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Conformal scattering and the Goursat problem

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Abstract

We work on a class of non-stationary vacuum space-times admitting a conformal compactification that is smooth at null and timelike infinity. Via a conformal transformation, the existence of a scattering operator for field equations is interpreted as the well-posedness of a Goursat problem on null infinity. We solve the Goursat problem in the case of Dirac and Maxwell fields. The case of the wave equation is also discussed and it is shown why the method cannot be applied at present. Then the conformal scattering operator is proved to be equivalent to an analytical scattering operator defined in terms of classical wave operators.

1 Introduction

Scattering theory, used in the framework of general relativity, is a powerful tool for studying the long-time influence of the geometry of space-time on the behaviour of fields. Stationary scattering was in particular used by S. Chandrasekhar *et al.* [8] to calculate the quasi-nomal modes (resonances) of field equations, of spin lower than or equal to two, on black hole geometries such as Scharzschild and Kerr. Time-dependent scattering was first used by J. Dimock [15] and J. Dimock and B. Kay [16, 17, 18] to describe the asymptotic behaviour of classical and quantum scalar fields on the Schwarzschild metric. Since then, many works in the same spirit have appeared, by A. Bachelot [1, 2, 3, 4, 5], A. Bachelot and A. Motet-Bachelot [6], D. Häfner [22, 23], D. Häfner and J.-P. Nicolas [24], F. Melnyk [31, 32], J.-P. Nicolas [33]. As a result, we now have a rather detailed understanding of the classical and quantum scattering by a spherically symmetric eternal black hole, including proofs of the Hawking effect, and our grasp of the Kerr case is slowly improving.

All the methods used in the works just cited, however successful they may have been, suffer from a common drawback : they rely heavily on spectral methods and therefore can only be applied in time independent situations (space-times with a Killing vector that can be interpreted as the time coordinate vector field, which does not necessarily mean

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stationary, e.g. Kerr). This constraint is purely technical. The relevent information for the development of a scattering theory in relativity is the asymptotics of the metric. These asymptotics are encoded in those of the field equations and then extracted by means of spectral techniques to obtain a scattering theory. It is these spectral techniques that are incompatible with generic time-dependence. The purpose of this paper is to show that, using more geometrical methods, it is possible to extract directly from the asymptotics of the metric, information on the scattering properties of fields ; the results obtained are completely equivalent to those obtained by analytical scattering techniques.

The main tool is the Penrose conformal compactification : a space-time $(\mathscr{M}, g)$ is represented by an "un-physical" space-time $(\widehat{\mathscr{M}}, \hat{g})$, $\hat{g}$ being a conformal rescaling of g, whose boundary (made of two null hypersurfaces $\mathscr{I}^{\pm}$ and "points" $i^{\pm}$ and i^{0}) describes the infinity in the original space-time. The asymptotics of the physical metric g are encoded into the regularity of $\hat{g}$ at the boundary of $\widehat{\mathscr{M}}$. The asymptotic behaviour of solutions of covariant hyperbolic and conformally invariant field equations is then entirely described by the trace of the solutions of the rescaled equations on $\mathscr{I}^{\pm}$. The idea is then to interpret the notion of scattering operator, at the level of the rescaled space-time, as an operator that to the trace of the rescaled field on $\mathscr{I}^{-}$ associates its trace on $\mathscr{I}^{+}$. To show that this operator is well-defined and is an isomorphism between adequate function spaces, it is necessary to solve the Goursat problem (or characteristic Cauchy problem) on $\mathscr{I}^{\pm}$ for the rescaled equation.

The essential ideas concerning the Goursat problem on $\mathscr{I}^{\pm}$ were developed by R. Penrose in [35]. The first use of these techniques for constructing a scattering theory is due to F.G. Friedlander [20, 21] in the context of static space-times with regular conformal structure at space-like infinity. The techniques were taken up again by J. Baez, I. Segal and S. Zhou [7] ; this is a construction in Minkowski space for nonlinear, conformally invariant wave equations, showing that the existence of a conformal scattering operator (that to traces on $\mathscr{I}^{-}$ associates traces on $\mathscr{I}^{+}$) is equivalent to a scattering theory defined in terms of classical wave operators. Immediately after this work, L. Hörmander published a short paper giving a rigorous proof of the solution of the Goursat problem for generic wave equations on spatially compact space-times [26]. Since then, as far as we know, the idea seems not to have been pushed further.

We work with space-times, referred to as smooth asymptotically simple space-times, whose Penrose compactification is regular at timelike and null infinity. The notion of asymptotically simple space-times was first introduced by R. Penrose (see for example [36] vol. 2) and since the works of Corvino in 2000 [13], Chrusciel and Delay in 2002 and 2003 [11, 12] and Corvino and Schoen in 2003 [14], it is known that there are solutions of the Einstein vacuum equations with full functional degrees of freedom in the initial data that are asymptotically simple with prescribable regularity at null and timelike infinity. This provides a generic framework of non stationary vacuum space-times with regular conformal compactification. On such space-times, we construct a conformal scattering operator for the Dirac and Maxwell equations and we prove the equivalence with analytical scattering theories defined in terms of classical wave operators. The case of the wave equation requires some extra details and will be treated fully in a separate paper; here we indicate some of the main steps but dont give a complete treatment. The paper is organized as follows :

- Section 2 contains a description of the geometry of the space-times we work with and their conformal compactification. The three equations we study are also described, including the Cauchy problem, conserved quantities and conformal invariance. Subsection 2.2 contains the important definitions (definitions 2.1 and 2.2) of the spin-frames we shall use in the physical and rescaled space-times.

- The next section (section 3) describes the strategy of construction of the conformal scattering operator (following the ideas of Hörmander in [26]) and contains the two main theorems for Dirac and Maxwell fields.

- In section 4, we prove that the conformal scattering theories obtained are equivalent to analytical scattering theories where the scattering operator is defined in terms of classical wave operators. The comparison dynamics are as simple and visual as can be since they are given by the flows of congruences of null geodesics near $\mathscr{I}^{\pm}$; this gives a description of the scattering operator in terms of asymptotic profiles rather than the more complex comparison dynamics. This part shows that the conformal technique allows one to construct genuine scattering theories in generic non-stationary frameworks where spectral methods cannot be applied. The technical complexity of spectral methods is also far greater than that of conformal techniques.

- The technical issues are then treated in details in appendices : crucial estimates used for the construction of the conformal scattering operator are given in appendix A, appendix B contains the proofs of the theorems and the detailed construction of a solution to the Goursat problem is done in appendix C.

This paper is the first step in a programme centred on conformal scattering in general relativity. Subsequent studies will include a detailed treatment of the wave equation as well as an alternative solution of the Goursat problem using the techniques of Hadamard and Leray (as Friedlander did for the wave equation in [19]). A key issue that will also be addressed is that of the peeling : what are the minimal conditions to impose on initial data on a spacelike hypersurface to ensure that the trace of the solution on $\mathscr{I}^{\pm}$ is smooth?

2 Geometrical and analytical frameworks

2.1 Asymptotically simple space-times and the space-times of Corvino-Schoen and Chrusciel-Delay

We work with the space-times constructed by Chrusciel and Delay [11, 12], Corvino [13], and Corvino and Schoen [14]. These space-times are asymptotically simple with specifiable regularity for null and timelike infinities and are diffeomorphic to the Schwarzschild or Kerr space-time in a neighbourhood of spacelike infinity. We concentrate our attention on the most regular case, where null and timelike infinities are smooth (meaning here $\mathcal{C}^{\infty}$). Our theorems can be extended to less regular cases but we do not wish to blur the ideas by too many technical remarks. The minimum regularity of future and null infinity compatible with our constructions will be discussed in subsequent papers.

Let us first recall the definition of asymptotically simple space-times in the smooth, asymptotically flat case. A 4-dimensional, globally hyperbolic, Lorentzian space-time $(\mathscr{M}, g)$, $\mathscr{M} \simeq \mathbb{R}^4$, is called asymptotically simple if there exists another globally hyperbolic, Lorentzian space-time $(\widehat{\mathscr{M}}, \hat{g})$ and a smooth scalar function Ω on $\widehat{\mathscr{M}}$ such that :

(i) $\mathscr{M}$ is an open submanifold of $\widehat{\mathscr{M}}$ whose boundary is the union of two points i^- and i^+ and a smooth null hypersurface (denoted $\mathscr{I}$ and pronounced scri, for "sript i") ; $\mathscr{I}$ is the disjoint union of the past light-cone $\mathscr{I}^+$ of i^+ and of the future light-cone $\mathscr{I}^-$ of i^- ; $\mathscr{I}^\pm$ are referred to as future and past null infinities respectively and $i^\pm$ as future and past timelike infinities respectively ;

(ii) $\Omega > 0$, $\hat{g} = \Omega^2 g$ on $\mathscr{M}$, $\Omega = 0$ and $\mathrm{d}\Omega \neq 0$ on $\partial\mathscr{M}$;

(iii) every null geodesic in $\mathscr{M}$ acquires a future endpoint on $\mathscr{I}^+$ and a past endpoint on $\mathscr{I}^-$.

It follows from the definition above that $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) is the set of future (resp. past) end-points of null geodesics and that i^+ (resp. i^-) is the set of future (resp. past) end-points of uniformly timelike curves. We also define i^0, or spacelike infinity, as the set of boundary points of uniformly spacelike Cauchy hypersurfaces in $\mathscr{M}$.

We denote by ∇ the Levi-Civita connection associated with the metric g and by $\hat{\nabla}$ the Levi-Civita connection associated with $\hat{g}$.

2.2 3+1 decompositions

In order to have a natural formulation of the Cauchy problem in the physical space-time, we perform a $3+1$ decomposition of the geometry that will allow us to express the field equations as evolution equations. We choose on $\mathscr{M}$ a global smooth time function t, such that $\nabla^a t$ is uniformly timelike on $\mathscr{M}$, i.e.

$$\exists\ 0 < C_1 < C_2 < +\infty\ ;\ C_1 \leq g_{ab} \nabla^a t \nabla^b t \leq C_2 \text{ at each point of } \mathscr{M}\,,$$

and the second derivative of t tends to zero at infinity along any direction (timelike, null or spacelike).

The foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ of $\mathscr{M}$ by the level hypersurfaces of t is a foliation by smooth Cauchy spacelike hypersurfaces, all diffeomorphic to $\Sigma = \mathbb{R}^3$. Moreover, $(\Sigma, h(t))$ is asymptotically flat for each t. We consider T^a the future-oriented normal vector field to the foliation, normalized so that $T^a T_a = 2$. This normalization is convenient in the context of spinors ; $T_{AA'}$ induces a Hermitian form on spin space and so we can choose a unitary spin frame that is "adapted to the foliation" as defined in [34] meaning that $T^{AA'} = o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'}$ in addition to $\iota^A o_A = 1$. In such a spin-frame we have

$$T_{\mathbf{AA'}} = T^{\mathbf{AA'}} = \mathrm{Id}_2\,,\ \text{i.e. } T_{AA'} \phi^A \bar{\phi}^{A'} = \left|\phi^0\right|^2 + \left|\phi^1\right|^2\ ,\ T^{AA'} \phi_A \bar{\phi}_{A'} = |\phi_0|^2 + |\phi_1|^2\ .$$

(Here we adopt the abstract index convention in which ordinary indices are abstract markers specifying the vector bundle to which a given (spin-)tensor belongs and bold-face indices referring to concrete indices labelling components in a given frame.) The metric g can be decomposed as follows :

$$g_{ab} = \frac{1}{2} T_a T_b - h_{ab} \text{ i.e. } g_{ab} \mathrm{d}x^a \mathrm{d}x^b = \frac{1}{2} N^2 \mathrm{d}t^2 - h \,, \tag{1}$$

where the lapse function N is defined by $T_a \mathrm{d}x^a = N \mathrm{d}t$ and h is for each time t a smooth Riemannian metric on Σ_t, it can therefore be considered as a smooth time-dependent riemannian metric on $\mathbb{R}^3$. The connection ∇_a can likewise be decomposed along T^a and along $(T^a)^\perp$:

$$\nabla_a = \frac{1}{2} T_a T^b \nabla_b - h_a{}^b \nabla_b = \frac{1}{2} T_a \nabla_T + \mathrm{D}_a \,, \tag{2}$$

where $\nabla_T = T^a \nabla_a$ is the covariant derivative along T^a and $\mathrm{D}_a = -h_a{}^b \nabla_b$ is the part of ∇_a orthogonal to T^a : $T^a \mathrm{D}_a = 0$. D_a is the four-dimensional covariant derivative restricted to act tangent to Σ_t. It differs from the Levi-Civita connection on $(\Sigma_t, h(t))$ by a combination of the extrinsic curvature (or second fundamental form) of the leaves of the foliation. In particular $\mathrm{D}_a T_b = K_{ab} = K_{(ab)}$ is $\sqrt{2}$ times the extrinsic curvature. More precisely we have

$$K_{ab} = \mathrm{D}_a T_b = h_a{}^c h_b{}^d \nabla_c T_d$$

and obviously $T^a K_{ab} = 0$. We introduce the following slightly different form of the space-like covariant derivative

$$\mathrm{D}_{AB} = T_{(A}^{A'} \nabla_{B)A'} = T_A^{A'} \mathrm{D}_{BA'} \,,\ \mathrm{D}^{AB} = T_{B'}^{(A} \nabla^{B)B'} = T_{B'}^{A} \mathrm{D}^{BB'} \,.$$

The product structure $\mathscr{M} \simeq \mathbb{R} \times \Sigma$ is fixed by identifying the points on different hypersurfaces Σ_t along the integral lines of T^a. This defines the vector field $\partial/\partial t$ as

$$\frac{\partial}{\partial t} = \frac{N}{\sqrt{2}} T^a \frac{\partial}{\partial x^a} \,,$$

independently of the choice of local coordinates on Σ.

On the rescaled space-time, we shall use two different foliations. One given by the hypersurfaces Σ_t, that approach $\mathscr{I}^\pm$ as $t \to \pm\infty$ (see figure 1). We define a second foliation as follows : we choose a global smooth time function τ on $(\widehat{\mathscr{M}}, \hat{g})$, such that $\hat{\nabla}^a \tau$ is a uniformly timelike vector field and $\tau(i^\pm) = \pm T$, $T > 0$, and we consider the level hypersurfaces of τ, $\mathscr{H}_\tau$. We assume in addition that $\mathscr{H}_0 = \Sigma_0$ (see figure 2). The hypersurfaces $\mathscr{H}_\tau$ correspond, in the physical space-time, to hyperboloids that are asymptotically null. In the rescaled space-time, they are smooth uniformly spacelike hypersurfaces, with boundary $\mathscr{I} \cap \mathscr{H}_\tau$. Note that $\mathscr{H}_{\pm T}$ is reduced to $i^\pm$.

