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Scattering of massless Dirac fields by a Kerr black hole

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Abstract

For the massless Dirac equation outside a slow Kerr black hole, we prove asymptotic completeness. We introduce a new Newman-Penrose tetrad in which the expression of the equation contains no artificial long-range perturbations. The main technique used is then a Mourre estimate. The geometry near the horizon requires to apply a unitary transformation before we find ourselves in a situation where the generator of dilations is a good conjugate operator. The results are eventually re-interpreted geometrically as providing the solution to a Goursat problem on the Penrose compactified exterior.

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Contents

1 Introduction

Black holes are the cosmological objects in which the effects of gravity are the most extreme. In the 1960's and the 1970's, some striking phenomena related to black holes were discovered, among which the Hawking radiation and superradiance. A complete mathematical understanding of these phenomena is far from being achieved yet ; it requires a detailed study of the propagation and scattering properties of classical and quantum fields on black hole space-times.

The first and simplest solution of the Einstein vacuum equations describing a black hole is the Schwarzschild metric, discovered in 1916 by Karl Schwarzschild [53]. It represents an asymptotically flat space-time containing nothing but a static, spherically symmetric, uncharged black hole. The Kerr family of metrics, discovered in 1963 by Roy Patrick Kerr [35], is a set of solutions of the Einstein vacuum equations generalizing the Schwarzschild metric. A subset of this family, referred to as slow Kerr metrics, describes an asymptotically flat space-time containing nothing but an eternal, uncharged, rotating black hole. This provides the realistic model for the exterior of a black hole (all cosmological objects are in rotation).

The scattering properties of classical and quantum fields outside a Schwarzschild black hole have been thoroughly studied. The first results on the subject were obtained by J. Dimock in 1985 [18] and by J. Dimock and B. Kay in 1986 and 1987 [19, 20, 21] for classical and quantum scalar fields. This work was pushed further by A. Bachelot in the 1990's ; his important series of papers starts with scattering theories for classical fields, Maxwell in 1991 [2], Klein-Gordon in 1994 [3] and culminates with a rigourous mathematical description of the Hawking effect for a spherical gravitational collapse in 1997 [4], 1999 [5] and 2000 [6]. Meanwhile other authors contributed to the subject, such as J.-P. Nicolas in 1995 with a scattering theory for classical massless Dirac fields [44], W.M. Jin in 1998 with a construction of wave operators in the massive case [34] and F. Melnyk in 2003 who obtained a complete scattering for massive charged Dirac fields [39] and the Hawking effect for charged, massive spin 1/2 fields [40]. Note that in [6, 39, 40, 44], the cases of Reissner-Nordstrøm (charged) and de Sitter (with a cosmological horizon) black holes are also treated ; these geometries do not fundamentally change the analytic difficulties in the construction of classical or quantum scattering theories. All these works use trace class perturbation methods and therefore cannot be extended to the Kerr case because of the lack of symmetry of the geometry (see below). One paper using different techniques appeared in 1992, due to S. De Bièvre, P. Hislop and I.M. Sigal [14] : by means of a Mourre estimate, they study the wave equation on non compact Riemannian manifolds ; possible applications are therefore static situations, such as the Schwarzschild case, which they treat, but the Kerr geometry is not even stationary and the results cannot be applied. A complete scattering theory for the wave equation, on stationary, asymptotically flat space-times, was subsequently obtained by D. Häfner in 2001 using the Mourre theory [31]. The theory of resonances is well understood in the Schwarzschild geometry, thanks to works by A. Bachelot and A. Motet-Bachelot in 1993 [7] and A. Sá Barreto and M. Zworski in 1998 [52]. There is also a work on a non linear Klein-Gordon equation on the Schwarzschild metric (and other similar geometries) with partial scattering results obtained by conformal methods, due to J.-P. Nicolas in 1995 [43].

In the more realistic framework of Kerr black holes, the analysis of the scattering properties of fields is faced with three fundamental difficulties, not present in the Schwarzschild framework.

1. Lack of symmetry. The Kerr solutions possess only two commuting Killing vector fields. In the Boyer-Lindquist coordinate system (t, r, θ, φ), based on these Killing vector fields, they are interpreted as the time coordinate vector field $\partial/\partial t$ and the longitude coordinate vector field $\partial/\partial\varphi$. Kerr space-time therefore has cylindrical, but not spherical, spatial symmetry. This prevents a straightforward decomposition in spin-weighted spherical harmonics, that reduces the problem to the study of a $(1+1)$-dimensional evolution system with potential. The trace-class perturbation methods used in the Schwarzschild case are in consequence not applicable. Another effect of the lack of spherical symmetry is the presence of artificial long-range terms at infinity in the field equations. To get rid of these terms, it is necessary to have a deeper understanding of the geometry, and of the dynamics naturally associated with the conformal structure, than what is required in the Schwarzschild case.

2. The point of view of scattering theory is that of an observer static at infinity. Such an observer perceives the propagation of a field outside the black hole as an evolution on a cylindrical manifold $\Sigma \simeq \mathbb{R} \times \mathbb{S}^2$, with one asymptotically flat end corresponding to infinity and one exponentially large (i.e. asymptotically hyperbolic, see remark 3.3) end representing the horizon. In the absence of spherical symmetry, the exponentially large end is awkward for scattering theory, more particularly for the choice of a conjugate operator in the framework of Mourre theory. The generator of dilations, that is the usual conjugate operator, cannot be used here.

3. Kerr space-time is not stationary ; there exists no globally defined timelike Killing vector field outside the black hole. In particular, the vector $\partial/\partial t$ is spacelike in a toroidal region, called the ergosphere, surrounding the horizon. For field equations of integral spin, such as the wave equation, Klein-Gordon or Maxwell, this means that no positive definite conserved energy exists, which allows fields to extract energy from the ergosphere, a phenomenon referred to as superradiance. For field equations of half integral spin (Weyl, Dirac or Rarita-Schwinger), we have a conserved L^2 norm, there is no superradiance and the lack of stationarity is not in itself a serious difficulty. This conserved L^2 norm is usually interpreted as a conserved charge. It is the good conserved quantity to work with because the field energy, which is the quadratic form associated with the Hamiltonian operator, is not positive definite for these equations (see also remark 2.1 for the Dirac case).

Because of the geometric complexity of the Kerr metric and the three difficulties mentioned above, analytic studies of the propagation of fields outside a Kerr black hole are few. In particular the complete understanding of superradiance in terms of time-dependent scattering is a major open problem. S. Chandrasekhar's fundamental work [12] uses systematically the Newman-Penrose formalism to develop stationary scattering theories and describe superradiance in terms of transmission and reflexion coefficients. As for time-dependent scattering, to our knowledge, the only result in the Kerr framework is D. Häfner's paper [32] ; it is a proof of asymptotic completeness for the non superradiant modes of the Klein-Gordon equation. In this work, the first two difficulties are present, but the third is avoided by the restriction to non superradiant modes. Some analytic results have also been obtained outside the scope of scattering theory : the existence of smooth solutions for Dirac's and Maxwell's equations was shown on generic space-times by A. De Vries [16, 17] with application to the Kerr metric where the existence of superradiance for Maxwell and its absence for Dirac are obtained ; one of us (J.-P. Nicolas) has published a generic analytic study of the evolution of Dirac fields in Sobolev and weighted Sobolev spaces, with applications to the Kerr metric and its maximal analytic extension [46], as well as a work on a non linear Klein-Gordon equation, proving the well-posedness of the minimum regularity Cauchy problem and, by means of a Penrose compactification, the existence of smooth asymptotic profiles for smooth solutions [47] ; there is also a paper by F. Finster, N. Kamran, J. Smoller and S.-T. Yau [23] on the time decay of Dirac fields.

