using theorem 7.1 and (7.8), prove theorem 7.2.*

Chapter 8

The wave equation on the Schwarzschild spacetime

8.1 The Cauchy problem

8.2 Energy current and its conservation law

8.3 Some decay results

8.4 The conformal scattering construction

8.5 Peeling

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