Using this new foliation, we can perform a $3+1$ decomposition of $\hat{g}$ and $\hat{\nabla}$. We denote by $\hat{T}^a$ the future-oriented normal vector field to $\mathscr{H}_\tau$ (for the metric $\hat{g}$), normalized so that $\hat{g}_{ab} \hat{T}^a \hat{T}^b = 2$. We have

$$\hat{g}_{ab} = \frac{1}{2} \hat{T}_a \hat{T}_b - \hat{h}_{ab} \text{ i.e. } \hat{g}_{ab} \mathrm{d}x^a \mathrm{d}x^b = \frac{1}{2} \hat{N}^2 \mathrm{d}\tau^2 - \hat{h} \,, \tag{3}$$

$$\hat{\nabla}_a = \frac{1}{2} \hat{T}_a \hat{T}^b \hat{\nabla}_b - \hat{h}_a^b \hat{\nabla}_b = \frac{1}{2} \hat{T}_a \hat{\nabla}_T + \hat{\mathrm{D}}_a \,, \tag{4}$$

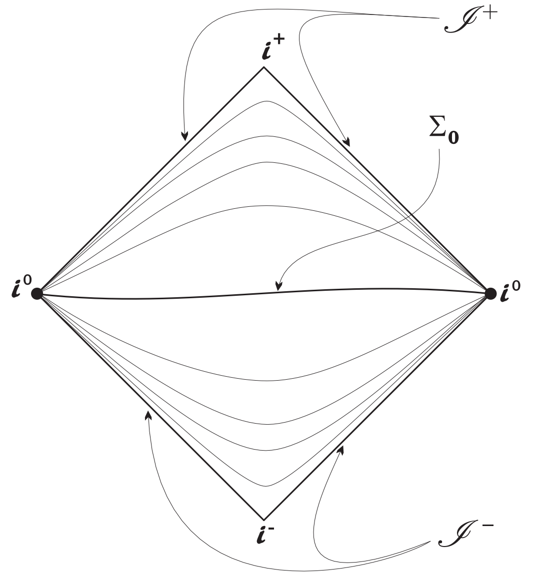

Figure 1: The foliation $\{\Sigma_t\}$ represented in the rescaled space-time.

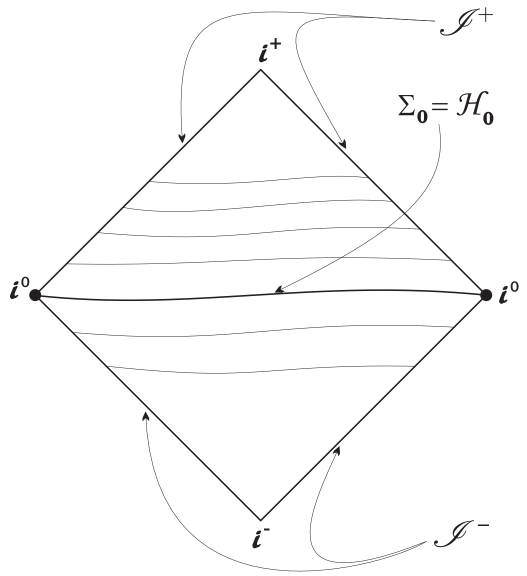

Figure 2: The foliation $\{\mathscr{H}_\tau\}_\tau$ whose leaves are transverse to $\mathscr{I}$.

where $\hat{T}_a \mathrm{d}x^a = \hat{N}\mathrm{d}\tau$ and $\hat{h}_{ab}$ is for each time $\tau \in]-T, T[$, a smooth Riemannian metric on $\mathscr{H}_\tau$.

2.2.1 Spin frames

We can shoose a global smooth spin frame over all of $\mathscr{M}$ whose derivative tends to zero at infinity ($\mathcal{M}$ is topologically trivial). Similarly one can choose a global smooth spin-frame on $\widehat{\mathscr{M}}$. However, we will wish to adapt the spin-frames to the geometry of $\mathscr{I}$, and this cannot be done globally.[3] These topological technicalities require us to use at least 3 spin-frames. However, they are all related by smooth bounded functions to a suitably chosen global smooth spin-frame. Furthermore, the formulae that we use do not depend on the choice of a spin-frame; densities for norms can be calculated locally on the separate patches, but will agree on overlaps and so the local integrals can be added together with a partition of unity to give the appropriate global norm. With these provisos, we choose spin-frames on two neighbourhoods near $\mathscr{I}$ as follows:

Definition 2.1. *On $\widehat{\mathscr{M}}$, near $\mathscr{I}$, we choose a spin-frame $(\hat{o}^A, \hat{\iota}^A)$, smooth except at i^- and i^+, such that $\hat{l}^a = \hat{o}^A \bar{\hat{o}}^{A'}$ is tangent to $\mathscr{I}^-$, $\hat{n}^a = \hat{\iota}^A \bar{\hat{\iota}}^{A'}$ is tangent to $\mathscr{I}^+$. Note that*

[3] To be more specific, we wish to choose spin-frames in the unphysical space-time so that $\hat{n}^a = \hat{\iota}^A \hat{\iota}^{A'}$ is tangent to $\mathscr{I}^+$, and $\hat{l}^a = \hat{o}^A \hat{o}^{A'}$ is orthogonal to the cross-sections of $\mathscr{I}^+$ given by the intersections with $\mathscr{H}_\tau$. However, the complex vector $\hat{m}^a = \hat{o}^A \hat{\iota}^{A'}$ is determined by its real part (the real and imaginary parts have the same length and are perpendicular) and so cannot be non-zero globally on the S^2s. Furthermore, such spin frames cannot continue smoothly over the interior of the hypersurfaces as $\hat{n}^a$ defines a map from the sphere to the light cone with nontrivial winding number and so cannot be extended smoothly over the interior of some $\mathscr{H}_\tau$ as this would deform the map to the trivial one. In total, then, we need three sets to cover the space-time, two to cover the S^2 factor near infinity on the complement of some world tube $K \times \mathbb{R}$ with K compact in each $\mathscr{H}_\tau$, and one to cover the interior of the space-time.

such a spin-frame is necessarily singular at $i^{\pm}$. We also assume that $\hat{\iota}^A = \iota^A$ near $\mathscr{I}^+$ and $\hat{o}^A = o^A$ near $\mathscr{I}^-$.

This will be enough for our treatment of Dirac's equation. In the Maxwell case, however, we will wish to choose the spin-frames (o^A, ι^A) in the physical space-time and $(\hat{o}^A, \hat{\iota}^A)$ so that they are related by a rescaling :

Definition 2.2. *We choose on $\mathscr{M}$, near $\mathscr{I}$, a spin-frame (o^A, ι^A) such that the associated tetrad $l^a = o^A \bar{o}^{A'}$, $n^a = \iota^A \bar{\iota}^{A'}$, $m^a = o^A \bar{\iota}^{A'}$, $\bar{m}^a = \iota^A \bar{o}^{A'}$, is a normalized Newman-Penrose tetrad for the metric g and satisfies in addition :*

- *l^a extends to become a tangent to $\mathscr{I}^-$ and n^a extends to become tangent to $\mathscr{I}^+$ in $\widehat{\mathscr{M}}$;*
- *$l^a + n^a = T^a$;*
- *$\Omega^{-2} l^a$ and $\Omega^{-1} m^a$ are smooth on $\mathscr{I}^+$ and $\Omega^{-2} n^a$ and $\Omega^{-1} m^a$ are smooth on $\mathscr{I}^-$.*

On $\widehat{\mathscr{M}}$, the spin-frame $(\hat{o}^A, \hat{\iota}^A)$, such that the associated tetrad $\hat{l}^a$, $\hat{n}^a$, $\hat{m}^a$, $\bar{\hat{m}}^a$ is a normalized Newman-Penrose tetrad for the metric $\hat{g}$, is then defined as a rescaling of (o^A, ι^A). Consider two functions λ_1 and λ_2 on $\mathscr{M}$ such that, for $t_0 > 0$ given,

- *$\lambda_1 = \Omega^{-1}$ in the future of the hypersurface Σ_{t_0}, $\lambda_2 = \Omega^{-1}$ in the past of Σ_{-t_0}, $\lambda_1 \lambda_2 = \Omega^{-1}$ on $\mathscr{M}$;*
- *λ_1 and λ_2 are positive and smooth on $\mathscr{M}$,*

then we define $(\hat{o}^A, \hat{\iota}^A)$ by $\hat{o}^A = \lambda_1 o^A$, $\hat{\iota}^A = \lambda_2 \iota^A$.

2.3 The Cauchy problem in the physical space-time

On each Σ_t, we consider the standard Sobolev spaces :

Definition 2.3. *Given $\mathcal{B}$ a vector bundle on Σ and $k \in \mathbb{N}$, we define $H^k(\Sigma_t\,;\,\mathcal{B})$ as the completion of the space $\mathcal{C}_0^\infty(\Sigma_t\,;\,\mathcal{B})$ of smooth compactly supported sections of $\mathcal{B}$ in the norm :*

$$\|\psi\|^2_{H^k(\Sigma_t)} = \int_\Sigma \sum_{|\alpha| \leq k} \langle \nabla^\alpha \psi, \nabla^\alpha \psi \rangle_g \, \mathrm{dVol}_g \,,$$

where $\langle .,. \rangle_g$ and dVol_g are the inner product and the volume form on Σ_t induced by the metric g (i.e. by the metric $h(t)$).

Implicit in this definition is a choice of metric on the bundle $\mathcal{B}$. In this paper, the bundles will be tensor bundles over the 2-component spinors and these do not have a natural positive definite metric. However, when working on a space-like hypersurface, the normal vector $T^{AA'}$ provides a good choice although any other normalized uniformly time-like vector field will do.

Remark 2.1. *For any $t_1, t_2 \in \mathbb{R}$, the metrics $h(t_1)$ and $h(t_2)$ are uniformly equivalent on Σ. Moreover, asymptotic simplicity and the way the function t was chosen entail that this equivalence is also uniform in time. Consequently, the Sobolev spaces $H^k(\Sigma_t\,;\ \mathcal{B})$ considered as Hilbert spaces on Σ are all uniformly isomorphic to $H^k(\Sigma_0\,;\ \mathcal{B})$. For simplicity, we shall denote $H^k(\Sigma_0\,;\ \mathcal{B})$ by $H^k(\Sigma\,;\ \mathcal{B})$ and the hermitian product and volume form on Σ_0 by $\langle .,. \rangle$ and dVol. When we wish to use explicitly the norm on a given Σ_t, we shall come back to the complete notations $H^k(\Sigma_t\,;\ \mathcal{B})$, $\langle .,. \rangle_g$ and dVol_g.*

Remark 2.2. *For further simplicity, we shall often use formal notations : H_t will denote a given Hilbert space on Σ_t and H the corresponding Hilbert space on Σ, equipped with the H_0 norm.*

2.3.1 The wave equation

The conformally invariant scalar wave equation

$$\left(\Box + \frac{1}{6} R_g \right) \phi = 0\,,\ \Box = \nabla_a \nabla^a\,, \tag{5}$$

where R_g is the scalar curvature of the metric g, is expressed as an evolution equation as

$$\left(\frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial t} \log N \right) \frac{\partial}{\partial t} - \frac{1}{2} N^4 \tilde{\Delta} + \frac{1}{12} N^2 R_g \right) \phi = 0\,,\ \tilde{\Delta} = \Delta_{\tilde{h}}\,,\ \tilde{h} = N^2 h\,.$$

The energy for the standard wave equation

$$\Box_g \phi = 0\,,$$

associated with the timelike vector field T^a is

$$\begin{aligned} E(\phi, t) &= \int_{\Sigma_t} T^a \mathbf{T}_{a0} \mathrm{dVol}_g \\ &= \int_{\Sigma_t} \left(|\partial_t \phi|^2 + N^2 h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \bar{\phi} + |\phi|^2 \right) \frac{1}{N} \mathrm{dVol}_g \\ \text{where } \ 8\pi \mathbf{T}_{ab} &= 2 \frac{\partial \phi}{\partial x^a} \frac{\partial \bar{\phi}}{\partial x^b} - g_{ab} g^{cd} \frac{\partial \phi}{\partial x^c} \frac{\partial \bar{\phi}}{\partial x^d}\,. \end{aligned}$$

The Cauchy problem for equation (5) is well-posed in any Sobolev space of integral order on Σ. Cauchy data consists of the field on Σ_0, ϕ_{Σ_0}, together with its first derivative $\psi_{\Sigma_0} = \partial \phi / \partial t|_{\Sigma_0}$. We have :

Lemma 2.1. *For any $k \in \mathbb{N}^*$, for any $(\phi_{\sigma_0}, \psi_{\Sigma_0}) \in H^k(\Sigma) \oplus H^{k-1}(\Sigma)$, there exists a unique solution*

$$\phi \in \mathcal{C}^0 \left(\mathbb{R}_t\,;\ H^k(\Sigma) \right) \cap \mathcal{C}^1 \left(\mathbb{R}_t\,;\ H^{k-1}(\Sigma) \right)$$

of (5) such that $\phi(0) = \phi_{\Sigma_0}$ and $\partial_t \phi(0) = \psi_{\Sigma_0}$. In addition,

$$\phi \in \bigcap_{l=0}^{k} \mathcal{C}^l \left(\mathbb{R}_t\,;\ H^{k-l}(\Sigma) \right)\,.$$

This result is proved for general globally hyperbolic space-times in Y. Choquet-Bruhat, D. Christodoulou and M. Francaviglia [10].

Characteristic data for the wave equation consists simply of ϕ restricted to the initial null hypersurface (see [35]).

2.3.2 Dirac fields

The massless Dirac equation reduces to the Weyl anti-neutrino equation

$$\nabla^{AA'}\phi_A = 0\,. \tag{6}$$

Its $3+1$ decomposition is given by

$$\nabla_t \phi_A = -\sqrt{2} N\, \mathrm{D}_A^B \phi_B$$

where ∇_t denotes $\nabla_{\frac{\partial}{\partial t}}$. Although the energy momentum tensor does not lead to any positive definite conserved quantity, a special feature of Dirac's equation is that it admits a closed 3-form, referred to as the charge or probability density

$$\omega = {}^*\phi_A \bar{\phi}_{A'} \mathrm{d}x^{AA'}\,,\ \nabla^{AA'}\left(\phi_A \bar{\phi}_{A'}\right) = 0\,. \tag{7}$$

This leads to an exactly conserved L^2 "energy"

$$\|\phi_A\|^2_{L^2(\Sigma_t)} = \int_{\Sigma_t} T^{AA'}\phi_A\bar{\phi}_{A'}\mathrm{dVol}_g = \int_{\Sigma_t}\left(|\phi_0|^2 + |\phi_1|^2\right)\mathrm{dVol}_g = \|\phi_A\|^2_{L^2(\Sigma_0)}\,, \tag{8}$$

(the expression in terms of components of ϕ_A is valid for any spin-frame (o^A, ι^A) adapted to the foliation $\{\Sigma_t\}_t$) and to estimates for higher derivatives. The Cauchy data for (6) consists of ϕ_A restricted to the initial data surface. The Cauchy problem for this equation is well posed in all Sobolev spaces (see [34] for a detailed proof for generic symmetric hyperbolic systems) :

Lemma 2.2. *For any $k \in \mathbb{N}$ and $\psi_A \in H^k(\Sigma;\ \mathbb{S}_A)$, there exists a unique solution*

$$\phi_A \in \mathcal{C}\left(\mathbb{R}_t;\ H^k(\Sigma;\ \mathbb{S}_A)\right)$$

of (6) such that $\phi_A(0) = \psi_A$. Moreover,

$$\phi_A \in \bigcap_{l=0}^{k} \mathcal{C}^l\left(\mathbb{R}_t;\ H^{k-l}(\Sigma;\ \mathbb{S}_A)\right)\,.$$

The characteristic data for a null hypersurface with null normal $n^a = \iota^A\bar{\iota}^{A'}$ is $\phi_1 := \phi_A\iota^A$, [35].