In this work we develop a complete scattering theory for massless Dirac fields outside a slow Kerr black hole ; this is, to our knowledge, the first complete scattering theory on the Kerr background. The choice of Dirac fields, with their conserved L^2 norm, has the advantage of avoiding the third difficulty. The spinorial aspect, however, requires to obtain a better understanding of the first two difficulties than what is necessary for the Klein-Gordon equation.

The paper is organized as follows :

- Section 2 is devoted to the presentation of the Kerr metric, of the Dirac equation on it and of our main results. We begin with a brief description of the Kerr metric, then we give the expression of Dirac's equation in the two-spinor formalism of R. Penrose and W. Rindler (see [50]). The Newman-Penrose formalism allows us to transform this intrinsic expression into a system of partial differential equations with respect to a coordinate basis. For this purpose, we choose Kinnersley's tetrad, which is the one commonly used. The resulting system contains artificial long-range terms. In order to get rid of these terms, we introduce a new tetrad closely related to the local rotation of space-time. The section ends with the statement of the main theorems of this work ; they express the existence and completeness of classical wave operators for two types of simplified dynamics : asymptotic profiles and Hamiltonians of Dirac type involving the Dirac operator on the 2-sphere.

Sections 3 to 7 contain the proofs of the theorems of section 2.

- In section 3, we define an abstract analytic framework, generalizing Dirac's equation outside a Kerr black hole, by retaining only the analytic features relevant to scattering theory. Then some simplified asymptotic comparison Hamiltonians are defined, for both asymptotic regions, in the general setting.

- Section 4 contains intermediate technical results necessary for the scattering theory.

- In section 5, after recalling the basic principles of Mourre theory, we prove the fundamental Mourre estimate. The generator of dilations cannot be used as conjugate operator because of the difficulty related to the asymptotic end corresponding to the horizon. However, it is possible to define a unitary transformation leading to a situation where it is a good conjugate operator. The correct conjugate operator is then defined by conjugating the generator of dilations by this unitary transformation ; it is similar to the operator introduced by R. Froese and P. Hislop [24], but the arguments used to prove the Mourre estimate are different (Froese and Hislop's argument is not adapted to Dirac's equation).

- Once the Mourre estimate is established, the asymptotic completeness follows by standard arguments described in section 6.

- Section 7 opens with a proof of the absence of eigenvalues for the Hamiltonian of the massless Dirac equation on the Kerr metric ; this is a straightforward consequence of Teukolski's separation of variables in the equation. Then, we construct asymptotic velocities using the asymptotic completeness results of section 6. Finally, the theorems of section 2 are obtained as consequences of this construction, the absence of eigenvalues and the results of section 6.

- Section 8 is a re-interpretation of the results of section 2 in geometrical terms. The inverse wave operators are understood as trace operators on smooth null hypersurfaces at the boundary of the Penrose compactification of the exterior of the black hole. The full scattering theory is thus realized as the solution of a Goursat problem on the compactified exterior, with null data specified on a union of two such smooth hypersurfaces, singular at the junction.

In the massive charged case on Kerr-Newman backgrounds for the classical equation, the full dynamics is a short-range perturbation of an intermediate spherically symmetric dynamics. This intermediate dynamics is a one dimensional Dirac equation with a long-range (at infinity) matrix-valued potential. This will require to introduce a Dollard modification in the wave operators. This case is currently under study. The purpose of the present paper is to solve the geometrical difficulties of the scattering of Dirac fields outside a rotating black hole. All such difficulties are already present in the case of massless Dirac fields on a Kerr background. In particular, the Mourre theory developed here should hold without modification in the charged massive case. Note however that the geometrical interpretation of section 8 is highly dependent on the massless, chargeless aspect. Indeed, in the massive, or charged case, the equation is no longer conformally invariant and the conformal constructions fail. The reverse problem, consisting of solving the Goursat problem on a compactified space-time in order to extract a scattering theory, is under study with a first work on asymptotically simple space-times [38]. The results of the present paper will be used in a subsequent work to develop a quantum scattering theory for the Dirac equation on the Kerr metric. It is at this quantum level that the effects of the non stationarity of space-time will appear (see remark 2.1).

Notations. Many of our equations will be expressed using the two-component spinor notations and abstract index formalism of R. Penrose and W. Rindler [50].

Abstract indices are denoted by light face latin letters, capital for spinor indices and lower case for tensor indices. Abstract indices are a notational device for keeping track of the nature of objects in the course of calculations, they do not imply any reference to a coordinate basis, all expressions and calculations involving them are perfectly intrinsic. For example, g_{ab} will refer to the space-time metric as an intrinsic symmetric tensor field of valence $\left[\begin{array}{c}0\\2\end{array}\right]$, i.e. a section of $\mathrm{T}^*\mathcal{M}\odot\mathrm{T}^*\mathcal{M}$ and g^{ab} will refer to the inverse metric as an intrinsic symmetric tensor field of valence $\left[\begin{array}{c}2\\0\end{array}\right]$, i.e. a section of $\mathrm{T}\mathcal{M}\odot\mathrm{T}\mathcal{M}$ (where $\odot$ denotes the symmetric tensor product, $\mathrm{T}\mathcal{M}$ the tangent bundle to our space-time manifold $\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ its cotangent bundle).

Concrete indices defining components in reference to a basis are represented by bold face latin letters. Concrete spinor indices, denoted by bold face capital latin letters, take their values in $\{0,1\}$ while concrete tensor indices, denoted by bold face lower case latin letters, take their values in $\{0,1,2,3\}$. Consider for example a basis of $\mathrm{T}\mathcal{M}$, that is a family of four smooth vector fields on $\mathcal{M}:\mathcal{B}=\{e_0,e_1,e_2,e_3\}$ such that at each point p of $\mathcal{M}$ the four vectors $e_0(p),e_1(p),e_2(p),e_3(p)$ are linearly independent, and the corresponding dual basis of $\mathrm{T}^*\mathcal{M}:\mathcal{B}^*=\left\{e^0,e^1,e^2,e^3\right\}$ such that $e^{\mathbf{a}}\left(e_{\mathbf{b}}\right)=\delta^{\mathbf{a}}_{\mathbf{b}}$, $\delta^{\mathbf{a}}_{\mathbf{b}}$ denoting the Kronecker symbol ; $g_{\mathbf{ab}}$ will refer to the components of the metric g_{ab} in the basis $\mathcal{B}:\ g_{\mathbf{ab}}=g(e_{\mathbf{a}},e_{\mathbf{b}})$ and $g^{\mathbf{ab}}$ will denote the components of the inverse metric g^{ab} in the dual basis $\mathcal{B}^*$, i.e. the 4×4 real symmetric matrices $(g_{\mathbf{ab}})$ and $\left(g^{\mathbf{ab}}\right)$ are the inverse of one another. In the abstract index formalism, the basis vectors $e_{\mathbf{a}}$, $\mathbf{a}=0,1,2,3$, are denoted ${e_{\mathbf{a}}}^a$ or ${g_{\mathbf{a}}}^a$. In a coordinate basis, the basis vectors $e_{\mathbf{a}}$ are coordinate vector fields and will also be denoted by $\partial_{\mathbf{a}}$ or $\frac{\partial}{\partial x^{\mathbf{a}}}$; the dual basis covectors $e^{\mathbf{a}}$ are coordinate 1-forms and will be denoted by $\mathrm{d}x^{\mathbf{a}}$.