2.3.3 Maxwell's equations

We work with the field version of the equation here

$$\nabla^{AA'}\phi_{AB} = 0\,,\ \phi_{AB} = \phi_{(AB)}\,, \tag{9}$$

where the curvature 2-form of the electromagnetic field $F_{ab} = \phi_{AB}\varepsilon_{A'B'} + \bar{\phi}_{A'B'}\varepsilon_{AB}$. The energy momentum tensor is

$$\mathbf{T}_{ab} = \phi_{AB}\bar{\phi}_{A'B'}$$

and this is conserved, $\nabla^a T_{ab} = 0$ leading to energy estimates of the standard form for

$$\begin{aligned} E(\phi, t) &= \int_{\Sigma_t} T^a \mathbf{T}_{a0} \mathrm{dVol}_g \\ &= \frac{1}{\sqrt{2}} \int_{\Sigma_t} \left(|\phi_{00}|^2 + 2|\phi_{01}|^2 + |\phi_{11}|^2 \right) \mathrm{dVol}_g \end{aligned} \tag{10}$$

where in the above we have used a spin-frame in which $T^{AA'} = o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'}$.

The Cauchy data for the Maxwell equations consists of $\phi_{AB}|_{\Sigma_0}$ subject to the constraint equations $D^{AB} \left(\phi_{AB}|_{\Sigma_0} \right) = 0$. It is easy to see that this constraint is preserved by the evolution as a consequence of the identity $\nabla^{AA'} \nabla^B_{A'} \phi_{AB} = 0$. (There are 4 equations for the three components of ϕ_{AB} so the system is overdetermined. However, this identity shows that, in effect, the constraint equation is a consequence of the other three equations if it is imposed on the initial data, so the system is consistent.)

In order to apply the theory of symmetric hyperbolic systems, we need to extract three of the four equations that form one. To do this, we choose a time-like vector field, T^a, with $T^a T_a = 2$ and consider the equations $T_{A'(A} \nabla^{CA'} \phi_{B)C} = 0$. These are equivalent to the evolution equations $T^c \nabla_c \phi_{AB} = D^C_{(A} \phi_{B)C}$. These evolution equations admit energy estimates for the energy (10) with the same T^a ; they are a symmetric hyperbolic system so that the Cauchy problem is well posed in H^k.

Lemma 2.3. *For any $k \in \mathbb{N}$ and for any $\psi_{AB} \in H^K \left(\Sigma_0 \, ; \; \mathbb{S}_{(AB)} \right)$ such that $D^{AB} \psi_{AB} = 0$, there exists a unique solution*

$$\phi_{AB} \in \mathcal{C} \left(\mathbb{R}_t \, ; \; H^k \left(\Sigma \, ; \; \mathbb{S}_{(AB)} \right) \right)$$

of (9) such that $\phi_{AB}(0) = \psi_{AB}$; that is to say, in particular, that $D^{AB} \left(\phi_{AB}|_{\Sigma_t} \right) = 0$ for all $t \in \mathbb{R}$. Moreover

$$\phi_{AB} \in \bigcap_{l=0}^{k} \mathcal{C}^l \left(\mathbb{R}_t \, ; \; H^{k-l} \left(\Sigma \, ; \; \mathbb{S}_{(AB)} \right) \right) .$$

Remark 2.3. *Note that the constraint on the hypersurfaces Σ_t take the form*

$$\left(T^{AA'} \nabla^B_{A'} \phi_{AB} \right) \Big|_{\Sigma_t} = 0 \, .$$

We shall denote the space of constrained L^2 data on Σ_t by $L^2_{\mathrm{Maxwell}} \left(\Sigma_t \, ; \; \mathbb{S}_{(AB)} \right)$ or simply $L^2_{\mathrm{Maxwell}} \left(\Sigma_t \right)$.

The characteristic data for a null hypersurface with null normal $n^a = \iota^A \bar{\iota}^{A'}$ is $\phi_2 = \phi_{AB} \iota^A \iota^B$, [35].

2.4 The field equations on the rescaled space-time

We first define function spaces on the rescaled space-time, based on the ones we use on the physical space-time. To a given Hilbert space H_t on Σ_t, we associate $\hat{H}_t$ its image under the isometry $\phi \mapsto \Omega^{-1} \phi$; that is to say

$$\| \phi \|_{H_t} = \left\| \Omega^{-1} \phi \right\|_{\hat{H}_t} .$$

If all the H_t norms are uniformly equivalent in time, then the $\hat{H}_t$ norms are all equivalent, but not uniformly. We likewise use the notation $\hat{H}$ for $\hat{H}_0$ considered as a Hilbert space on Σ.

All three equations are conformally invariant with $\phi \mapsto \Omega^{-1}\phi$. This means that if we denote formally the equation by

$$L_g \phi = 0\,, \tag{11}$$

and if the Cauchy problem for (11) is well posed in a given Hilbert space H, then $\phi \in \mathcal{C}(\mathbb{R}_t\,;\; H)$ is a solution of (11) if and only if $\hat{\phi} := \Omega^{-1}\phi \in \mathcal{C}(\mathbb{R}_t\,;\; \hat{H})$ is a solution of

$$L_{\hat{g}} \hat{\phi} = 0\,, \tag{12}$$

where (12) is the covariant equation (11) with the metric g and its associated covariant derivative replaced by $\hat{g}$ and $\hat{\nabla}$.

Remark 2.4. *An additional important property of Dirac's equation is that the closed 3-form*

$$\omega = {}^* \left(\phi_A \bar{\phi}_{A'} \mathrm{d}x^{AA'} \right)$$

is conformally invariant, where $*$ is the Hodge $*$-operator, which on a 1-form is given by $*\alpha_a \mathrm{d}x^a = \frac{1}{6} < \alpha^a \varepsilon_{abcd} \mathrm{d}x^b \mathrm{d}x^c \mathrm{d}x^d$. The same is true in the Maxwell case of the 3-form

$${}^* \left(T^a \mathbf{T}_{ab} \mathrm{d}x^b \right) = {}^* \left(T^a \phi_{AB} \bar{\phi}_{A'B'} \mathrm{d}x^{BB'} \right)$$

with T^a remaining unrescaled.

So we see that in the Dirac case, if H_t is taken to be $L^2(\Sigma_t\,;\; \mathbb{S}_A)$, the space $\hat{H}_t$ is simply the natural L^2 space on Σ_t associated to the metric $\hat{g}$, that we can denote $L^2_{\hat{g}}\left(\Sigma_t\,;\; \mathbb{S}_A\right)$, or $L^2\left(\Sigma_t\,;\; \hat{g}\right)$ if we do not wish to specify the bundle. In the Maxwell case however, since the vector T^a is not rescaled, if we take $H_t = L^2_{\mathrm{Maxwell}}\left(\Sigma_t\,;\; \mathbb{S}_{(AB)}\right)$, then the space $\hat{H}_t$ will be a weighted L^2 space on Σ_t associated with the metric $\hat{g}$, the weight being exactly Ω.

The rescaled space-time is smooth and globally hyperbolic. Using the foliation $\{\mathscr{H}_\tau\}_\tau$ of $\widehat{\mathscr{M}}$ and the $3+1$ decomposition of $\hat{g}$, we obtain a well-posed Cauchy problem in some natural Hilbert spaces defined on the hypersurfaces $\mathscr{H}_\tau$. These spaces depend on τ in that the metric and the hypersurface $\mathscr{H}_\tau$ both depend on τ, but they vary smoothly with τ.

Finally we prove an important property of the equations we consider. This property will entail the existence of trace operators on $\mathscr{I}^\pm$.

Lemma 2.4. *For $\mathcal{C}_0^\infty$ data on Σ_0 the corresponding solution $\hat{\phi}$ is smooth on $\widehat{\mathscr{M}}$ and its support remains away from i^0.*

Proof. The fact that the support of the solution remains away from i^0 is a straightforward consequence of the finite propagation speed (see figure 3 for the shape of the support). This allows us to deform, away from the support of the initial data, the spacelike hypersurface Σ_0 into a hypersurface $\hat{\Sigma}_0$ that is spacelike for the metric $\hat{g}$ but goes across $\mathscr{I}^+$ and remains away from i^0.

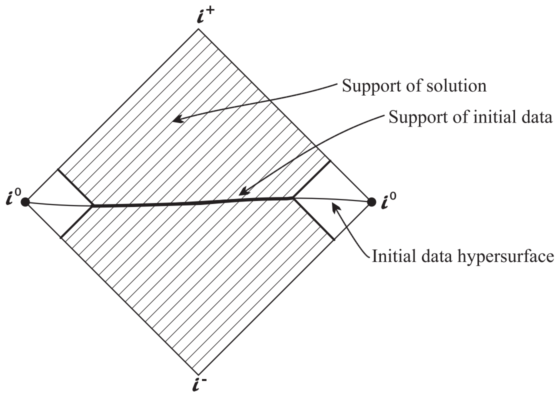

Figure 3: Support of solution for compactly supported data.

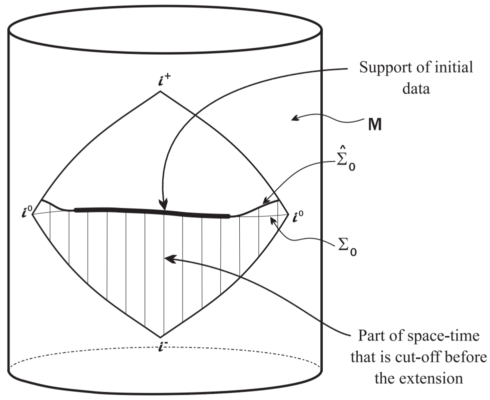

Figure 4: Construction of the hypersurface $\hat{\Sigma}_0$ and extension of the space-time lying in the future of this hypersurface as the smooth space-time $\mathbf{M}$.

The part of the space-time $\widehat{\mathscr{M}}$ that lies in the future of this hypersurface is completely regular and can be extended to a smooth Lorentzian globally hyperbolic space-time without boundary (see figure 4), say $(\mathbf{M} = \mathbb{R} \times S^3, G)$. The equation

$$L_G \Phi = 0$$

then admits a unique solution $\Phi \in \mathcal{C}^\infty(\mathbf{M})$. This is a consequence of Leray's theorem (see [29]) for symmetric hyperbolic systems on smooth globally hyperbolic space-times. This solution Φ coincides with $\hat{\phi}$ in the part of the future of $\hat{\Sigma}_0$ that lies inside of $\mathscr{M}$, by local uniqueness of solutions of (12), consequence of the finite propagation speed. Hence, $\hat{\phi}$ extends to a smooth function on $\widehat{\mathscr{M}}$. □

3 The conformal scattering operator : strategy of construction and main theorems

The construction of a conformal scattering operator is done in three main steps, following the general strategy of Friedlander, [20], using techniques based on those in Hörmander's work [26].

First step : trace operators $\mathfrak{T}^\pm$ on $\mathscr{I}^\pm$. In the conformal description of scattering, this step is the easiest ; it is the construction of a trace operator that, to the initial data on Σ_0 associates the characteristic data on $\mathscr{I}^\pm$ of the corresponding solution. We work with data in $\mathcal{C}_0^\infty(\Sigma_0)$. For such data, the solution $\hat{\phi}$ is smooth on $\widehat{\mathscr{M}}$. Therefore we can define the trace of $\hat{\phi}$ on $\mathscr{I}^\pm$. The equations being linear, the trace depends linearly on the initial data. Then, for a well chosen Hilbert space $\hat{H}^\pm_{\mathscr{I}}$ on $\mathscr{I}^\pm$, we prove an estimate of the form :

$$\exists C > 0\,;\ \forall \hat{\phi}_{\Sigma_0} \in \mathcal{C}_0^\infty(\Sigma_0)\,,\ \left\| \hat{\phi}^\pm_{\mathscr{I}} \right\|_{\hat{H}^\pm_{\mathscr{I}}} \leq C \left\| \hat{\phi}_{\Sigma_0} \right\|_{\hat{H}} \qquad (13)$$

where $\hat{\phi}^{\pm}_{\mathscr{I}}$ is not the complete trace of $\hat{\phi}$ but its characteristic data on $\mathscr{I}^{\pm}$: the estimate (13) is obtained by integrating a 3-form on a closed hypersurface ; the part integrated on $\mathscr{I}$ is degenerate and involves only one component of the trace of the solution on $\mathscr{I}^{\pm}$. This allows to define, by density of $\mathcal{C}_0^\infty(\Sigma_0)$ in $\hat{H}$, bounded operators $\mathfrak{T}^{\pm}$ from $\hat{H}$ to $\hat{H}^{\pm}_{\mathscr{I}}$, that to $\hat{\phi}_{\Sigma_0}$ associate $\hat{\phi}^{\pm}_{\mathscr{I}}$. These operators are the analogues, in this conformal construction, of inverse wave operators in usual analytical time-dependent scattering theories.

Second step : The trace operators $\mathfrak{T}^{\pm}$ are one-to-one. We prove the estimate reciproqual to (13) :

$$\exists C > 0\,;\ \forall \hat{\phi}_{\Sigma_0} \in \mathcal{C}_0^\infty(\Sigma_0)\,,\ \ \left\| \hat{\phi}_{\Sigma_0} \right\|_{\hat{H}} \leq C \left\| \hat{\phi}^{\pm}_{\mathscr{I}} \right\|_{\hat{H}^{\pm}_{\mathscr{I}}}\,. \tag{14}$$

By density of $\mathcal{C}_0^\infty(\Sigma)$ in $\hat{H}$ and by continuity of $\mathfrak{T}^{\pm}$, this shows that $\mathfrak{T}^{\pm}$ is one-to-one from $\hat{H}$ onto the closed sub-space $\mathfrak{T}^{\pm}\hat{H}$ of $\hat{H}^{\pm}_{\mathscr{I}}$.

Third step : $\mathfrak{T}^{\pm}$ is an isomorphism. We show that for any $\hat{\phi}^{\pm}_{\mathscr{I}} \in \mathcal{C}_0^\infty(\mathscr{I}^{\pm})$, where we define $\mathcal{C}_0^\infty(\mathscr{I}^{\pm})$ as the space of $\mathcal{C}^\infty$ functions on $\mathscr{I}^{\pm}$ whose support remains away from i^0 and $i^{\pm}$, there exists $\hat{\phi}^{\pm}_{\Sigma_0} \in \hat{H}$ such that $\mathfrak{T}^{\pm}\hat{\phi}^{\pm}_{\Sigma_0} = \hat{\phi}^{\pm}_{\mathscr{I}}$. This amounts to solving the Goursat problem on $\mathscr{I}^{\pm}$ with data $\hat{\phi}^{\pm}_{\mathscr{I}}$. We thus obtain a densely defined inverse to $\mathfrak{T}^{\pm}$ that is clearly continuous by estimate (14). Hence this densely defined inverse can be extended to a bounded operator from $\hat{H}^{\pm}_{\mathscr{I}}$ to $\hat{H}$ that is an inverse for $\mathfrak{T}^{\pm}$. The operators $(\mathfrak{T}^{\pm})^{-1}$ are the analogues of direct wave operators in analytical time-dependent scattering theories. The conformal scattering operator is then defined as

$$S := \mathfrak{T}^{+}\left(\mathfrak{T}^{-}\right)^{-1}\,. \tag{15}$$

It is the operator that takes characteristic data at $\mathscr{I}^{-}$ to the corresponding data at $\mathscr{I}^{+}$.

Using this strategy, we prove the following theorems (details of proofs are given in appendix B). The case of the wave equation remains open for the moment. We treat the Dirac and Maxwell cases completely. We work with the spin-frames $(\hat{o}^A, \hat{\iota}^A)$ on $\widehat{\mathscr{M}}$ that were introduced at the end of subsection 2.2 : that of definition 2.1 is enough for Dirac, but for Maxwell we need to use the spin-frame of definition 2.2.