We adopt Einstein's convention for the same index appearing twice, once up, once down, in the same term. For concrete indices, the sum is taken over all the values of the index. In the case of abstract indices, this signifies the contraction of the index, i.e. f_aV^a denotes the action of the 1-form f_a on the vector field V^a. The indexed 1-form $\mathrm{d}x^a\in\mathrm{T}^*\mathcal{M}\otimes\mathbb{S}^A\otimes\mathbb{S}^{A'}$ and the indexed vector $\partial_a\in\mathrm{T}\mathcal{M}\otimes\mathbb{S}_A\otimes\mathbb{S}_{A'}$ (see subsection 2.2 for the meaning of the notations $\mathbb{S}^A$, $\mathbb{S}^{A'}$, $\mathbb{S}_A$ and $\mathbb{S}_{A'}$) are used to suppress form and vector abstract indices : $\mathrm{d}x^a$ maps the 1-form ω_a as an indexed quantity to the same 1-form $\omega=\omega_a\mathrm{d}x^a$ with its index suppressed, ∂_a maps the vector field V^a to the same vector field $V=V^a\partial_a$ with its index suppressed.

For a manifold Y we denote by $C^\infty_b(Y)$ the set of all C^∞ functions on Y, that are bounded together with all their derivatives. We denote by $C_\infty(Y)$ the set of all continuous functions tending to zero at infinity.

2 The Kerr metric and Dirac's equation

2.1 The Kerr metric

Kerr's space-time is described in terms of Boyer-Lindquist coordinates as the manifold $\mathcal{M}=\mathbb{R}_t\times\mathbb{R}_r\times S^2_\omega$ equipped with the lorentzian metric

$$g=\left(1-\frac{2Mr}{\rho^2}\right)\mathrm{d}t^2+\frac{4aMr\sin^2\theta}{\rho^2}\,\mathrm{d}t\mathrm{d}\varphi-\frac{\rho^2}{\Delta}\mathrm{d}r^2-\rho^2\mathrm{d}\theta^2-\frac{\sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi^2, \tag{2.1}$$

$$\rho^2=r^2+a^2\cos^2\theta\,,\ \Delta=r^2-2Mr+a^2\,,$$

$$\sigma^2 = \left(r^2+a^2\right)\rho^2 + 2Mra^2\sin^2\theta = \left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta \,,$$

where M is the mass of the black hole and a its angular momentum per unit mass. If $|a|$ is not too large, $(\mathcal{M}, g)$ is an asymptotically flat universe containing nothing but an eternal, uncharged, rotating black hole. For no value of r is the sphere $\{r\} \times S^2_{\theta,\varphi}$ reduced to a point, which justifies the extension of the variable r to the whole real axis. The expression (2.1) of the Kerr metric has two types of singularities. The set of points $\{\rho^2 = 0\}$ (the equatorial ring $\{r = 0\,,\ \theta = \pi/2\}$ of the $\{r = 0\}$ sphere) is a true curvature singularity. The spheres where Δ vanishes, called horizons, are mere coordinate singularities. Using appropriate coordinate systems, they are understood as regular null hypersurfaces that can be crossed one way but would require speeds greater than that of light to be crossed the other way, hence their name : event horizons. The black hole is the part of our space-time lying beyond an event horizon. There are three types of Kerr space-times according to the number of horizons (which depends on the respective importance of M and a).

- Slow Kerr space-time for $0 < |a| < M$. Δ has two real roots

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,, \tag{2.2}$$

  so there are two horizons, the spheres $\{r = r_-\}$ and $\{r = r_+\}$, on either side of $\{r = M\}$. The case $a = 0$ reduces to Schwarzschild's space-time.

- Extreme Kerr space-time for $|a| = M$. M is then the double root of Δ and the sphere $\{r = M\}$ is the only horizon.

- Fast Kerr space-time for $|a| > M$. Δ has no real root and the space-time has no horizon. There is no black hole in this case ; the ring singularity is a naked singularity.

We only work with slow Kerr metrics ; they are usually considered as the generic description of a space-time containing simply a rotating uncharged black hole, since the extreme case is believed to be unstable. The two horizons separate $\mathcal{M}$ into three connected components called Boyer-Lindquist blocks : block I, denoted here $\mathcal{B}_I$, is the exterior of the black hole $\{r > r_+\}$; block II, $\{r_- < r < r_+\}$, is a dynamic region situated beyond the outer horizon and where the inertial frames are dragged towards the inner horizon ; block III, $\{r < r_-\}$, is the part of space-time located beyond the inner horizon, it contains the ring singularity and a time machine called Carter's time machine. No Boyer-Lindquist block is stationary, that is to say there exists no globally defined timelike Killing vector field on any given block. In particular, block I contains a toroidal region, called the ergosphere, surrounding the horizon,

$$\mathcal{E} = \left\{(t,r,\theta,\varphi)\,;\ r_+ < r < M + \sqrt{M^2 - a^2\cos^2\theta}\right\}\,,$$

where the vector $\partial/\partial t$ is spacelike.

An important feature of Kerr's space-time is that it has Petrov type D (see translation of Petrov's original paper [51], or standard general relativity textbooks, or [48]). This means that the Weyl tensor has two double roots at each point. These roots, referred to as the principal null directions of the Weyl tensor, are given by the two vector fields

$$V^\pm = \frac{\left(r^2+a^2\right)}{\Delta}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\varphi}\,. \tag{2.3}$$

Since V^+ and V^- are (twice) repeated null directions of the Weyl tensor, by the Goldberg-Sachs theorem (see for example [48]) their integral curves define geodesic shear-free null congruences.

We shall refer to the integral curves of V^+ (resp. V^-) as the outgoing (resp. incoming) principal null geodesics.

Since the quantities ρ^2 and σ^2 are positive on block I (in fact ρ^2 is positive on the whole space-time, but σ^2 is negative in the time machine in block III), we denote $\rho = \sqrt{\rho^2}$ and $\sigma = \sqrt{\sigma^2}$.

Our purpose in this paper is to describe the scattering of linear massless Dirac fields by a slow Kerr black hole from the point of view of an observer static at infinity. For such observers, the exterior of the black hole is the only visible part of space-time. Besides, their perception of time is well described by the time function t of the Boyer-Lindquist coordinates. The horizon will therefore appear to them as a singularity of the metric (for more details on the nature of this singularity, see for example [46] or [47]). One may tend to think that t is simply a bad choice of time coordinate since it makes a regular part of space-time appear as singular. However, our choice of observer is natural in that it is a good description of a distant observer (typically, a telescope on earth aimed in the direction of a black hole) and the choice of time coordinate describes the experience of such observers. Hence, we work on $\mathcal{B}_I = \mathbb{R}_t \times]r_+, +\infty[\times S^2_{\theta,\varphi}$ equipped with the metric (2.1) and we shall consider Dirac's equation as an evolution equation with respect to t. We denote Σ the generic spacelike slice : $\Sigma =]r_+, +\infty[\times S^2_{\theta,\varphi}$ and $\Sigma_t = \{t\} \times \Sigma$.

2.2 Dirac's and Weyl's equations in the Newman-Penrose formalism

The function t of Boyer-Lindquist coordinates is a globally defined time function on block I, i.e. its gradient $\nabla^a t$,

$$\nabla^a t = g^{ab} \nabla_b t \ , \quad \nabla_a t \mathrm{d}x^a = \mathrm{d}t \, ,$$

is a smooth, timelike, non vanishing vector field on block I (in spite of the fact that in Boyer-Lindquist coordinates $\partial/\partial t$ is not everywhere timelike). The time orientation of block I is defined by t, i.e. a timelike or null vector field is said to be future oriented if t is increasing along its integral lines. The foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ by the level hypersurfaces $\Sigma_t = \{t\} \times \Sigma$ of the function t, is a foliation of block I by Cauchy hypersurfaces. Block I is therefore globally hyperbolic. In dimension 4, this entails the existence of a spin-structure (see R.P. Geroch [27, 28, 29] and E. Stiefel [54]). We denote by $\mathbb{S}$ (or $\mathbb{S}^A$ in the abstract index formalism) the spin bundle over $\mathcal{B}_I$ and $\bar{\mathbb{S}}$ (or $\mathbb{S}^{A'}$) the same bundle with the complex structure replaced by its opposite. The dual bundles $\mathbb{S}^*$ and $\bar{\mathbb{S}}^*$ will be denoted respectively $\mathbb{S}_A$ and $\mathbb{S}_{A'}$. The complexified tangent bundle to $\mathcal{B}_I$ is recovered as the tensor product of $\mathbb{S}$ and $\bar{\mathbb{S}}$, i.e.

$$T\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S} \otimes \bar{\mathbb{S}} \text{ or } T^a\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}^A \otimes \mathbb{S}^{A'}$$

and similarly

$$T^*\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}^* \otimes \bar{\mathbb{S}}^* \text{ or } T_a\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}_A \otimes \mathbb{S}_{A'} \, .$$

An abstract tensor index a is thus understood as an unprimed spinor index A and a primed spinor index A' clumped together : $a = AA'$.

The spin bundle $\mathbb{S}$ is equipped with a canonical symplectic form, ε_{AB}, referred to as the Levi-Civita symbol. It is used to raise and lower spinor indices, but due to its skew-symmetry, the order is important :

$$\varepsilon^{AB}\kappa_B = \kappa^A \ , \quad \kappa^A \varepsilon_{AB} = \kappa_B \, .$$

The complex conjugate $\overline{\varepsilon_{AB}} = \bar{\varepsilon}_{A'B'}$, simply denoted $\varepsilon_{A'B'}$, plays a similar role on $\bar{\mathbb{S}}$. These symplectic structures are compatible with the metric, more precisely

$$g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'} \, .$$

The Dirac equation finds its simplest expression in terms of two-component spinors (sections of the bundles $\mathbb{S}^A$, $\mathbb{S}_A$, $\mathbb{S}^{A'}$ or $\mathbb{S}_{A'}$) :

$$\left\{ \begin{array}{l} \nabla^{AA'}\phi_A = \mu\chi^{A'}, \\ \nabla^{AA'}\chi_{A'} = \mu\phi^A\,,\ \mu = \frac{m}{\sqrt{2}}\,, \end{array} \right. \tag{2.4}$$

where $m \geq 0$ is the mass of the field. In the massless case, equation (2.4) reduces to the Weyl anti-neutrino equation

$$\nabla^{AA'}\phi_A = 0\,, \tag{2.5}$$

since the equation on χ (the Weyl neutrino equation),

$$\nabla^{AA'}\chi_{A'} = 0\,,$$

is the complex conjugate of the anti-neutrino equation

$$\nabla^{AA'}\bar{\chi}_A = 0\,.$$

Equation (2.5) is the object of this paper ; we shall refer to it as the Weyl equation.

The full Dirac equation (2.4) possesses a conserved current (see for example [46]) on general curved space-times, defined by the future oriented non-spacelike vector field, sum of two future oriented null vector fields :

$$V^a = \phi^A\bar{\phi}^{A'} + \bar{\chi}^A\chi^{A'}\,.$$

This implies that the total charge outside the black hole

$$C(t) = \int_{\Sigma_t} V_a T^a \mathrm{dVol} = \int_{\Sigma_t} \left(\phi_A\bar{\phi}_{A'} + \bar{\chi}_A\chi_{A'}\right) T^{AA'}\mathrm{dVol}\,, \tag{2.6}$$

is constant throughout time, where T^a is the future oriented normal vector field to Σ_t, normalized for convenience so that $T_aT^a = 2$, and dVol is the volume form induced on Σ_t by the Kerr metric, i.e.

$$\mathrm{dVol} = \sqrt{\frac{\sigma^2\rho^2}{\Delta}}\,\mathrm{d}r\mathrm{d}\omega\,. \tag{2.7}$$

The quantity $C(t)$ defines a norm for $(\phi_A, \chi_{A'})$ on Σ_t (in fact the natural L^2 norm, see for example [46]). This will be explained in more details in section 2.5.

Remark 2.1. *Thanks to this charge conservation, the non-stationarity of space-time is not seen as a difficulty for the scattering theory of classical Dirac fields. The effects, however, do appear at the level of the physical interpretation. Let us consider the so-called Klein paradox as a toy model to explain how they can be seen :*

$$i\partial_t\psi = (\alpha D_r + \beta m + V)\,\psi\,,$$

with

$$V \in \mathcal{C}_b^\infty(\mathbb{R})\ ,\quad \lim_{r\to-\infty} V(r) = 0\ ,\quad \lim_{r\to+\infty} V(r) = U > 0\,.$$

If $U > 2m$, then a particle whose energy is between m and $-m+U$ will propagate near $-\infty$ as an electron and near $+\infty$ as a positron.

It is therefore natural to ask whether there is creation of particles in this situation, i.e. whether eternal rotating black holes create particles. Most physicists claim that there is in fact creation of particles (see for example [13]), but the mathematical proof is still missing. It *is clear that such a mathematical proof can only be given in a second quantized, many particle framework, and it would require the use of the classical scattering results proved in this paper. The Klein paradox has been studied from a mathematical point of view by Bongaarts and Ruijsenaars [9, 10] ; they show that the classical scattering matrix cannot be implemented as a unitary operator in the Fock space of the free fields.*

Using the Newman-Penrose formalism, equation (2.4) can be expressed as a system of partial differential equations with respect to a coordinate basis. This formalism is based on the choice of a null tetrad, i.e. a set of four vector fields l^a, n^a, m^a and $\bar{m}^a$, the first two being real and future oriented, $\bar{m}^a$ being the complex conjugate of m^a, such that all four vector fields are null and m^a is orthogonal to l^a and n^a, that is to say

$$l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0\,. \tag{2.8}$$

The tetrad is said to be normalized if in addition

$$l_a n^a = 1\,,\ m_a \bar{m}^a = -1\,. \tag{2.9}$$

Such a null tetrad defines at each point a basis of the complexified tangent space to our manifold, in other words, the tetrad is a global section of the complexified principal bundle. The vectors l^a and n^a describe "dynamic" or scattering directions, i.e. directions along which light rays may escape towards infinity (or more generally asymptotic regions corresponding to scattering channels). The vector m^a tends to have, at least spatially, bounded integral curves, typically m^a and $\bar{m}^a$ generate rotations.

The principle of the Newman-Penrose formalism is to decompose the covariant derivative into directional covariant derivatives along the frame vectors. To each directional derivative corresponds a standard symbol :

$$D = l^a \nabla_a\,,\ D' = n^a \nabla_a\,,\ \delta = m^a \nabla_a\,,\ \delta' = \bar{m}^a \nabla_a\,.$$

The connection coefficients (first order derivatives of the metric) can be organized into combinations involving only derivatives of frame vectors along frame vectors. These combinations are referred to as spin coefficients. For a normalized tetrad, there are twelve spin coefficients defined as follows (see R. Penrose & W. Rindler [50], Vol 1, p. 226-228)

$$\kappa = m^a D l_a\,,\ \tilde{\rho} = m^a \delta' l_a\,,\ \tilde{\sigma} = m^a \delta l_a\,,\ \tau = m^a D' l_a\,, \tag{2.10}$$

$$\varepsilon = \tfrac{1}{2}\left(n^a D l_a + m^a D \bar{m}_a\right)\,,\ \alpha = \tfrac{1}{2}\left(n^a \delta' l_a + m^a \delta' \bar{m}_a\right) \tag{2.11}$$

$$\beta = \tfrac{1}{2}\left(n^a \delta l_a + m^a \delta \bar{m}_a\right)\,,\ \gamma = \tfrac{1}{2}\left(n^a D' l_a + m^a D' \bar{m}_a\right)\,, \tag{2.12}$$