Theorem 1 (Dirac's equation)**.** *Here we have $H_0 = L^2(\Sigma_0\,;\ \mathbb{S}_A)$. The trace operators $\mathfrak{T}^{\pm}$, that, to some initial data $\hat{\phi}_A\Big|_{\Sigma_0} \in \mathcal{C}_0^\infty(\Sigma_0)$ for the rescaled equation*

$$\hat{\nabla}^{AA'}\hat{\phi}_A = 0\,, \tag{16}$$

associate respectively the trace of $\hat{\phi}_1$ on $\mathscr{I}^{+}$ and the trace of $\hat{\phi}_0$ on $\mathscr{I}^{-}$, extend as isomorphisms from $\hat{H}$ onto $L^2(\mathscr{I}^{\pm}\,;\ \mathbb{C})$.

Theorem 2 (Maxwell's equation)**.** *Here the space H_0 is the space of constrained L^2 Maxwell data on Σ_0 :*

$$L^2_{\mathrm{Maxwell}}(\Sigma_0) = \left\{ \phi_{AB} \in L^2(\Sigma_0\,;\ \mathbb{S}_{(AB)})\,;\ \hat{\mathrm{D}}^{AB}\hat{\phi}_{AB} = 0 \right\}\,.$$

The trace operators $\mathfrak{T}^{\pm}$, that, to some constrained initial data $\hat{\phi}_{AB} \in \hat{H}_0$ for the rescaled equation

$$\hat{\nabla}^{AA'} \hat{\phi}_{AB} = 0 \,, \tag{17}$$

associate respectively the trace of $\hat{\phi}_2 = \hat{\iota}^A \hat{\iota}^B \hat{\phi}_{AB}$ on $\mathscr{I}^+$ and the trace of $\hat{\phi}_0 = \hat{o}^A \hat{o}^B \hat{\phi}_{AB}$ on $\mathscr{I}^-$, extend as isomorphisms from $\hat{H}_0$ onto $L^2(\mathscr{I}^{\pm}\,;\,\mathbb{C})$.

Remark 3.1 (Null data and the trace on $\mathscr{I}$ of the rescaled field)**.** *For each field equation considered here, the characteristic data on $\mathscr{I}^{\pm}$, i.e. the data necessary for solving the Goursat problem on $\mathscr{I}^{\pm}$, contain less information than the Cauchy data on a spacelike hypersurface or indeed the full trace of the field on $\mathscr{I}^{\pm}$: for the wave equation, it is the trace of $\hat{\phi}$ only with no information on $\partial_\tau \hat{\phi}$, for Dirac and Maxwell, it is the trace of only one component of the field. It is however important to understand that, if the field is regular enough, the remaining information is completely determined in terms of the characteristic data by the restriction to $\mathscr{I}$ of the equation and some choice of boundary condition. This restriction can be considered as a contraint equation on $\mathscr{I}$. We illustrate this remark in the case where the characteristic data is $\hat{\phi}^+_{\mathscr{I}} \in \mathcal{C}^\infty_0(\mathscr{I}^+)$.*

- *Dirac's equation : the null data $\hat{\phi}^+_{\mathscr{I}}$ is the trace of $\hat{\phi}_1$ on $\mathscr{I}^+$. We use the expression of the equation in terms of components*

$$\begin{aligned} \hat{n}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_0 - \hat{m}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_1 + (\mu - \gamma) \hat{\phi}_0 + (\tau - \beta) \phi_1 &= 0 \,, \\ \hat{l}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_1 - \bar{\hat{m}}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_0 + (\alpha - \pi) \hat{\phi}_0 + (\varepsilon - \rho) \phi_1 &= 0 \,, \end{aligned}$$

  μ, γ, τ, β, α, π, ε and ρ being spin coefficients (see [36] vol 1). The restriction of the equation to $\mathscr{I}^+$ is the first equation above ; it is an ordinary differential equation along the null generators of $\mathscr{I}^+$ that determines $\hat{\phi}_0$ in terms of $\hat{\phi}_1$ provided $\hat{\phi}_0$ is assumed to be zero at i^+.

- *Maxwell's equations : the null data $\hat{\phi}^+_{\mathscr{I}}$ is the trace of $\hat{\phi}_2$ on $\mathscr{I}^+$. This is similar to the Dirac case. The full system is written in terms of components as*

$$\begin{aligned} \hat{n}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_0 - \hat{m}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_1 + (\mu - 2\gamma)\, \hat{\phi}_0 + 2\tau \hat{\phi}_1 - \sigma \hat{\phi}_2 &= 0 \,, &(18)\\ \hat{n}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_1 - \hat{m}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_2 - \nu \hat{\phi}_0 + 2\mu \hat{\phi}_1 + (\tau - 2\beta)\, \hat{\phi}_2 &= 0 \,, &(19)\\ \hat{l}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_1 - \bar{\hat{m}}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_0 - (\pi - 2\alpha)\, \hat{\phi}_0 - 2\rho \hat{\phi}_1 + \kappa \hat{\phi}_2 &= 0 \,, &(20)\\ \hat{l}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_2 - \bar{\hat{m}}^{\mathbf{a}} \hat{\nabla}_{\mathbf{a}} \hat{\phi}_1 + \lambda \hat{\phi}_0 - 2\pi \hat{\phi}_1 - (\rho - 2\varepsilon)\, \hat{\phi}_2 &= 0 \,, &(21) \end{aligned}$$

  where $\hat{\phi}_0 = \hat{o}^A \hat{o}^B \hat{\phi}_{AB}$, $\hat{\phi}_1 = \hat{o}^A \hat{\iota}^B \hat{\phi}_{AB}$, $\hat{\phi}_2 = \hat{\iota}^A \hat{\iota}^B \hat{\phi}_{AB}$ and κ, ρ, σ, τ, ε, α, β, γ, π, λ, μ, ν are the spin-coefficients (see [36] vol. 1). The first two equations allow to calculate $\hat{\phi}_0$ and $\hat{\phi}_1$ on $\mathscr{I}^+$ in terms of $\hat{\phi}_2$ if we assume that $\hat{\phi}_0$ and $\hat{\phi}_1$ are zero at i^+. Note that equation (18) is the contraction of the system (17) with the frame vector $\hat{m}^a$, i.e. (18) is equivalent to $\hat{m}^a \nabla^B_{A'} \hat{\phi}_{AB} = 0$. Similarly, (19) is obtained by contracting $\hat{n}^a$ into (17), (20) by contraction with $-\hat{l}^a$ and (21) by contraction with $-\bar{\hat{m}}^a$. The two constraint equations on $\mathscr{I}^+$ (equations (18) and (19)) are therefore equivalent to $\hat{\iota}^{A'} \hat{\nabla}^A_{A'} \hat{\phi}_{AB} = 0$.

- *The wave equation. The null data* $\hat{\phi}^+_{\mathscr{I}}$ *is the trace of* ϕ *on* $\mathscr{I}^+$. *The trace of* $\partial_\tau \hat{\phi}$ *on* $\mathscr{I}^+$ *is here also determined in terms of the null data by an ordinary differential equation along the generators of* $\mathscr{I}^+$.

This remark will be crucial for finding a solution to the Goursat problem for Dirac's and Maxwell's equations. For the wave equation however, we shall not need it. This fact is further commented on in remark C.1.

4 Equivalence with conventional scattering theory

Let us first recall a very simple example of scattering theory defined in terms of classical wave operators.

4.1 Analytic scattering theory for spin 1/2 in flat space

Consider the massless Dirac equation on Minkowski space, expressed in spherical coordinates using a spin-frame (o^A, ι^A) associated with the Newman-Penrose tetrad

$$l = \frac{1}{\sqrt{2}} (\partial_t + \partial_r) \ , \ n = \frac{1}{\sqrt{2}} (\partial_t - \partial_r) \ , \ m = \frac{1}{r\sqrt{2}} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\varphi \right) \ :$$

$$\left(\partial_t - \sigma^1 \left(\partial_r + \frac{1}{r} \right) - \frac{1}{r} \sigma^2 \left(\partial_\theta + \frac{1}{2} \cot\theta \right) - \frac{1}{r\sin\theta} \sigma^3 \partial_\varphi \right) \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = 0 \, ,$$

where σ^1, σ^2, σ^3 are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ .$$

Replacing $\Phi = {}^t(\phi_0, \phi_1)$ by $\Psi = r\Phi$, we obtain the simpler form of the equation

$$\partial_t \Psi = iH\Psi \ , \quad H = -i\sigma^1 \partial_r - \frac{i}{r} \sigma^2 \left(\partial_\theta + \frac{1}{2} \cot\theta \right) - \frac{i}{r\sin\theta} \sigma^3 \partial_\varphi \, .$$

The Hamiltonian H is self-adjoint on $\mathcal{H} = L^2(\mathbb{R}^+ \times S^2 \, ; \ \mathrm{d}r\mathrm{d}\omega)$ and thus defines the unitary one-parameter group e^{itH} that solves the equation in $\mathcal{H}$. We introduce a simplified dynamics

$$H_0 = -i\sigma^1 \partial_r \, .$$

The operator H_0 is self-adjoint on $\mathcal{H}_0 = L^2(\mathbb{R} \times S^2 \, ; \ \mathrm{d}r\mathrm{d}\omega)$ and the associated one-parameter group e^{itH_0}, when restricted to act on the spaces of outgoing and incoming data, respectively

$$\mathcal{H}_0^+ = \left\{ \Psi = \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix} \in \mathcal{H}_0 \right\} \text{ and } \mathcal{H}_0^- = \left\{ \Psi = \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} \in \mathcal{H}_0 \right\} ,$$

reduces to a simple radial translation at speed ± 1 :

$$e^{itH_0} \big|_{\mathcal{H}_0^\pm} = e^{\mp t \partial_r} \, .$$

Given any function $\chi(r) \in \mathcal{C}^\infty([0,+\infty[)$ such that $\chi \equiv 0$ in the neighbourhood of 0, $\chi \equiv 1$ for r large enough and $0 \leq \chi \leq 1$, we define the identifying operator

$$\begin{array}{rccl} \mathcal{J}_\chi \, : & \mathcal{H}_0 & \longrightarrow & \mathcal{H} \\ & \Psi & \longmapsto & (\chi\Psi)|_{\mathbb{R}^+ \times S^2} \, . \end{array}$$

The wave operators

$$W^\pm \Psi := \lim_{t\to\pm\infty} e^{-itH} \mathcal{J}_\chi e^{itH_0} \Psi \, , \; \Psi \in \mathcal{H}_0^\pm \, ,$$

exist, are independent of the choice of χ and are complete, i.e. the inverse wave operators

$$\tilde{W}^\pm \; := \; \text{S-lim}_{t\to\pm\infty} e^{-itH_0} \mathcal{J}_\chi^* e^{itH} \, ,$$

$$\mathcal{J}_\chi^* \, : \begin{array}{ccc} \mathcal{H} & \to & \mathcal{H}_0 \\ \Psi & \mapsto & \widetilde{\chi\Psi} \end{array} , \qquad \text{where } \widetilde{\chi\Psi} = \left| \begin{array}{l} \chi\Psi \text{ on } \mathbb{R}_r^+ \times S^2 \, , \\ 0 \text{ on } \mathbb{R}_r^- \times S^2 \, , \end{array} \right.$$

exist (where S-lim denotes the strong limit) and satisfy

$$\tilde{W}^\pm W^\pm = \mathrm{Id}_{\mathcal{H}_0^\pm} \, , \; W^\pm \tilde{W}^\pm = \mathrm{Id}_{\mathcal{H}} \, , \; \tilde{W}^\pm = \left(W^\pm \right)^* \, .$$

The scattering operator that describes the complete evolution of the field from its past scattering data to its future scattering data is

$$S := \tilde{W}^+ W^- \, .$$

A proof of these results can be found in [33]. It turns out that this example, however simple it may appear, is a perfect model for the interpretation of our conformal scattering operators in terms of wave operators.

The simplified dynamics e^{itH_0} restricted to act on $\mathcal{H}_0^\pm$ is simply the flow of the outgoing/incoming radial null vector field

$$v^\pm = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial r} \, .$$

The scattering data associated to such comparison dynamics are referred to as asymptotic profiles. They provide a very visual and natural description of the scattering properties of an equation : the solutions of the equation, as time becomes large, behave like a given profile pushed along the flow of a simple vector field.

4.2 Wave operators for spin 1/2

In our asymptotically simple situation, for K a large enough compact set, we can find a congruence of outgoing null geodesics, defining $\mathscr{I}^+$ (i.e. each point of $\mathscr{I}^+$ is reached by one and only one of these geodesics), on $\mathbb{R}_t^+ \times (\Sigma \setminus K)$. Similarly, we can choose a congruence of incoming null geodesics, defining $\mathscr{I}^-$, on $\mathbb{R}_t^- \times (\Sigma \setminus K)$. This is represented on the compactified space-time $\widehat{\mathscr{M}}$ in figure 5.

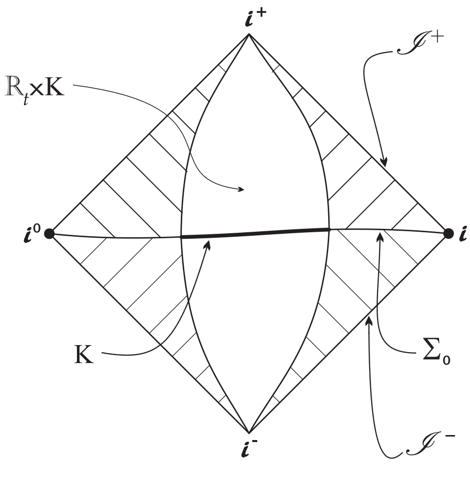

Figure 5: Congruences of null geodesics outside $\mathbb{R}_t \times K$.

As is done in appendix C (see figure 7), we extend the future and past null congruences by timelike congruences inside $[0, +\infty[_t \times K$ (resp. $]-\infty, 0]_t \times K$) to obtain two smooth weakly timelike congruences : one of $\mathbb{R}_t^+ \times \Sigma$, denoted $\mathscr{C}^+$, the other of $\mathbb{R}_t^- \times \Sigma$, denoted $\mathscr{C}^-$.

We use these congruences to define the analogues of $\mathcal{J}_\chi e^{itH_0}$ and $e^{-itH_0}\mathcal{J}_\chi^*$. First, we choose a conformal factor Ω such that the metric $\hat{g}$ and its inverse can be extended continuously to i^0 (and are therefore uniformly bounded on $\widehat{\mathscr{M}}$). This is possible in the Schwarzschild case (see [25]) and the result in the Kerr case should follow from the fact that the Kerr metric is a short-range perturbation of the Schwarzschild metric ; the proof however is not known to this day (as far as we are aware) although the possibility of choosing such a conformal factor Ω should be seen as a (fairly safe) conjecture[4].

Remark 4.1. *For spinor fields in $L^2(\mathscr{I}^+)$, the norm is given by*

$$\left\| \hat{\phi}_A \right\|^2_{L^2(\mathscr{I}^+)} = \int_{\mathscr{I}^+} \hat{\phi}_A \bar{\hat{\phi}}_{A'} \hat{n}^a \mathrm{dVol}_{\mathscr{I}^+} = \int_{\mathscr{I}^+} |\hat{\phi}_1|^2 \mathrm{dVol}_{\mathscr{I}^+} ,$$

and the information is reduced to that of the component $\hat{\phi}_1$. The information of the component $\hat{\phi}_0$ is lost and is irrelevant to the definition of the Hilbert space. It can therefore be projected to 0 in the definition of the analogues of $\mathcal{J}_\chi$. For $\hat{\phi}_A \in L^2(\mathscr{I}^-)$ it is the other component that is relevant and $\hat{\phi}_1$ may therefore be taken to vanish in the analogues of $\mathcal{J}_\chi$.