$$\pi = -\bar{m}^a D n^a\,,\ \lambda = -\bar{m}^a \delta' n^a\,,\ \mu = -\bar{m}^a \delta n^a\,,\ \nu = -\bar{m}^a D' n^a\,, \tag{2.13}$$

where we have denoted by $\tilde{\rho}$ and $\tilde{\sigma}$ the spin coefficients usually denoted ρ and σ, in order to avoid confusion with the functions $\rho = \sqrt{\rho^2}$ and $\sigma = \sqrt{\sigma^2}$ appearing in the expression (2.1) of the Kerr metric. The spin coefficients can also be expressed in terms of the Ricci rotation coefficients $\gamma_{(a)(b)(c)}$ (see for example S. Chandrasekhar [12]). For this definition, the frame vectors are denoted by

$$l^a = {e_{(1)}}^a\,,\ n^a = {e_{(2)}}^a\,,\ m^a = {e_{(3)}}^a\,,\ \bar{m}^a = {e_{(4)}}^a\,,$$

the dual 1-forms by

$$l_a = {e^{(1)}}_a\,,\ n_a = {e^{(2)}}_a\,,\ m_a = {e^{(3)}}_a\,,\ \bar{m}_a = {e^{(4)}}_a\,,$$

and the components of tensors with respect to this frame and co-frame are denoted by light-face latin indices within brackets, e.g. :

$$R^{(a)}{}_{(b)(c)(d)} = R^a{}_{bcd}\, e^{(a)}{}_a\, e_{(b)}{}^b\, e_{(c)}{}^c\, e_{(d)}{}^d\,.$$

The Ricci rotation coefficients are defined by

$$\gamma_{(a)(b)(c)} = \frac{1}{2}\left[\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}\right] ,$$

$$\lambda_{(a)(b)(c)} = \left[\frac{\partial}{\partial x^j} e_{(b)i} - \frac{\partial}{\partial x^i} e_{(b)j}\right] e_{(a)}{}^i e_{(c)}{}^j$$

and the expression of the spin-coefficients in terms of the $\gamma_{(a)(b)(c)}$ is :

$$\kappa = \gamma_{(3)(1)(1)}\,,\ \tilde{\rho} = \gamma_{(3)(1)(4)}\,,\ \varepsilon = \tfrac{1}{2}\left(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}\right) , \tag{2.14}$$

$$\tilde{\sigma} = \gamma_{(3)(1)(3)}\,,\ \mu = \gamma_{(2)(4)(3)}\,,\ \gamma = \tfrac{1}{2}\left(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\right) , \tag{2.15}$$

$$\lambda = \gamma_{(2)(4)(4)}\,,\ \tau = \gamma_{(3)(1)(2)}\,,\ \alpha = \tfrac{1}{2}\left(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\right) , \tag{2.16}$$

$$\nu = \gamma_{(2)(4)(2)}\,,\ \pi = \gamma_{(2)(4)(1)}\,,\ \beta = \tfrac{1}{2}\left(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\right) . \tag{2.17}$$

We can now express equation (2.4) as a system of partial differential equations, involving partial derivatives along the frame vectors ; this system acts on the components of ϕ_A and $\chi_{A'}$ in a unitary spin-frame (o^A, ι^A), defined uniquely up to an overall sign factor by the requirements that

$$o^A\bar{o}^{A'} = l^a\,,\ \iota^A\bar{\iota}^{A'} = n^a\,,\ o^A\bar{\iota}^{A'} = m^a\,,\ \iota^A\bar{o}^{A'} = \bar{m}^a\,,\ o_A\iota^A = 1\,. \tag{2.18}$$

We denote by ϕ_0 and ϕ_1 the components of ϕ_A in (o^A, ι^A), and $\chi_{0'}$ and $\chi_{1'}$ the components of $\chi_{A'}$ in $(\bar{o}^{A'}, \bar{\iota}^{A'})$:

$$\phi_0 = \phi_A o^A\ ,\quad \phi_1 = \phi_A \iota^A\ ,\quad \chi_{0'} = \chi_{A'}\bar{o}^{A'}\ ,\quad \chi_{1'} = \chi_{A'}\bar{\iota}^{A'}\,.$$

Dirac's equation then takes the form (see for example [12])

$$\left\{\begin{array}{l} n^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_0 - m^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_1 + (\mu-\gamma)\phi_0 + (\tau-\beta)\phi_1 = \frac{m}{\sqrt{2}}\chi_{1'}\,, \\[2ex] l^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_1 - \bar{m}^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_0 + (\alpha-\pi)\phi_0 + (\varepsilon-\tilde{\rho})\phi_1 = -\frac{m}{\sqrt{2}}\chi_{0'}\,, \\[2ex] n^{\mathbf{a}}\partial_{\mathbf{a}}\,\chi_{0'} - \bar{m}^{\mathbf{a}}\partial_{\mathbf{a}}\,\chi_{1'} + (\bar{\mu}-\bar{\gamma})\chi_{0'} + (\bar{\tau}-\bar{\beta})\chi_{1'} = \frac{m}{\sqrt{2}}\phi_1\,, \\[2ex] l^{\mathbf{a}}\partial_{\mathbf{a}}\,\chi_{1'} - m^{\mathbf{a}}\partial_{\mathbf{a}}\,\chi_{0'} + (\bar{\alpha}-\bar{\pi})\chi_{0'} + (\bar{\varepsilon}-\bar{\tilde{\rho}})\chi_{1'} = -\frac{m}{\sqrt{2}}\phi_0 \end{array}\right.$$

and the Weyl equation is simply :

$$\left\{\begin{array}{l} n^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_0 - m^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_1 + (\mu-\gamma)\phi_0 + (\tau-\beta)\phi_1 = 0\,, \\[2ex] l^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_1 - \bar{m}^{\mathbf{a}}\partial_{\mathbf{a}}\,\phi_0 + (\alpha-\pi)\phi_0 + (\varepsilon-\tilde{\rho})\phi_1 = 0\,. \end{array}\right. \tag{2.19}$$

2.3 A choice of null tetrad and the calculation of the spin coefficients

The description of Kerr's space-time in the framework of the Newman-Penrose formalism has been used before by S.A. Teukolski [56] and W.G. Unruh [57] to calculate the expression of the massless Dirac equation in Boyer-Lindquist coordinates (note that Unruh, although his calculations used the Newman-Penrose formalism, described his results in terms of Dirac matrices), and subsequently by S. Chandrasekhar for the full Dirac equation (see [11] for the original work, but also [12]). The tetrad used in all these references is due to Kinnersley [36]. It is naturally inherited from the type D structure. The two real null vectors are chosen along the principal null directions V^+ and V^- :

$$l^a \frac{\partial}{\partial x^a} = \lambda V^+ \,,\; n^a \frac{\partial}{\partial x^a} = \mu V^- \,,$$

the normalization condition $l_a n^a = 1$ then gives

$$\lambda \, \mu \, g \left(V^+ , V^- \right) = 1 \,,$$

whence, after calculation,

$$\lambda \, \mu \, \frac{2\rho^2}{\Delta} = 1 \,.$$

Kinnersley's choice was simply to take $\lambda = 1$. Once the directions of l^a and n^a are chosen, the complex vector fields are uniquely determined, modulo a phase factor $e^{i\theta}$, by (2.8) and (2.9). This gives Kinnersley's tetrad, which we denote L^a, N^a, m^a, $\bar{m}^a$:

$$L^a \frac{\partial}{\partial x^a} = \frac{1}{\Delta} \left[\left(r^2 + a^2 \right) \frac{\partial}{\partial t} + \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right] \,, \tag{2.20}$$

$$N^a \frac{\partial}{\partial x^a} = \frac{1}{2\rho^2} \left[\left(r^2 + a^2 \right) \frac{\partial}{\partial t} - \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right] \,, \tag{2.21}$$

$$m^a \frac{\partial}{\partial x^a} = \frac{1}{p\sqrt{2}} \left[ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right] \,, \tag{2.22}$$

$$\bar{m}^a \frac{\partial}{\partial x^a} = \frac{1}{\overline{p}\sqrt{2}} \left[-ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} - \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right] \,, \tag{2.23}$$

where

$$p = r + ia\cos\theta \,.$$

In this tetrad, the real null vectors L^a and N^a have very different behaviours near the horizon because $1/\Delta$ blows up there while $1/(2\rho^2)$ remains bounded. The consequence will be that the two components ϕ_0 and ϕ_1 of the spinor ϕ, solution to the massless Dirac equation, will not be on an equal footing near the horizon. This would break the time symmetry of our scattering construction (the components would need to be rescaled near the horizon in different manners for future and past scattering data). We prefer to modify this tetrad so that the real vectors behave similarly at the horizon. We define a normalized Newman-Penrose tetrad l^a, n^a, m^a, $\bar{m}^a$ by a simple modification of Kinnersley's tetrad : we choose

$$l^a \frac{\partial}{\partial x^a} = \lambda V^+ \,,\; n^a \frac{\partial}{\partial x^a} = \mu V^- \,,\; \lambda = \mu$$

and the vectors m^a and $\bar{m}^a$ remain unchanged. This gives us

$$l^a \frac{\partial}{\partial x^a} = \frac{1}{\sqrt{2\Delta\rho^2}} \left[\left(r^2 + a^2 \right) \frac{\partial}{\partial t} + \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right] \,, \tag{2.24}$$

$$n^a \frac{\partial}{\partial x^a} = \frac{1}{\sqrt{2\Delta\rho^2}} \left[\left(r^2+a^2\right) \frac{\partial}{\partial t} - \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right] , \tag{2.25}$$

$$m^a \frac{\partial}{\partial x^a} = \frac{1}{p\sqrt{2}} \left[ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right] , \tag{2.26}$$

$$\bar{m}^a \frac{\partial}{\partial x^a} = \frac{1}{\bar{p}\sqrt{2}} \left[-ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} - \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right] . \tag{2.27}$$

The dual tetrad of 1-forms is

$$l_a \mathrm{d}x^a = \sqrt{\frac{\Delta}{2\rho^2}} \left[\mathrm{d}t - \frac{\rho^2}{\Delta} \mathrm{d}r - a \sin^2\theta \, \mathrm{d}\varphi \right] , \tag{2.28}$$

$$n_a \mathrm{d}x^a = \sqrt{\frac{\Delta}{2\rho^2}} \left[\mathrm{d}t + \frac{\rho^2}{\Delta} \mathrm{d}r - a \sin^2\theta \, \mathrm{d}\varphi \right] , \tag{2.29}$$

$$m_a \mathrm{d}x^a = \frac{1}{p\sqrt{2}} \left[ia \sin\theta \, \mathrm{d}t - \rho^2 \mathrm{d}\theta - i \left(r^2+a^2\right) \sin\theta \, \mathrm{d}\varphi \right] , \tag{2.30}$$

$$\bar{m}_a \mathrm{d}x^a = \frac{1}{\bar{p}\sqrt{2}} \left[-ia \sin\theta \, \mathrm{d}t - \rho^2 \mathrm{d}\theta + i \left(r^2+a^2\right) \sin\theta \, \mathrm{d}\varphi \right] . \tag{2.31}$$

To the tetrad (2.24)-(2.27), we associate a spin-frame (o^A, ι^A) satisfying (2.18). The calculation of the spin-coefficients gives

$$\kappa = \tilde{\sigma} = \lambda = \nu = 0 , \tag{2.32}$$

$$\tilde{\rho} = \mu = -\frac{1}{\bar{p}} \sqrt{\frac{\Delta}{2\rho^2}} \, , \quad \tau = -\frac{ia \sin\theta}{\sqrt{2}\, \rho^2} \, , \quad \pi = \frac{ia \sin\theta}{\sqrt{2}\, \bar{p}^2} \, , \quad \varepsilon = \frac{(r-M)\rho^2 - r\Delta}{2\rho^2 \sqrt{2\Delta\rho^2}} \, , \tag{2.33}$$

$$\alpha = \frac{ia \sin\theta}{\sqrt{2}\, \bar{p}^2} - \frac{\cot\theta}{2\sqrt{2}\, \bar{p}} + \frac{a^2 \sin\theta \cos\theta}{2\rho^2 \sqrt{2}\, \bar{p}} \, , \quad \beta = \frac{\cot\theta}{2\sqrt{2}\, p} + \frac{a^2 \sin\theta \cos\theta}{2\rho^2 \sqrt{2}\, p} \, , \tag{2.34}$$

$$\gamma = \frac{(r-M)\rho^2 - r\Delta}{2\rho^2 \sqrt{2\Delta\rho^2}} - \sqrt{\frac{\Delta}{2\rho^2}} \frac{ia \cos\theta}{\rho^2} . \tag{2.35}$$

2.4 Calculation and first simplifications of Weyl's equation

Replacing in equation (2.19) the expressions of the frame vectors and of the spin-coefficients gives us the following explicit expression of the Weyl equation on the Kerr metric in terms of Boyer-Lindquist coordinates :

$$\frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} \partial_t \phi_0 - \sqrt{\frac{\Delta}{2\rho^2}} \partial_r \phi_0 + \frac{a}{\sqrt{2\Delta\rho^2}} \partial_\varphi \phi_0 - \frac{ia \sin\theta}{p\sqrt{2}} \partial_t \phi_1 - \frac{1}{p\sqrt{2}} \partial_\theta \phi_1 - \frac{i}{p\sqrt{2}\sin\theta} \partial_\varphi \phi_1$$
$$- \frac{(r-M)\rho^2 + r\Delta}{2\rho^2 \sqrt{2\Delta\rho^2}} \phi_0 - \left(\frac{\cot\theta}{2p\sqrt{2}} + \frac{ia \sin\theta}{\sqrt{2}\rho^2} + \frac{a^2 \sin\theta \cos\theta}{2\rho^2 \sqrt{2} p} \right) \phi_1 = 0 \, , \tag{2.36}$$

$$\frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} \partial_t \phi_1 + \sqrt{\frac{\Delta}{2\rho^2}} \partial_r \phi_1 + \frac{a}{\sqrt{2\Delta\rho^2}} \partial_\varphi \phi_1 + \frac{ia \sin\theta}{\bar{p}\sqrt{2}} \partial_t \phi_0 - \frac{1}{\bar{p}\sqrt{2}} \partial_\theta \phi_0 + \frac{i}{\bar{p}\sqrt{2}\sin\theta} \partial_\varphi \phi_0$$
$$+ \left(\frac{-\cot\theta}{2\bar{p}\sqrt{2}} + \frac{a^2 \sin\theta \cos\theta}{2\rho^2 \sqrt{2} \bar{p}} \right) \phi_0 + \left(\frac{(r-M)\rho^2 + r\Delta}{2\rho^2 \sqrt{2\Delta\rho^2}} + \frac{ia\Delta \cos\theta}{\rho^2 \sqrt{2\Delta\rho^2}} \right) \phi_1 = 0 \, . \tag{2.37}$$

Multiplying the spinor by the measure density associated with an adequate radial variable will get rid of some long-range potentials (the same technique was used in [44] for the Dirac equation on the Schwarzschild metric). Other long-range potentials do remain in the equation. The method used to eliminate them is described in subsection 2.5.1.