Definition 4.1 (of $\mathbb{P}_\chi^+(t)$)**.** *We consider a function $\chi \in \mathcal{C}^\infty(\Sigma)$ such that $\chi \equiv 0$ on K, $\chi \equiv 1$ outside a given bounded open set $\mathscr{O}$ containing K and $0 \le \chi \le 1$. We introduce the operator $\mathbb{P}_\chi^+(t)$ that to $\hat{\phi}_A = \hat{\phi}_1 \hat{o}_A \in L^2(\mathscr{I}^+)$ associates its projection onto Σ_t along the congruence $\mathscr{C}^+$, multiplied by χ. The operator $\mathbb{P}_\chi^-(t)$ is defined on elements of $L^2(\mathscr{I}^-)$ in a symmetrical manner interchanging o_A and ι_A to define $\mathscr{C}^-$ analagously to $\mathscr{C}^+$.*

[4]It has so far not been possible in the Schwarzschild case to choose Ω such that $\hat{g}$ is Lipschitz in a neighbourhood of i^0 (again, see [25]), although the question is not yet fully understood. The situation should be similar in Kerr.

Remark 4.2. *It is important to note that the operator $\mathbb{P}^+_\chi(t)$ could be defined in an equivalent manner using merely the congruence of null geodesics outside $\mathbb{R}^+_t \times K$ and not the congruence $\mathscr{C}^+$. For a spinor field $\hat{\phi}_A \in L^2(\mathscr{I}^+)$, $\mathbb{P}^+_\chi(t)\hat{\phi}_A$ is the spinor field $\hat{\psi}_A$ on Σ_t defined as follows :*

- *$\hat{\psi}_0$ is identically zero ;*
- *$\hat{\psi}_1$ is zero inside of K ;*
- *outside of K, $\hat{\psi}_1$ is equal to the projection onto $\Sigma_t \setminus K$, along the congruence of outgoing null geodesics, of $\hat{\phi}_1$, multiplied by χ.*

A similar symmetrical definition is valid for $\mathbb{P}^-_\chi(t)$. This clearly decomposes $\mathbb{P}^\pm_\chi(t)$ into the action of a dynamics defined by the flow of a null geodesic congruence, followed by a smooth cut-off. These operators are therefore the analogues of $\mathcal{J}_\chi e^{itH_0}$ in our simple example.

Lemma 4.1. *For our choice of conformal factor, the operator $\mathbb{P}^\pm_\chi(t)$ is bounded from $L^2(\mathscr{I}^\pm)$ to $L^2(\Sigma_t)$, the norm being uniformly bounded in $t \geq 0$ (resp. $t \leq 0$).*

Proof. It follows from the fact that the metric is uniformly bounded on $\widehat{\mathscr{M}}$ and that $\mathscr{C}^\pm$ is a smooth congruence. □

We are now ready to define our direct and inverse wave operators and to state the main theorem of this section. We denote by $\mathcal{U}(t,s)$ the propagator for equation (16) that to data $\hat{\phi}(s) \in L^2(\Sigma_s)$ associates the restiction $\hat{\phi}(t)$ of the associated solution to Σ_t.

Theorem 3. *The wave operators*

$$W^\pm = \text{S-lim}_{t\to\pm\infty}\mathcal{U}(0,t)\mathbb{P}^\pm_\chi(t)\,, \tag{22}$$

$$\tilde{W}^\pm = \text{S-lim}_{t\to\pm\infty}\left(\mathbb{P}^\pm_\chi(t)\right)^*\mathcal{U}(t,0)\,, \tag{23}$$

exist, i.e.

$$W^\pm \in \mathcal{L}\left(L^2(\mathscr{I}^\pm)\,;\ \hat{H}_0\right)\ ,\quad \tilde{W}^\pm \in \mathcal{L}\left(\hat{H}_0\,;\ L^2(\mathscr{I}^\pm)\right)\,,$$

where $H_0 = L^2\left(\Sigma_0\,;\ \mathbb{S}_A\right)$, are independent of the choices of K large enough and the function χ and satisfy

$$\tilde{W}^\pm = \left(W^\pm\right)^*\ ,\ \tilde{W}^\pm W^\pm = \mathrm{Id}_{L^2(\mathscr{I}^\pm)}\,,\ W^\pm\tilde{W}^\pm = \mathrm{Id}_{\hat{H}_0}\,.$$

Remark 4.3. *The operator*

$$\left(\mathbb{P}^\pm_\chi(t)\right)^*\ :\ \hat{H}_t \longmapsto L^2(\mathscr{I}^\pm)$$

acts by first multiplying a spinor field $\hat{\phi}_A \in \hat{H}_t$ by χ and then by projecting it onto $\mathscr{I}^\pm$ along the congruence $\mathscr{C}^\pm$. Since the result is an element of $L^2(\mathscr{I}^\pm)$, only its component $\left[\left(\mathbb{P}^+_\chi(t)\right)^*\hat{\phi}_A\right]\hat{\imath}^A$ (resp. $\left[\left(\mathbb{P}^-_\chi(t)\right)^*\hat{\phi}_A\right]\hat{o}^A$) is relevent, the other one may be taken to be identically zero.

The proof of theorem 3, given in appendix B, shows in fact more than what is stated in the theorem.

Theorem 4. *We have in addition : $\tilde{W}^\pm = \mathfrak{T}^\pm$. Consequently, $S = \tilde{W}^+W^-$.*

4.3 The Maxwell case

Similarly to the Dirac case, we denote by $\mathcal{U}(t,s)$ the propagator for the rescaled equation (17) that to constrained data on Σ_s associates the restriction to Σ_t of the corresponding solution. We also consider $\chi \in \mathcal{C}^\infty(\Sigma)$ such that $\chi \equiv 0$ on K, $\chi \equiv 1$ outside a given bounded open set $\mathscr{O}$ containing K and $0 \leq \chi \leq 1$.

Definition 4.2. *We introduce the operator $\mathbb{P}^\pm_\chi(t) = \chi P^\pm_K(t)$, the composition of the multiplication by χ with $P^\pm_K$ (for the definition of $P^\pm_K$ in the Maxwell case, see definition C.4).*

Theorem 5. *The wave operators*

$$\begin{aligned} W^\pm &= \text{S-lim}_{t\to\pm\infty}\mathcal{U}(0,t)\mathbb{P}^\pm_\chi(t)\,, \\ \tilde{W}^\pm &= \text{S-lim}_{t\to\pm\infty}\left(\mathbb{P}^\pm_\chi(t)\right)^*\mathcal{U}(t,0)\,, \end{aligned}$$

exist, i.e.

$$W^\pm \in \mathcal{L}\left(L^2(\mathscr{I}^\pm);\ \hat{H}_0\right)\ ,\quad \tilde{W}^\pm \in \mathcal{L}\left(\hat{H}_0\,;\ L^2(\mathscr{I}^\pm)\right)\,,$$

where $H_0 = L^2_{\mathrm{Maxwell}}(\Sigma_0)$, are independent of the choices of K large enough and the function χ and satisfy

$$\tilde{W}^\pm = \left(W^\pm\right)^*\ ,\ \tilde{W}^\pm W^\pm = \mathrm{Id}_{L^2(\mathscr{I}^\pm)}\,,\ W^\pm\tilde{W}^\pm = \mathrm{Id}_{\hat{H}_0}\,.$$

We have in addition : $\tilde{W}^\pm = \mathfrak{T}^\pm$. Consequently, $S = \tilde{W}^+ W^-$.

Remark 4.4. *We see that the situation here is slightly more complicated than for Dirac. The operator $\mathbb{P}^\pm_\chi(t)$ is not merely the composition of a projection along null geodesics and a cut-off. Here is also involved the calculation of the component $\hat{\phi}_1$ on Σ_t, so that $\mathbb{P}^\pm_\chi(t)\hat{\phi}^\pm_{\mathscr{I}}$ can satisfy the constraints.*

Note also that the adjoint of $P^\pm_K$ is just as simple as in the Dirac case. The action of $\left(P^\pm_K\right)^*$ on a Maxwell field $\phi_{AB}|_{\Sigma_t} \in L^2_{\mathrm{Maxwell}}(\Sigma_t)$, $t>0$ (resp. $t<0$), is, first, to multiply it by χ, second to keep only its component $(\chi\phi_2)|_{\Sigma_t}$ (resp. $(\chi\phi_0)|_{\Sigma_t}$), and finally to project this component on $\mathscr{I}^\pm$ along the congruence $\mathscr{C}^\pm$. It is therefore just as trivial as in the Dirac case to see that $\tilde{W}^\pm = \mathfrak{T}^\pm$. The rest of the proof is essentially analogous to the proof in the Dirac case and we omit it.

A Estimates from characteristic data on $\mathscr{I}$

In order to prove these, we will use special features of $\mathscr{I}$. In particular, these estimates will not be true in such a simple form in the context of a characteristic initial value problem from a finite light cone (although analogues will, of course, exist). We first obtain an explicit representation of the metric in a neighbourhood of $\mathscr{I}^+$ and i^+.

We deal with three regions separately, a neighbourhood U^+ of i^+, a neighbourhood of U^0 of i^0 on which the space-time agrees with the Kerr solution, and a neighbourhood U of $\mathscr{I}^+$ which covers the rest of $\mathscr{I}^+$ but is bounded away from i^0 and i^+. On U^0 the strategy is to use the (exact) energy estimate using the time-like Killing vector. On U and U^+ we use an approximate Killing vector that generalises the dilation from Minkowski space.

A.1 The U^0 region

We simply evolve the hypersurface Σ_0 in this region leaving it fixed elsewhere ; we obtain a surface $\hat{\Sigma}_0$ very much like that of figure 4, such that the part that differs from Σ_0 lies entirely in the U^0 region. The time-like Killing vector K^a of the Kerr solution leads to exactly conserved quantities $K^a \mathbf{T}_{ab}$, $\nabla^b (K^a \mathbf{T}_{ab}) = 0$. Thus, the divergence theorem can be applied in the unphysical space-time in which $\mathscr{I}$ is a finite surface to give that

$$\int_{\Sigma_0} K^a \mathbf{T}_{ab}{}^* dx^b = \int_{\hat{\Sigma}_0} K^a \mathbf{T}_{ab}{}^* dx^b + \int_{\mathscr{I}_0^+} K^a \mathbf{T}_{ab}{}^* dx^b \tag{24}$$

where $\mathscr{I}_0^+$ is the part of $\mathscr{I}^+$ to the past of the hypersurface $\hat{\Sigma}_0$.

This is sufficient to move on to the next step in the case of the Maxwell equations (recall that the Dirac case is straightforward because of the exactly conserved quantity on the rescaled space-time) because the energy integrands are conformally invariant and the formulae in the unphysical space-time are equal to their counterparts in the physical space-time.

Remark A.1. *In order that there be no flux at i^0, we work with compactly supported data ; for these, the solutions are bounded away from i^0 ; then we extend the equality (24) by density to finite energy data on Σ_0.*

However, we need to do a bit more work in the case of the wave equation because in that case, the standard positive definite stress-energy tensor is not conformally invariant : firstly the energy in the unphysical space-time will not be equal to that in the physical space-time, and furthermore, it will not be exactly conserved unless the scalar curvature vanishes in the unphysical space-time. The latter point is easy to remedy if we choose our conformal scale for the unphysical space-time by first choosing an appropriate solution to the conformally invariant wave equation, and rescaling in such a way that this solution is reduced to 1. Such a conformal scale is guaranteed to have vanishing scalar curvature by virtue of the fact that '1' is a solution to the conformally invariant wave equation implying that $(\Box + R/6)1 = R/6 = 0$. In the Kerr solution, an appropriate choice is

$$\phi = \frac{1}{r_- - r_+} \log(1 - \frac{r_+ - r_-}{r - r_-}) \,, \qquad r_\pm = m \pm \sqrt{m^2 - a^2}$$

which behaves like $1/r$ as $r \to \infty$. Restricting to Scharzschild for simplicity, we see that the unphysical metric becomes

$$\hat{ds}^2 = \frac{\log(1-2mR)^2}{R^2} (R^2(1-2mR)du^2 + 2dudR + d\sigma^2) \,,$$

where $R = 1/r$, $u = t - r - 2m \log(r - 2m)$, and $d\sigma^2$ denotes the round unit sphere metric. The Killing vector $K = \partial/\partial t$ becomes the Killing vector $K = \partial/\partial u$ in the unphysical space-time and so $K^a \mathbf{T}_{ab}$ is conserved in the unphysical space-time. The hypersurface Σ_0 is $u = -1/R - 2m \log(r - 2m)$ and $\hat{\Sigma}_0$ can be taken to agree with Σ_0 for $R > R_0$ but for $R < R_0$, we can take $u = \alpha(R - R_0) - 1/R_0 - \log(R_0 - 2m)$ for some α with $0 < \alpha < (1 + 2mR_0)/R_0^2$.

We note that, although this rescaling brings $\mathscr{I}$ into a finite coordinate range into the future, i^0 is still at $u = -\infty$. This does not lead to any problems of leakage of flux in equation (24) on account of remark A.1.

The distinction in the wave equation case from Maxwell and Dirac is that the norm that is preserved is not the original energy norm from the physical space-time.

A.2 the U^+ region

The strategy here is to examine in detail the proof of the standard energy estimate near $\mathscr{I}$. We will use a foliation that, near $\mathscr{I}$, is given by $\Omega =$ constant where Ω is the conformal factor relating the physical metric g to the unphysical one $\hat{g} = \Omega^2 g$. In the standard proof of the energy estimate, one uses Gronwall's inequality starting with

$$\frac{\partial}{\partial s}\int_{\Omega=s} K^a T_{ab}\hat{\nabla}^b\Omega \mathrm{dVol}_s = \int (\hat{\nabla}^a K^b) T_{ab} \mathrm{dVol}_s$$

where dVol_s is defined by ${}^*\mathrm{d}x^a = \nabla^a\Omega\mathrm{dVol}_s$ on $\Omega = s$. Gronwall's inequality can be invoked if we know that

$$(\hat{\nabla}^a K^b)T_{ab} \leq cK^a T_{ab}\nabla^b\Omega \tag{25}$$

for some uniform constant c. Although $\hat{\nabla}^{(a}K^{b)}$ clearly remains bounded, the difficulty we have is that as $\Omega \to 0$, $\hat{\nabla}^a\Omega$ will become null, and so the right hand side of (25) no longer controls all the components of T_{ab}. Indeed, we will wish to take for K^a a vector field that also becomes null on $\mathscr{I}$ (in fact we will take $K^a = \hat{\nabla}^a\Omega$) in order to get a bound involving only the characteristic data and this makes our task harder. The proof of (25) relies on special properties of the geometry of $\mathscr{I}$, and will not be true for a generic lightcone.

The space-times in question are smooth at $\mathscr{I}^+$ and i^+ and in particular have the property that the Weyl tensor vanishes on $\mathscr{I}^+$ and at i^+.

Lemma A.1. *There exists a conformal scale near i^+ so that the Ricci curvature $\hat{R}_{ab}$ vanishes at i^+ and on $\mathscr{I}$ $\hat{R} = 0 = n^a\hat{R}_{ab}$.*

Proof: First we choose a conformal scale that brings i^+ to a finite point. The operator

$$L_{ab}\omega = (\nabla_{A(A'}\nabla_{B')B} + \frac{1}{2}R_{A(A'B')B})_0\omega$$

is conformally invariant when ω is taken to have weight 1. We obtain ω on $\mathscr{I}^+$ by setting $\omega(i^+) = 1$ and solving the conformally invariant second order ordinary differential equation $n^a n^b L_{ab}\Omega = 0$ up the generators of $\mathscr{I}^+$. There is a freedom in choice of the first derivative of ω at i^+ that will be useful to us later. We can then, as a power series, extend this scale off $\mathscr{I}$ by using $1/\omega$ thus obtained, as characteristic data for a solution to the conformally invariant wave equation near i^+ (this can be done at least at the level of formal power series, [35], and extended as a smooth function to a full neighbourhood of i^+ in such a way that the equations hold to all orders at $\mathscr{I}^+$). If we now choose the conformal scale in which $\omega = 1$, we have that $n^a n^b\hat{R}_{ab} = 0 = R$ along $\mathscr{I}^+$. In particular, in this conformal scale, the full Ricci curvature vanishes at i^+, and $\hat{R}$ and $\hat{R}_{ab}n^a n^b$ vanish along $\mathscr{I}^+$.

The Bianchi identities now give, using the vanishing of the Weyl Curvature along $\mathscr{I}$ together with $\hat{R}_{ab}n^a n^b = 0 = R$, the equations $D\hat{\Phi}_{21} - 2\bar{\rho}' = 0$ from the primed version of equ 4.12.36 of [36]. Since Φ_{21} vanishes at i^+, this implies that it vanishes along $\mathscr{I}$. Similarly Φ_{11} can be seen to vanish on $\mathscr{I}$ using the primed versions of equations 4.12.37 and 4.12.40 of [36]. These together imply that $\hat{R}_{ab}n^a = 0$. $\square$

We will use, for the approximate conformal Killing vector, $K^a = \hat{\nabla}^a\Omega$ near $\mathscr{I}$, where Ω is the conformal factor relating the physical metric to the one derived above. This has conformal Killing form $\nabla_{A(A'}\nabla_{B')B}\Omega$. We have, from the conformal rescaling properties of the trace free part of the Ricci tensor Φ_{ab}

$$\Phi_{ab} = \hat{\Phi}_{ab} + \Omega^{-1}\nabla_{A(A'}\nabla_{B')B}\Omega\,.$$

We assume that $\Phi_{ab} = O(\Omega^2)$ (as will be the case, for example, for Einstein-Maxwell fields), then this implies that the conformal Killing form of K^a is

$$\hat{\nabla}_{A(A'}\hat{\nabla}_{B')B}\Omega = \Omega\hat{\Phi}_{ab} + O(\Omega^2)\,. \tag{26}$$

Introduce Gaussian normal coordinates x^a for this metric taken from the exponential map from linear coordinates on the tangent space at i^+. We find that, firstly $\mathscr{I}$ is given by $x^a x^b\eta_{ab} = 0$ where η_{ab} is the flat metric on the tangent space at i^+ (this follows since the null vectors will necessarily map onto $\mathscr{I}$). Furthermore, the vanishing of the Weyl tensor and $\hat{R}_{ab}n^a$ and $\hat{R}$ on $\mathscr{I}$ implies that $n^c n^d R_{acbd} = 0$ on $\mathscr{I}$ and so the geodesic deviation equations implies that a parallel propagated frame from i^+ down $\mathscr{I}^+$ is also Lie derived. Thus, in these coordinates, the metric at $\mathscr{I}$ agrees with the flat one (the parallel propagated frame from i^+ up the generators of $\mathscr{I}^+$ is also Lie derived up $\mathscr{I}^+$ and so agrees with the coordinate frame). We can therefore write

$$\hat{ds}^2 = \eta_{ab}dx^a dx^b + \eta_{cd}x^c x^d h$$

where h is a smooth rank-2 symmetric tensor near $\mathscr{I}$. In these coordinates we have $\Omega = x^a x^b\eta_{ab} + O((x^a x^b\eta_{ab})^3)$ as can be seen by comparing (26) to the expression for the lie derivative of $\hat{ds}^2$ along $\hat{\nabla}^a\Omega$.

We introduce coordinates (t, r, θ) related to the coordinates x^a in the standard way with θ being coordinates on the unit sphere and r and t being the usual radial and timelike coordinates. $\mathscr{I}^+$ is then given by $t = -r$ and $\hat{n}^a\partial_a = (\partial_t - \partial_r)/\sqrt{2}$ is tangent to $\mathscr{I}^+$ and $\hat{l}^a\partial_a = (\partial_t + \partial_r)/\sqrt{2}$ is transverse to $\mathscr{I}^+$. We can write

$$\sqrt{2}x^a = (t - r)\hat{l}^a + (t + r)\hat{n}^a\,.$$

We can choose a spin frame so that $\hat{l}^a = \hat{o}^A\bar{\hat{o}}^{A'}$ and $\hat{n}^a = \hat{\iota}^A\bar{\hat{\iota}}^{A'}$.

A.3 The Maxwell case

For Maxwell fields, $\mathbf{T}_{ab} = \phi_{AB}\phi_{A'B'}$. We wish to see that

$$(\hat{\nabla}^a\hat{\nabla}^b\Omega)\mathbf{T}_{ab} = \Omega\Phi^{ab}\mathbf{T}_{ab} \leq c\hat{\nabla}^a\Omega\hat{\nabla}^b\Omega\mathbf{T}_{ab} \tag{27}$$

uniformly on U^+. We note, first of all, that $\Phi^{ab}(i^+) = 0$ and that $x^a\Phi_{ab} = 0$ on $\mathscr{I}^+$. This in particular implies that, since Φ_{ab} is smooth at i^+, Φ_{ab} actually vanishes to second order: $\Phi_{ab} = x^c x^d \Phi^+_{acbd}$ where Φ^+_{abcd} has the symmetries and trace properties of a Weyl tensor. We can therefore write $\Phi_{ab} = (t-r)^2\Phi^+_{ab}$ for some bounded Φ^+_{ab}.

Then we have that in terms of the spin frame above

$$\mathbf{T}_{ab}\hat{\nabla}^a\Omega\hat{\nabla}^b\Omega = (t-r)^2|\phi_{11}|^2 + (t^2-r^2)|\phi_{01}|^2 + (t+r)^2|\phi_{00}|^2$$

whereas, writing $\Phi^{+ab} = \Phi^+_{00}n^a n^b + \Re(\Phi^+_{01}n^{(a}m^{b)} + \Phi^+_{02}m^a m^b)$ we see that

$$\Omega\Phi^{ab}\mathbf{T}_{ab} = (t+r)(t-r)^3\left(\Phi^+_{00}|\phi_{11}|^2 + \Re(\Phi^+_{01}\phi_{11}\bar{\phi}_{01} + \Phi^+_{02}\phi_{00}\bar{\phi}_{11})\right) + O((t^2-r^2)^2)$$

It is clear from these formulae that (27) holds, firstly near i^+, because the left hand side vanishes to 4th order whereas the right hand side only to second order, and secondly, near $\mathscr{I}^+$ because $(t+r)\phi_{00}$ and $\sqrt{t+r}\phi_{01}$ are controlled by the right hand side, and the terms that appear on the left hand side are clearly bounded by these.

B Proof of the main theorems

We do the constructions for $\mathscr{I}^+$, the case of $\mathscr{I}^-$ can be treated analogously.

B.1 Proof of theorem 1

The Dirac/Weyl equation (6) admits a conserved current in any globally hyperbolic space-time, given by

$$V^a = \phi^A\bar{\phi}^{A'}\,, \qquad \nabla_a V^a = 0\,,$$

the conservation following directly from the field equation. In particular, this gives rise to a closed 3-form $\omega = {}^*V_a dx^a$ and the conformal rescaling weights of ϕ_A together with that of the $*$-operator on 3-forms means that this 3-form is the same on the rescaled space-time :

$$\omega = \hat{\omega} = {}^*\hat{V}_a \mathrm{d}x^a\,,\ \hat{V}^a = \hat{\phi}^A\bar{\hat{\phi}}^{A'}\,. \tag{28}$$

On restriction to a space-like hypersurface Σ, the indexed 3-form ${}^*\mathrm{d}x^a$ is proportional to the unit normal T^a to Σ, and we can set $*\mathrm{d}x^a = T^a\mathrm{dVol}_\Sigma$ and this defines the volume form dVol_Σ for Σ. However, on restriction to $\mathscr{I}$, we will still have ${}^*\mathrm{d}x^a = n^a\mathrm{dVol}_{\mathscr{I}}$ for some choice of normal n^a to $\mathscr{I}$ and 3-form $\mathrm{dVol}_{\mathscr{I}}$ on $\mathscr{I}$ but there is no invariant normalization for n^a and hence neither also for $\mathrm{dVol}_{\mathscr{I}}$. However, the product ${}^*\mathrm{d}x^a = n^a\mathrm{dVol}_{\mathscr{I}}$ is invariant. The quantities n^a and $\mathrm{dVol}_{\mathscr{I}}$ can be understood more abstractly as spin-weighted quantities : tensors with values in certain line-bundles. In the spin coefficent formalism, one assumes that one has chosen a spinor dyad (o^A, ι^A), but only up to scale. On $\mathscr{I}$ we can choose ι^A up to scale by requiring that $\iota^A\bar{\iota}^{A'} = n^a$ be normal to $\mathscr{I}$. Under $\iota^A \to \lambda\iota^A$, we will have $n^a \to \lambda\bar{\lambda}n^a$ and $\mathrm{dVol}_{\mathscr{I}} \to \lambda^{-1}\bar{\lambda}^{-1}\mathrm{dVol}_{\mathscr{I}}$ and we say that n^a has weight $(1;1)$ and $\mathrm{dVol}_{\mathscr{I}}$ has weight $(-1;-1)$ (more generally quantities might also have weights under rescaling of o^A but that will not be needed here). A quantity has weight

$(-1;0)$ can also be taken to mean that it takes values in the line-subbundle of the spin-bundle spanned by ι^A and this line bundle together with its complex conjugate generate the appropriate line bundles for spin-weighted quantities.

The null data on $\mathscr{I}$ for the Dirac field is $\hat{\phi}_1 = \hat{\phi}_A \iota^A$ and has spin-weight $(1;0)$ where $\hat{\phi}_A$ is the rescaled Dirac field in the unphysical space-time and $\hat{n}^a = \iota^A \bar{\iota}^{A'}$ also in the unphysical space-time (on $\mathscr{I}^+$). The 3-form above then becomes

$$\omega = |\hat{\phi}_1|^2 \mathrm{dVol}_{\mathscr{I}}$$

and it can be seen that this is invariant both under conformal rescalings and rescalings of ι^A.

The 3-form (28) is closed provided the solution $\hat{\phi}^A$ is regular enough to allow differentiation. Essentially, $\hat{\phi}^A \in H^1(\widehat{\mathscr{M}}\,;\, \mathbb{S}^A)$ is enough. This is the case of solutions $\hat{\phi}^A \in \mathcal{C}^\infty(\widehat{\mathscr{M}})$ associated with smooth, compactly supported initial data on Σ_0. Since the support of such solutions remains away from i^0, we can use Stokes' theorem on the hypersurface constituted of Σ_0 and $\mathscr{I}^+$ to obtain

$$\int_{\Sigma_0} \frac{1}{\sqrt{2}} \hat{T}^a \hat{V}_a \widehat{\mathrm{dVol}}_{\Sigma_0} = \int_{\mathscr{I}^+} \hat{n}^a \hat{V}_a \mathrm{dVol}_{\mathscr{I}^+} \, ,$$

i.e.

$$\left\| \hat{\phi} \right\|^2_{L^2(\Sigma_0\,;\, \widehat{\mathrm{dVol}})} = \int_{\mathscr{I}^+} \left| \hat{\phi}_1 \right|^2 \mathrm{dVol}_{\mathscr{I}^+} \, . \tag{29}$$

This gives us estimates (13) and (14) for Dirac fields. In addition we recover that the characteristic data on $\mathscr{I}^+$ is $\hat{\phi}_1 = \hat{\iota}^A \hat{\phi}_A$. A similar construction at past null infinity would reveal the null data there to be $\hat{\phi}_0 = \hat{o}^A \hat{\phi}_A$, since the normal (and tangent) vector field to $\mathscr{I}^-$ is $l^a = o^A \bar{o}^{A'}$.

We now suppose that we are given some null data $\hat{\phi}^+_{\mathscr{I}} \in \mathcal{C}_0^\infty(\mathscr{I}^+)$.[5] Appendix C.2 gives us the third step of the construction and concludes the proof of theorem 1. □

B.2 Proof of theorem 3

The existence of $\tilde{W}^+$ is straightforward. We consider $\hat{\phi}_A \big|_{\Sigma_0} \in \mathcal{C}_0^\infty(\Sigma_0)$, then by smoothness on $\widehat{\mathscr{M}}$ of the associated solution $\hat{\phi}_A$, the spinor field

$$\left(\mathbb{P}^+_\chi(t)\right)^* \mathcal{U}(t,0) \left(\hat{\phi}_A \Big|_{\Sigma_0} \right)$$

[5] We need also to impose the condition that it arises from a spinor field that is smooth at i^+. Thus there exists a smooth ψ^A on $\widehat{\mathscr{M}}$ supported away from i^0 such that $\hat{\phi}^+_{\mathscr{I}} = \psi_A \hat{\iota}^A = \psi_1$. A priori, there is considerable freedom in the choice of such a ψ_A: we can add on any smooth spinor field whose restriction to $\mathscr{I}^+$ is a multiple of ι_A. The construction of appendix C.2 shows in fact that we have no freedom at all in the trace of ψ_A on $\mathscr{I}^+$; it is completely determined by the constraint equation on $\mathscr{I}^+$ and the requirement that the support of ψ_A should remain away from i_+ (needed for technical reasons because our choice of spin-frame $(\hat{o}^A, \hat{\iota}^A)$ is singular at i^+).

converges pointwise on $\mathscr{I}^+$ to $\mathfrak{T}^+ \left(\hat{\phi}_A \Big|_{\Sigma_0} \right)$. The convergence in $L^2(\mathscr{I}^+)$ for such data is a direct consequence of Lebesgue's dominated convergence theorem. Since $\tilde{W}^+$ and $\mathfrak{T}^+$ coincide on a dense subset, they coincide on all of $L^2(\Sigma_0)$ and the convergence in $L^2(\mathscr{I}^+)$ likewise remains true for all L^2 data by density. Besides, $\tilde{W}^+$ is clearly independent of the choices of K large enough and the function χ for smooth compactly supported data ; this remains true for all L^2 data by density. The existence of $W^\pm$ and the fact that $\tilde{W}^\pm = (W^\pm)^{-1}$ is proved by our resolution of the Goursat problem. More precisely, let $\hat{\phi}^+_{\mathscr{I}} \in \mathcal{C}^\infty_0(\mathscr{I}^+)$ considered as a spinor field element of $L^2(\mathscr{I}^+)$. Since $\hat{\phi}^+_{\mathscr{I}}$ is supported away from i^+, $\mathbb{P}^+_\chi(t)\phi^+_{\mathscr{I}}$ will coincide on all Σ_t (for t large enough) with the projection on Σ_t along $\mathscr{C}^+$ of $\hat{\phi}^+_{\mathscr{I}}$. It follows from definition C.3 and proposition C.1 that $\mathcal{U}(0,t)\mathbb{P}^+_\chi(t)\hat{\phi}^+_{\mathscr{I}}$ converges in $L^2(\Sigma_0)$ and that the image under $\mathfrak{T}^+$ of the limit is $\hat{\phi}^+_{\mathscr{I}}$. By density, W^+ is therefore well-defined as the strong limit (22) in $L^2(\Sigma_0)$ and is a right-inverse of $\mathfrak{T}^+ = \tilde{W}^+$. The trace operator $\mathfrak{T}^+$ being an isomorphism, we conclude $W^+ = \left(\tilde{W}^+\right)^{-1} = (\mathfrak{T}^+)^{-1}$. Finally $\tilde{W}^+ = (W^+)^*$ is trivial. □

C Finding a solution to the Goursat problem

We treat the Goursat problem on $\mathscr{I}^+$; the case of $\mathscr{I}^-$ is dealt with similarly.