We introduce the "good" radial variable for time-dependent scattering : a variable r_* (already used in [12] and more recently [32]) such that the principal null geodesics have radial speed ± 1 with respect to this coordinate, i.e. such that

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{r^2+a^2}{\Delta} \,. \tag{2.38}$$

In the Schwarzschild case, r_* is the Regge-Wheeler coordinate $r + 2M\mathrm{Log}(r-2M)$. On the exterior of a slow Kerr black hole, we have

$$r_* = r + M\mathrm{Log}\left(r^2 - 2Mr + a^2\right) + \frac{2M^2}{\sqrt{M^2-a^2}}\mathrm{Log}\sqrt{\frac{r-r_+}{r-r_-}} + R_0 \,, \tag{2.39}$$

where $R_0 \in \mathbb{R}$ is arbitrary.

The measure dVol has the following expression with respect to the coordinates r_*, θ and φ :

$$\mathrm{dVol} = \sqrt{\frac{\Delta\,\sigma^2\rho^2}{\left(r^2+a^2\right)^2}}\,\mathrm{d}r_*\mathrm{d}\omega\,,\;\; \mathrm{d}\omega = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi\,.$$

We define the "density spinor"

$$\tilde{\phi}_A = \left(\frac{\Delta\,\sigma^2\rho^2}{\left(r^2+a^2\right)^2}\right)^{1/4}\;\phi_A\,. \tag{2.40}$$

The only differences between the equation satisfied by $\tilde{\phi}$ and (2.36)-(2.37) come from the terms

$$\begin{aligned}
\left(\frac{\Delta\,\sigma^2\rho^2}{\left(r^2+a^2\right)^2}\right)^{\frac14}\frac{\partial}{\partial r}\left(\frac{\Delta\,\sigma^2\rho^2}{\left(r^2+a^2\right)^2}\right)^{-\frac14} &= -\frac{(r-M)\rho^2 + r\Delta}{2\Delta\rho^2} \\
&\quad + \frac{\left((r-M)\left(r^2+a^2\right) - 2r\Delta\right)a^2\sin^2\theta}{2\sigma^2\left(r^2+a^2\right)}\,, \\
\left(\frac{\Delta\,\sigma^2\rho^2}{\left(r^2+a^2\right)^2}\right)^{\frac14}\frac{\partial}{\partial \theta}\left(\frac{\Delta\,\sigma^2\rho^2}{\left(r^2+a^2\right)^2}\right)^{-\frac14} &= \frac{a^2\sin\theta\cos\theta}{2\rho^2} + \frac{\Delta a^2\sin\theta\cos\theta}{2\sigma^2}\,.
\end{aligned}$$

Hence, the vector $\tilde{\Phi} = {}^t\left(\tilde{\phi}_0\,,\,\tilde{\phi}_1\right)$ satisfies the following system of equations

$$M_t\partial_t\tilde{\Phi} + M_r\partial_r\tilde{\Phi} + M_\theta\left(\partial_\theta + \frac12\cot\theta\right)\tilde{\Phi} + M_\varphi\frac{1}{\sin\theta}\partial_\varphi\tilde{\Phi} + P\tilde{\Phi} = 0\,, \tag{2.41}$$

$$M_t = \begin{pmatrix} \frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} & -\frac{ia\sin\theta}{p\sqrt{2}} \\ \frac{ia\sin\theta}{\bar{p}\sqrt{2}} & \frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} \end{pmatrix}\,,\;\; M_r = \sqrt{\frac{\Delta}{2\rho^2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

$$M_\theta = \begin{pmatrix} 0 & \frac{-1}{p\sqrt{2}} \\ \frac{-1}{\bar{p}\sqrt{2}} & 0 \end{pmatrix}\,,\;\; M_\varphi = \begin{pmatrix} \frac{a\sin\theta}{\sqrt{2\Delta\rho^2}} & \frac{-i}{p\sqrt{2}} \\ \frac{i}{\bar{p}\sqrt{2}} & \frac{a\sin\theta}{\sqrt{2\Delta\rho^2}} \end{pmatrix}\,,$$

$$P = \begin{pmatrix} -\frac{(r-M)\left(r^2+a^2\right)-2r\Delta}{2\sigma^2\left(r^2+a^2\right)}a^2\sin^2\theta\sqrt{\frac{\Delta}{2\rho^2}} & -\frac{ia\sin\theta}{\sqrt{2}\rho^2} - \frac{a^2\sin\theta\cos\theta}{\rho^2\sqrt{2}p} - \frac{\Delta a^2\sin\theta\cos\theta}{2\sigma^2 p\sqrt{2}} \\ -\frac{\Delta a^2\sin\theta\cos\theta}{2\sigma^2\sqrt{2}\bar{p}} & \frac{(r-M)\left(r^2+a^2\right)-2r\Delta}{2\sigma^2\left(r^2+a^2\right)}a^2\sin^2\theta\sqrt{\frac{\Delta}{2\rho^2}} + \frac{ia\sqrt{\Delta}\cos\theta}{\rho^2\sqrt{2\rho^2}} \end{pmatrix}\,.$$

2.5 Further simplifications of the equation

Multipling equation (2.41) by the matrix M_t^{-1}, we obtain the evolution system :

$$\partial_t \tilde{\Phi} + M_t^{-1} M_r \partial_r \tilde{\Phi} + M_t^{-1} M_{S^2}\, i\,\not{D}_{S^2} \tilde{\Phi} + \frac{a}{\sqrt{2\Delta\rho^2}} M_t^{-1} \partial_\varphi \tilde{\Phi} + M_t^{-1} P \tilde{\Phi} = 0\,,$$

$$M_{S^2} = \begin{pmatrix} \frac{-1}{p\sqrt{2}} & 0 \\ 0 & \frac{-1}{\bar{p}\sqrt{2}} \end{pmatrix},\ i\not{D}_{S^2} = \begin{pmatrix} 0 & \partial_\theta + \frac{1}{2}\cot\theta + \frac{i}{\sin\theta}\partial_\varphi \\ \partial_\theta + \frac{1}{2}\cot\theta - \frac{i}{\sin\theta}\partial_\varphi & 0 \end{pmatrix},$$

where the angular terms have been decomposed into the Dirac operator $\not{D}_{S^2}$ on the 2-sphere and a remainder involving only derivatives with respect to φ. A first advantage of this decomposition is that the operators $\not{D}_{S^2}$ and ∂_φ are regular on the whole 2-sphere ; the singularities appearing in $\cot\theta$ and $\frac{1}{\sin\theta}\partial_\varphi$ are thus understood as coordinate singularities. The other advantage is that the Dirac operator $\not{D}_{S^2}$ is spherically symmetric. Hence the lack of spherical symmetry (that is to say, the effects of rotation) is materialized first by the term in ∂_φ and second by the lack of symmetry in the matrix $M_t^{-1}M_{S^2}$. We see that the matrix $M_t^{-1}M_{S^2}$ behaves as r^{-1} near infinity, whereas $\frac{a}{\sqrt{2\Delta\rho^2}}M_t^{-1}$ falls off as r^{-2}. Thus, the term in ∂_φ can be understood as a long-range perturbation of the "principal" part involving $\not{D}_{S^2}$:

$$\frac{a}{\sqrt{2\Delta\rho^2}} M_t^{-1}\partial_\varphi = \frac{1}{r}\, O\left(M_t^{-1} M_{S^2}\, i\,\not{D}_{S^2}\right) \text{ as } r \to +\infty\,.$$

This however is of no matter since the term in ∂_φ will be treated as a potential (falling off as r^{-2} and therefore short-range) using the cylindrical symmetry. The real problem comes from the matrix $M_t^{-1}M_{S^2}$: we have

$$M_t^{-1}M_{S^2} = \begin{pmatrix} -\frac{(r^2+a^2)\sqrt{\Delta\rho^2}}{p\sigma^2} & -\frac{ia\Delta\sin\theta}{\sigma^2} \\ \frac{ia\Delta\sin\theta}{\sigma^2} & -\frac{(r^2+a^2)\sqrt{\Delta\rho^2}}{\bar{p}\sigma^2} \end{pmatrix} \simeq \frac{-1}{r}\,\mathrm{Id}_2 \text{ as } r\to+\infty$$

and there exists no "spherically symmetric" matrix M_0 (meaning that the coefficients of M_0 depend solely on r), falling off as r^{-1} at infinity, such that $M_t^{-1}M_{S^2} - M_0 = O\left(r^{-2-\varepsilon}\right)$ as $r \to +\infty$. This is obvious when we consider the fact that $ia\Delta\sin\theta/\sigma^2$ is zero on the axis and falls off as r^{-2} at infinity on the equator ; no spherically symmetric matrix can make up for such a behaviour. This shows that $M_t^{-1}M_{S^2}\not{D}_{S^2}$ is a long-range perturbation of $M_0\not{D}_{S^2}$ for any spherically symmetric matrix M_0 falling-off as r^{-1} at infinity.