On $\mathscr{I}^+$, we consider a null data $\hat{\phi}^+_{\mathscr{I}} \in \mathcal{C}^\infty_0(\mathscr{I}^+)$ and we look for a solution of each equation that has finite energy on Σ_0, such that the image of the data on Σ_0 by $\mathfrak{T}^+$ is $\hat{\phi}^+_{\mathscr{I}}$. There are two steps in this construction. The first is to find a solution in the future $\widehat{\mathscr{M}}^+_\varepsilon$ of a hypersurface $\mathscr{H}_\varepsilon$ (where $0 < \varepsilon << 1$ is chosen so that the boundary of $\mathscr{H}_\varepsilon$ is in the past of the support of $\hat{\phi}^+_{\mathscr{I}}$) such that the trace of its relevent component on $\mathscr{I}^+$ is $\hat{\phi}^+_{\mathscr{I}}$. The second is, by means of energy estimates, to show that this solution can be extended uniquely to $\widehat{\mathscr{M}}$ and that the extension has finite energy on Σ_0.

The first step is exactly finding a solution to the Goursat problem on a finite light-cone (that of i_+) on the rescaled space-time. For a generic wave equation

$$\Box_{\hat{g}}\hat{\phi} + \langle a, \hat{\nabla}\rangle\hat{\phi} + b\hat{\phi} = 0 \,, \tag{30}$$

where a is a smooth vector field on $\widehat{\mathscr{M}}$ and b a smooth scalar field, two methods are well known. One is due to Hörmander [26] and uses energy estimates, weak convergence and compactness. The other is due to Friedlander [19] ; it is based on the works of Hadamard and Leray and gives an integral representation of the solution. Both can be easily adapted to the Dirac case, although some care is needed. The Maxwell case is more subtle. We describe the first approach briefly in the case of equation (30), then we explain how it is used for Dirac and Maxwell. The second approach will be the subject of a subsequent paper.

C.1 The wave equation

We modify very slightly Hörmander's technique : instead of slowing down the propagation speed so that $\mathscr{I}^+$ becomes spacelike, we approach $\mathscr{I}^+$ by spacelike hypersurfaces without changing the equation. This proves extremely useful in the reformulation of our conformal scattering theory in terms of wave operators. The obvious choice for the spacelike hypersurfaces approaching $\mathscr{I}^+$ is the foliation $\{\Sigma_t\}_t$.

Next, we choose in $\widehat{\mathscr{M}}^+$ (denoting the future of Σ_0 in $\widehat{\mathscr{M}}$) a congruence $\mathscr{C}$ of timelike or weakly timelike lines, to allow us to project the null data on $\mathscr{I}^+$ as regular initial data on Σ_t for any $t \geq 0$. There are two natural choices :

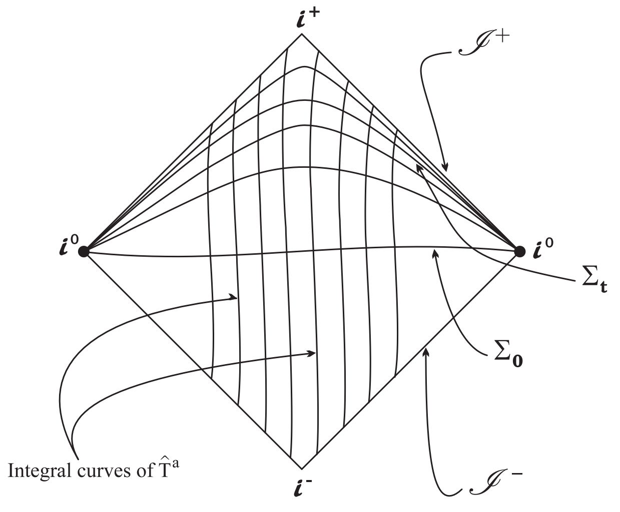

Figure 6: Timelike congruence normal to the foliation $\{\mathscr{H}_\tau\}_\tau$.

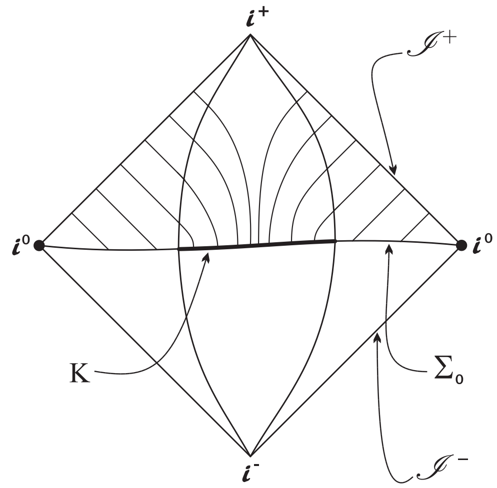

Figure 7: A weakly timelike congruence, coinciding with a congruence of outgoing null geodesics outside a compact in space.

1. The integral curves of the normal vector to the foliation $\{\mathscr{H}_\tau\}_\tau$ (see figure 6).

2. Given K a large enough compact subset of Σ, we can choose, in $[0,+\infty[_t\times(\Sigma\setminus K)$, a congruence of outgoing null geodesics defining $\mathscr{I}^+$ (i.e. such that each part of $\mathscr{I}^+$ is met by one and only one of these lines). Then, these lines can be extended smoothly inside $[0,+\infty[_t\times K$ as a timelike congruence (see figure 7).

Either choice is good and so would be any congruence of smooth timelike curves on $\widehat{\mathscr{M}}$. The second choice will be excessively useful for the reformulation of conformal scattering in terms of wave operators, so we adopt it.

Definition C.1. *We denote by $P^+_K(t)$ the operator that, to $\hat{\phi}^+_{\mathscr{I}}$ associates the initial data for (30) defined as follows :*

- $\hat{\phi}\Big|_{\Sigma_t}$ *is the projection of $\hat{\phi}^+_{\mathscr{I}}$ on Σ_t along the congruence $\mathscr{C}$;*

- $\partial_T\hat{\phi}\Big|_{\Sigma_t} = 0\,.$

We denote by ${}_t\hat{\phi}$ the solution of (30) on $\widehat{\mathscr{M}}^+$ associated with the data $P_K^+(t)\hat{\phi}^+_{\mathscr{I}}$ on Σ_t.

The energy estimates established in [26] prove that ${}_t\hat{\phi}$ is uniformly bounded[6] with respect to t in $H^1(\widehat{\mathscr{M}_\varepsilon^+}\,;\ \hat{g})$, the natural Sobolev space on $\widehat{\mathscr{M}_\varepsilon^+}$ associated with the metric $\hat{g}$. Hence, we can extract a sequence $\{\,{}_{t_n}\hat{\phi}\}_n$ that converges weakly in $H^1(\widehat{\mathscr{M}_\varepsilon^+}\,;\ \hat{g})$ and hence strongly, by the Rellich theorem, in $H^{1-s}(\widehat{\mathscr{M}_\varepsilon^+}\,;\ \hat{g})$ for any $s > 0$. The limit, denoted $\hat{\phi}$, belongs to $H^1(\widehat{\mathscr{M}_\varepsilon^+}\,;\ \hat{g})$ and still satisfies equation (30). Moreover, by standard trace theorems, weak convergence in H^1 entails strong convergence on any hypersurface of a given Lipschitz foliation, with continuous dependence of the parameter of the leaves. We conclude that the trace of $\hat{\phi}$ on $\mathscr{I}^+$ is equal to $\hat{\phi}^+_{\mathscr{I}}$.

Remark C.1. *Note that this method does not guarantee the smoothness of the solution unless we use estimates in $H^k(\widehat{\mathscr{M}_\varepsilon^+}\,;\ \hat{g})$ for all $k \in \mathbb{N}$. For such estimates, some care must be taken in the choice of $\partial_T\hat{\phi}\Big|_{\Sigma_t}$ (this is similar to the phenomenon that we encounter in the definition of the data on Σ_t approaching $\hat{\phi}^+_{\mathscr{I}}$ for Dirac or Maxwell ; see below).*

Remark C.2. *In the case of the wave equation, although the Goursat problem can be solved, the second step of the construction (uniform estimate from $\mathscr{H}_\varepsilon$ to Σ_0) runs into difficulties for exactly the same reasons as the estimates from characteristic data on $\mathscr{I}^+$ in appendix A.*

C.2 Dirac's equation

In these last two sections, we shall use spin-frames (o^A, ι^A) and $(\hat{o}^A, \hat{\iota}^A)$ as introduced in definition 2.2. The fact that they are only defined outside a fixed compact in space is not a problem here, since we only use explicit projections onto the spin-frames when working with data that are zero in a given compact set (as the projections onto hypersurfaces Σ_t along $\mathscr{C}$ of fields on $\mathscr{I}^+$ that are zero near i^+). More precisely, we consider a neighbourhood $\mathcal{O}_\infty$ of i^+ on $\mathscr{I}^+$ (slightly smaller than the region in which $\hat{\phi}^+_{\mathscr{I}}$ is assumed to vanish) and we define a region $\mathcal{O}$ on $\widehat{\mathscr{M}}$ as the set of points whose projection on $\mathscr{I}^+$ along the congruence $\mathscr{C}$ lies inside $\mathcal{O}_\infty$. The intersection of $\mathcal{O}$ with a hypersurface Σ_t will be denoted $\mathcal{O}_t$. We shall simply assume that the spin-frame (o^A, ι^A) is defined globally outside $\mathcal{O}$ (modulo the topological problem on the 2-sphere that is solved using two patches and that is irrelevant here).

The definition of $P_K^+(t)$ is more subtle in this case. We look for a way of defining initial data ${}_t\hat{\phi}_A\Big|_{\Sigma_t}$ that allows us to follow the same strategy as for the wave equation. When trying to use this scheme, some retrictions naturally appear for the choice of data. Let us assume that we have defined ${}_t\hat{\phi}_A\Big|_{\Sigma_t}$, we denote by ${}_t\hat{\phi}_A$ the associated solution of (16) in $\mathscr{M}$. A first natural constraint is that ${}_t\hat{\phi}_1\Big|_{\Sigma_t}$ converges to $\hat{\phi}^+_{\mathscr{I}}$ as $t \to +\infty$, so as to guarantee that the second component of the solution we construct has the right

[6] The fact that we remain away from the singularity of $\hat{g}$ at i^0 by working on $\widehat{\mathscr{M}_\varepsilon^+}$ is crucial here ; the uniform boundedness in $H^1(\widehat{\mathscr{M}}^+\,;\ \hat{g})$ could not be obtained as a consequence of [26].

trace on $\mathscr{I}^+$. Now, in order to be able to construct this solution, we need to control the family of solutions ${}_t\hat{\phi}$ uniformly in H^1 on $\widehat{\mathscr{M}_\varepsilon^+}$. For ${}_t\hat{\phi}$ to be bounded uniformly in t on $L^2(\widehat{\mathscr{M}_\varepsilon^+}\,;\,\hat{g})$, thanks to the conserved current, we simply need to assume that

$$\exists C>0\,;\ \forall t\geq 0\,,\ \left\| {}_t\hat{\phi}\Big|_{\Sigma_t}\right\|_{L^2(\widehat{\mathscr{M}_\varepsilon^+}\,;\,\hat{g})}\leq C\,.$$

Let us now see what further constraints a uniform H^1 control on $\widehat{\mathscr{M}_\varepsilon^+}$ entails.

In the following, we use Dirac 4-component spinors, and denote by ϕ or $\hat{\phi}$ the chiral 4-component spinors corresponding to the 2-component spinors ϕ_A and $\hat{\phi}_A$ resppectively. Using the parallelizability of $\widehat{\mathscr{M}}$, we choose a global section $\{\hat{e}_{\mathbf{a}}{}^a\}_{\mathbf{a}=0,1,2,3}$ of the principal bundle of orthonormal frames, such that $\hat{e}_0{}^a=\frac{1}{\sqrt{2}}\hat{T}^a$. The rescaled equation (16) then takes the form

$$\sum_{\mathbf{a}=0}^{3}\hat{e}_{\mathbf{a}}.\hat{\nabla}_{\mathbf{a}}\,{}_t\hat{\phi}=0\,, \tag{31}$$

where "." denotes the Clifford product and

$$\hat{\nabla}_{\mathbf{a}}=\hat{\nabla}_{\hat{e}_{\mathbf{a}}}=\hat{e}_{\mathbf{a}}{}^a\,\hat{\nabla}_a\,.$$

Each covariant derivative $\hat{\nabla}_{\mathbf{a}}\,{}_t\hat{\phi}$ satisfies an equation similar to (31), but with a right-hand side :

$$\sum_{\mathbf{a}=0}^{3}\hat{e}_{\mathbf{a}}.\hat{\nabla}_{\mathbf{a}}\left(\hat{\nabla}_{\mathbf{b}}\,{}_t\hat{\phi}\right)=-\sum_{\mathbf{a}=0}^{3}\left\{\left(\hat{\nabla}_{\mathbf{b}}\hat{e}_{\mathbf{a}}\right).\hat{\nabla}_{\mathbf{a}}\,{}_t\hat{\phi}+\hat{e}_{\mathbf{a}}.\left[\hat{\nabla}_{\mathbf{b}},\hat{\nabla}_{\mathbf{a}}\right]\,{}_t\hat{\phi}\right\}\,,\ \mathbf{b}=0,...,3\,. \tag{32}$$

Hence, an H^1 control over ${}_t\hat{\phi}$ will be obtained as a consequence of the natural L^2 control given on each unknown by the five coupled Dirac equations (31)-(32). This L^2 control will be uniform in t provided the L^2 norm of the restriction to Σ_t of each of the functions ${}_t\hat{\phi}$, $\hat{\nabla}_{\mathbf{a}}\,{}_t\hat{\phi}$, $\mathbf{a}=0,1,2,3$, is bounded uniformly in t. A useful thing to do at this stage is to decompose equation (31) on Σ_t in terms of $\hat{\nabla}_0$ and derivatives that are tangent to Σ_t. In order to express these tangential derivatives, we define the family of smooth functions $\{f_t\}_t$ on Σ as the restriction to Σ_t of τ projected back to Σ_0 along the integral curves of $\hat{T}^a$: i.e., for each $x\in\Sigma$, $f_t(x)$ is the only $\tau\in\mathbb{R}$ such that $(\tau,x)\in\Sigma_t$ (using the product structure on $\widehat{\mathscr{M}}$ induced by the time function τ and the integral lines of $\hat{T}^a$). The tangential derivatives of ${}_t\hat{\phi}$ on Σ_t then take the form :

$$\left[\left(\hat{\nabla}_\alpha+\hat{\nabla}_\alpha f_t(x)\hat{\nabla}_0\right)\,{}_t\hat{\phi}\right](f_t(x),x)\ ,\quad \alpha=1,2,3.$$

In order to control all the directional covariant derivatives, it is sufficient to control the tangential ones and $\hat{\nabla}_0\,{}_t\hat{\phi}$, since the functions f_t are Lipschitz on Σ uniformly in t. The $\hat{\nabla}_0$ time derivative of ${}_t\hat{\phi}$ on Σ_t can be expressed in terms of its tangential derivatives, re-writing the restriction of Dirac's equation to Σ_t as follows :

$$\begin{aligned}&\left[\left(\hat{e}_0-\sum_{\alpha=1}^{3}\hat{\nabla}_\alpha f_t(x)\hat{e}_\alpha\right).\hat{\nabla}_0\,{}_t\hat{\phi}\right](f_t(x),x)\\&\qquad=-\left[\sum_{\alpha=1}^{3}\hat{e}_\alpha.\left(\hat{\nabla}_\alpha+\hat{\nabla}_\alpha f_t(x)\hat{\nabla}_0\right)\,{}_t\hat{\phi}\right](f_t(x),x)\,.\end{aligned}$$

Clifford multiplying by the vector

$$\hat{e}_0 - \sum_{\alpha=1}^{3} \hat{\nabla}_\alpha f_t(x) \hat{e}_\alpha \,, \tag{33}$$

that is normal to Σ_t, we obtain

$$\begin{aligned} &\left(1 - \sum_{\alpha=1}^{3} |\nabla_\alpha f_t(x)|^2\right) \hat{\nabla}_0 \, {}_t\hat{\phi}(f_t(x), x) \\ &= - \left(\hat{e}_0 - \sum_{\alpha=1}^{3} \hat{\nabla}_\alpha f_t(x) \hat{e}_\alpha\right) \cdot \left[\sum_{\beta=1}^{3} \hat{e}_\beta . \left(\hat{\nabla}_\beta + \hat{\nabla}_\beta f_t(x) \hat{\nabla}_0\right) {}_t\hat{\phi}\right] (f_t(x), x) \,. \end{aligned}$$

Therefore, in order to guarantee that the L^2 norm on Σ_t of $\hat{\nabla}_0 \, {}_t\hat{\phi}_A$ is bounded uniformly in t, the $L^2(\Sigma_t \, ; \, \hat{g})$ norm of

$$\left(\hat{e}_0 - \sum_{\alpha=1}^{3} \hat{\nabla}_\alpha f_t(x) \hat{e}_\alpha\right) \cdot \left[\sum_{\beta=1}^{3} \hat{e}_\beta . \left(\hat{\nabla}_\beta + \hat{\nabla}_\beta f_t(x) \hat{\nabla}_0\right) {}_t\hat{\phi}\right] (f_t(x), x) \tag{34}$$

must tend to zero as $t \to +\infty$ at least as fast as $1 - \sum_{\alpha=1}^{3} |\nabla_\alpha f_t(x)|^2$.

The quantity (34) involves only tangential derivatives of ${}_t\hat{\phi}$ and is therefore quite easy to control in our choice of ${}_t\hat{\phi}_A\Big|_{\Sigma_t}$. In particular, if ${}_t\hat{\phi}$ is the restriction of the same solution of the Dirac equation (i.e., is independent of t) the convergence will be trivially guaranteed. On $\mathscr{I}^+$ (i.e. as $t \to +\infty$), (34) reduces to the constraint equation (see remark 3.1) and if this is not satisfied, it will be an obstruction to uniform control in H^1. We are now in position to specify a good choice of ${}_t\hat{\phi}_A\Big|_{\Sigma_t}$.

Definition C.2. *Starting from the null datum $\hat{\phi}_1|_{\mathscr{I}^+} = \hat{\phi}^+_{\mathscr{I}}$, we can integrate the constraint equation to first obtain $\hat{\phi}_0|_{\mathscr{I}^+}$ to obtain a spinor field $\hat{\phi}_A$ on $\mathscr{I}^+$. We can then integrate the full Dirac field equations component by component and order by order to obtain as many terms in the Taylor series as we desire for the solution with given null datum. This is a question of integrating coupled ODE's up the generators of $\mathscr{I}$, see, for example, Penrose [35] for details. We note that in the term by term integration we need to use the fact that $\hat{\phi}_A$ should vanish in some small neighbourhood of i^+ to fix the constants of integration. The resulting spinor field $\hat{\phi}_A\Big|_{\mathscr{I}^+}$ is in $\mathcal{C}^\infty(\mathscr{I}^+)$ in such a way that its support remains away from i^+.*

We can choose ${}_t\hat{\phi}_A$ to be a smooth extension of the given $\hat{\phi}^+_{\mathscr{I}^+}$ on $\mathscr{I}^+$ that agrees with the associated Taylor series to arbitrary order (although agreement to 2nd order will be sufficient for control in H^1).

With this ${}_t\hat{\phi}_A\Big|_{\Sigma_t}$, the quantity (34) can be made to converge as fast as we like in $L^2(\Sigma_t)$ as $t \to +\infty$. Indeed, we have

$$1 - \sum_{\alpha=1}^{3} \left|\hat{\nabla}_\alpha f_t\right|^2 = \left\|\hat{e}_0 - \sum_{\alpha=1}^{3} \hat{\nabla}_\alpha f_t(x) \hat{e}_\alpha\right\|_{\hat{g}} \,. \tag{35}$$

The rate at which this quantity tends to zero as $t \to +\infty$ measures the speed at which the hypersersurfaces Σ_t approach $\mathscr{I}^+$ with respect to the metric $\hat{g}$. We note that for the chosen foliation by t =constant surfaces, this will actually vanish to second order in $1/t$ at $\mathscr{I}$ since Ω is of the same order as $1/t$ near $\mathscr{I}$ and $g^{ab}\nabla_a t \nabla_b t = O(1)$ implies that $\hat{g}^{ab}\nabla_a(1/t)\nabla_b(1/t) = O(\Omega^2)$. Now, the convergence of (34) towards

$$\left(\hat{e}_0 - \sum_{\alpha=1}^{3} \hat{\nabla}_\alpha f_\infty(x)\hat{e}_\alpha\right) \left[\sum_{\beta=1}^{3} \hat{e}_\beta . \left(\hat{\nabla}_\beta + \hat{\nabla}_\beta f_\infty(x)\hat{\nabla}_0\right)\right] {}_t\hat{\phi}\Big|_{\mathscr{I}^+} = 0\,,$$

where f_∞ is the function defining $\mathscr{I}^+$ in the same way as f_t defines Σ_t, is controlled by the speed at which the tangent vector fields $\hat{e}_\alpha + \hat{\nabla}_\alpha f_t(x)\hat{e}_0$, and the normal vector field (33), on Σ_t, approach their trace on $\mathscr{I}^+$, which in turn is controlled by (35). This therefore guarantees a uniform H^1 control for the family ${}_t\hat{\phi}_A$ on $\widehat{\mathscr{M}_\varepsilon^+}$ and the rest of the construction follows the same steps as for the wave equation.

Remark C.3. *Note that we have been forced into imposing that the trace of the solution on $\mathscr{I}^+$ satisfies the constraints on $\mathscr{I}^+$, and not merely that the trace of the first component of the solution is equal to $\hat{\phi}^+_{\mathscr{I}}$.*

The choice of data given by definition C.2 has been made for the sole purpose of gaining a uniform H^1 control in order to be able to use compactness arguments for guaranteeing the existence of a trace. Once this is obtained, we can in fact simplify the data on Σ_t to construct another family of solutions. This new family will not be bounded in H^1 but its restrictions to Σ_ε will nevertheless converge in L^2 towards the same data, whose image under $\mathfrak{T}^+$ is precisely $\hat{\phi}^+_{\mathscr{I}}$. We define this new family, and the operator $P^+_K(t)$, as follows :

Definition C.3. *For each $t > 0$, we define $P^+_K(t)$ as the operator that to $\hat{\phi}^+_{\mathscr{I}}$ associates the spinor field ${}_t\hat{\psi}_A\Big|_{\Sigma_t}$ on Σ_t, whose component ${}_t\hat{\psi}_0\Big|_{\Sigma_t}$ is identically zero and such that ${}_t\hat{\psi}_1\Big|_{\Sigma_t}$ is the projection of $\hat{\phi}^+_{\mathscr{I}}$ on Σ_t along $\mathscr{C}$. We denote by ${}_t\hat{\psi}_A$ the associated solution of (16).*

Proposition C.1. *The families ${}_t\hat{\phi}_A$ and ${}_t\hat{\psi}_A$, defined using definitions C.2 and C.3 respectively, converge in $L^2(\mathscr{H}_\varepsilon\,;\ \hat{g})$ and in $L^2(\widehat{\mathscr{M}_\varepsilon^+}\,;\ \hat{g})$ towards the same limit.*

Proof. The L^2 norm on Σ_t of ${}_t\hat{\phi}_A - {}_t\hat{\psi}_A$ tends to zero as $t \to +\infty$ because the normal vector to Σ_t approaches n^a and the contribution of the first component to the norm decreases to zero (the second components of ${}_t\hat{\phi}_A$ and ${}_t\hat{\psi}_A$ coincide on Σ_t, while the first components differ but their difference is uniformly bounded). The proposition is then a direct consequence of the charge conservation for (16). □

For the last step of the construction, we need the following result :

Lemma C.1. *Given a spinor field ${}_\varepsilon\hat{\phi}_A \in L^2(\mathscr{H}_\varepsilon\,;\ \hat{g})$, there exists a unique solution $\hat{\phi}_A$, continuous in τ with values in $L^2(\mathscr{H}_\tau\,;\ \hat{g})$, such that $\hat{\phi}_A\Big|_{\mathscr{H}_\varepsilon} = {}_\varepsilon\hat{\phi}_A$. Moreover, there exists $C > 0$ independent of ${}_\varepsilon\hat{\phi}_A$ such that*

$$\left\| \hat{\phi}_A\Big|_{\Sigma_0} \right\|_{L^2(\Sigma_0\,;\ \hat{g})} \leq C \left\| {}_\varepsilon\hat{\phi}_A \right\|_{L^2(\mathscr{H}_\varepsilon\,;\ \hat{g})} .$$

Proof. This is a direct consequence of the charge conservation for Dirac's equation and the density of $\mathcal{C}_0^\infty(\mathscr{H}_\varepsilon)$ (denoting smooth spinor fields on $\mathscr{H}_\varepsilon$ whose support remains away from $\mathscr{I}^+$) in $L^2(\mathscr{H}_\varepsilon\,;\ \hat{g})$. □

Finally we conclude, using proposition C.1 and lemma C.1 :

Proposition C.2. *The family ${}_t\hat{\psi}_A$ converges in $L^2(\Sigma_0\,;\ \hat{g})$ and in $L^2(\widehat{\mathscr{M}})$ towards a solution $\hat{\phi}_A$ of equation (16). This solution is continuous in τ with values in $L^2(\mathscr{H}_\tau\,;\ \hat{g})$ (also continuous in t with values in $L^2(\Sigma_t\,;\ \hat{g})$) and it satisfies*

$$\mathfrak{T}^+ \left(\hat{\phi}_A \Big|_{\Sigma_0} \right) = \hat{\phi}^+_{\mathscr{I}}\,.$$

C.3 Maxwell's equations

We consider on $\mathscr{I}^+$ a complete spinor field $\hat{\phi}_{AB}\Big|_{\mathscr{I}^+} = \hat{\phi}_{(AB)}\Big|_{\mathscr{I}^+}$, such that

- $\hat{\phi}_2\Big|_{\mathscr{I}^+} = \hat{\phi}^+_{\mathscr{I}}$,
- $\hat{\phi}_0\Big|_{\mathscr{I}^+}$ and $\hat{\phi}_1\Big|_{\mathscr{I}^+}$ are the solutions of the constraints on $\mathscr{I}^+$, determined by $\hat{\phi}_2\Big|_{\mathscr{I}^+}$ and the requirement that ${}_t\hat{\phi}_{AB}\Big|_{\mathscr{I}^+}$ vanishes in a neighbourhood of i^+.

The main difference between Dirac's and Maxwell's equations is that Maxwell's equations are overdetermined, whence the presence of constraints on spacelike hypersurfaces. These constraints need to be taken into accound in the definition of the data on Σ_t approaching the data on $\mathscr{I}^+$. We describe how the construction proposed for the Dirac case can be modified. Once these modifications are made, the proof can be done in the same manner and we omit it.

The Maxwell system on the physical space-time, decomposed on the spin-frame is written as

$$\begin{aligned}
m^a\nabla^B_{A'}\phi_{AB} = n^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_0 - m^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_1 + (\mu-2\gamma)\,\phi_0 + 2\tau\phi_1 - \sigma\phi_2 &= 0\,, &(36)\\
n^a\nabla^B_{A'}\phi_{AB} = n^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_1 - m^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_2 - \nu\phi_0 + 2\mu\phi_1 + (\tau-2\beta)\,\phi_2 &= 0\,, &(37)\\
-l^a\nabla^B_{A'}\phi_{AB} = l^{\mathbf{a}}\nabla_{\mathbf{a}}\hat{\phi}_1 - \bar{m}^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_0 - (\pi-2\alpha)\,\phi_0 - 2\rho\phi_1 + \kappa\phi_2 &= 0\,, &(38)\\
-\bar{m}^a\nabla^B_{A'}\phi_{AB} = l^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_2 - \bar{m}^{\mathbf{a}}\nabla_{\mathbf{a}}\phi_1 + \lambda\phi_0 - 2\pi\phi_1 - (\rho-2\varepsilon)\,\phi_2 &= 0\,, &(39)
\end{aligned}$$

where the spin-coefficients are those of the physical space-time, defined using the spin-frame (o^A,ι^A). The constraint equation $T^a\nabla^B_{A'}\phi_{AB}=0$ is therefore

$$\begin{aligned}
\mathbb{D}\cdot\phi := (l^a-n^a)\,\nabla_a\phi_1 \quad &+ \quad m^a\nabla_a\phi_2 - \bar{m}^a\nabla_a\phi_0 + (\nu+2\alpha-\pi)\,\phi_0\\
&- \quad 2\,(\rho+\mu)\,\phi_1 + (\kappa+2\beta-\tau)\,\phi_2 = 0\,. \qquad (40)
\end{aligned}$$

We could be tempted to define the data on Σ_t by projecting $\hat{\phi}_{AB}\Big|_{\mathscr{I}^+}$ onto Σ_t along the congruence $\mathscr{C}$. However, the data thus obtained would not satisfy the constraints on Σ_t. Instead, we define ${}_t\hat{\phi}_{AB}\Big|_{\Sigma_t}$ as follows :

We first choose a smooth extension of the (perhaps truncated) Taylor series solution to the Maxwell equations with the given null datum on $\mathscr{I}^+$, and then project it onto the solution of the constraint equations. This projection is implemented by means of a projection operator $\mathbb{P} = 1 - \mathbb{D}^*(\mathbb{D}\mathbb{D}^*)^{-1}\mathbb{D}$ which is a bounded operator, see [30]. Being a bounded operator, the convergence in H^1 is preserved by this construction.

This construction provides data on Σ_t that approach the full spinor field $\hat{\phi}_{AB}\Big|_{\mathscr{I}^+}$ on $\mathscr{I}^+$ and guarantee a uniform H^1 control of the associated solutions ${}_t\hat{\phi}$ on $\widehat{\mathscr{M}_\varepsilon^+}$. The uniform L^2 estimate between $\mathscr{H}_\varepsilon$ and Σ_0 is then obtained using the constructions of appendix A near i^0.

Similarly to the Dirac case, once a first construction of a solution to the Goursat problem has been obtained, it can be simplified by dropping the requirement of a uniform H^1 control. The new family of solutions will converge in L^2 (and no longer weakly in H^1) towards the same solution. This simplified definition of data on Σ_t is once again the one adopted for the construction of wave operators.

Definition C.4. *For each $t > 0$, the operator $P_K^+(t)$ is defined as associating to $\hat{\phi}_{\mathscr{I}}^+$ the Maxwell field ${}_t\hat{\phi}_{AB}\Big|_{\Sigma_t}$ on Σ_t such that : ${}_t\hat{\phi}_0\Big|_{\Sigma_t}$ is identically zero, ${}_t\hat{\phi}_2\Big|_{\Sigma_t}$ is the projection of $\hat{\phi}_{\mathscr{I}}^+$ on Σ_t along $\mathscr{C}$ and ${}_t\hat{\phi}_1\Big|_{\Sigma_t}$ is determined as the solution of (40) that vanishes inside $\mathcal{O}^+$. The operator $P_K^-(t)$ is defined in a symmetrical manner in the past.*

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