Remark 2.2. *This problem is caused by the rotation of Kerr's space-time. The natural way of minimizing the effects of rotation in the expression of an equation is to choose means of describing the geometry that are as closely tied in with this rotation as possible. We have essentially two possibilities :*

- *Change coordinates to follow locally non rotating observers ; this induces time-dependent expressions for the metric and the equation, and therefore entails even more serious analytic difficulties.*
- *Find a new Newman-Penrose tetrad in some sense associated with locally non-rotating observers. The next paragraph is devoted to the construction of such a tetrad.*

The upshot will be that Kinnersley's tetrad, although it is systematically used in detailed studies of the Kerr geometry, including Chandrasekhar's stationary scattering theories, is not adapted to the point of view of time-dependent scattering.

Note that we have not quite used Kinnersley's tetrad, but a rescaled version of it. Using the exact Kinnersley tetrad would produce similar long-range terms at infinity.

2.5.1 A new Newman-Penrose tetrad adapted to the foliation

Given a Newman-Penrose tetrad l^a, n^a, m^a, $\bar{m}^a$, the vector field $l^a + n^a$ is timelike future-oriented as the sum of two future-oriented null vectors. Hence, to any Newman-Penrose tetrad, we can associate a preferred timelike future-pointing vector field (or observer), given by the sum of the two real frame vectors. Besides, the norm of such a vector field must always be $\sqrt{2}$ since

$$(l_a + n_a)(l^a + n^a) = 2\,.$$

Locally non-rotating observers are described by the future-oriented normal to the hypersurfaces Σ_t. We consider T^a the future-oriented vector field normal to the Σ_t and normalized so that $T^a T_a = 2$. It is given in Boyer-Lindquist coordinates by (see [46])

$$T^a \frac{\partial}{\partial x^a} = \sqrt{\frac{2\sigma^2}{\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{2aMr}{\sigma^2} \frac{\partial}{\partial\varphi} \right) .$$

We are looking for a Newman-Penrose tetrad $\mathbf{l}^a$, $\mathbf{n}^a$, $\mathbf{m}^a$, $\bar{\mathbf{m}}^a$, that follows the local rotation of space-time. The first natural idea is to impose

$$\mathbf{l}^a + \mathbf{n}^a = T^a\,. \tag{2.42}$$

This is exactly the notion of a tetrad adapted to the foliation as it was defined in [46]. The way we choose to construct such a tetrad is guided by our wish to minimize the apparent influence of rotation in our equation. Requiring (2.42) is a first step in this direction, but there are many possible choices of $\mathbf{l}^a$ and $\mathbf{n}^a$ compatible with (2.42). We single out a pair of null vectors that are not accelerated in the angular directions ; i.e. we choose $\mathbf{l}^a$ and $\mathbf{n}^a$ in the plane spanned by T^a and ∂_r. Requiring that $\mathbf{l}^a$ should be outgoing, $\mathbf{n}^a$ incoming, and a similar behaviour of the two vectors near the horizon, we obtain

$$\mathbf{l}^a \frac{\partial}{\partial x^a} = \frac{1}{2} T^a \frac{\partial}{\partial x^a} + \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r} = \frac{\sigma}{\sqrt{2\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{2aMr}{\sigma^2} \frac{\partial}{\partial\varphi} \right) + \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r}\,, \tag{2.43}$$

$$\mathbf{n}^a \frac{\partial}{\partial x^a} = \frac{1}{2} T^a \frac{\partial}{\partial x^a} - \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r} = \frac{\sigma}{\sqrt{2\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{2aMr}{\sigma^2} \frac{\partial}{\partial\varphi} \right) - \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r}\,. \tag{2.44}$$

The choice of $\mathbf{m}^a$ is now imposed, except for the freedom of a complex factor of modulus 1. The vector fields $T^a\partial_a$, ∂_r, ∂_θ and ∂_φ define an orthogonal frame everywhere (except on the axis where ∂_θ is singular) ; since $\mathbf{m}^a$ is orthogonal to $\mathbf{l}^a$ and $\mathbf{n}^a$ and since these two vectors span the plane $\langle T^a, \partial_r \rangle$, $\mathbf{m}^a$ must be tangent to the 2-sphere. This gives (choosing the complex factor so as to obtain the simplest expression)

$$\mathbf{m}^a \frac{\partial}{\partial x^a} = \frac{1}{\sqrt{2\rho^2}} \left(\frac{\partial}{\partial\theta} + \frac{\rho^2}{\sigma} \frac{i}{\sin\theta} \frac{\partial}{\partial\varphi} \right) , \tag{2.45}$$

$$\bar{\mathbf{m}}^a \frac{\partial}{\partial x^a} = \frac{1}{\sqrt{2\rho^2}} \left(\frac{\partial}{\partial\theta} - \frac{\rho^2}{\sigma} \frac{i}{\sin\theta} \frac{\partial}{\partial\varphi} \right) . \tag{2.46}$$

We now recall some well-known facts about Newman-Penrose tetrads and spin-frames, then we shall see how they can be significant to us.

Properties. *We consider a Newman-Penrose tetrad l^a, n^a, m^a, $\bar{m}^a$ and a unitary spin-frame (o^A, ι^A) related to the tetrad by (2.18). We also denote the frame spinors o^A and ι^A by*

$$o^A = \varepsilon_0{}^A \ , \quad \iota^A = \varepsilon_1{}^A$$

and the dual dyad $(-\iota_A, o_A)$ by

$$-\iota_A = \varepsilon_A{}^0 \ , \quad o_A = \varepsilon_A{}^1 \, .$$

To any vector field X^a, we can associate the matrix $X^{\mathbf{AA'}}$ of the components of its spinor form $X^{AA'}$. More precisely

$$X^{\mathbf{AA'}} = \begin{pmatrix} X^{00'} & X^{01'} \\ X^{10'} & X^{11'} \end{pmatrix} ,$$

and, writing for example the first component in details,

$$X^{00'} = \varepsilon_A{}^0 \bar{\varepsilon}_{A'}{}^{0'} X^{AA'} = \iota_A \bar{\iota}_{A'} X^{AA'} = n_a X^a \, .$$

With similar calculations for the three other components, we obtain

$$X^{\mathbf{AA'}} = \begin{pmatrix} n_a X^a & -\bar{m}_a X^a \\ -m_a X^a & l_a X^a \end{pmatrix} .$$

Denoting $\mathbf{X}$ the matrix $X^{\mathbf{AA'}}$, the quadratic form on $\mathbb{S}_A$ associated with X^a :

$$\phi_A \mapsto \phi_A \bar{\phi}_{A'} X^{AA'} ,$$

is expressed in terms of $\mathbf{X}$ and the vector

$$\Phi = \begin{pmatrix} \phi_0 = \varepsilon_0{}^A \phi_A \\ \phi_1 = \varepsilon_1{}^A \phi_A \end{pmatrix}$$

as follows

$$\phi_A \bar{\phi}_{A'} X^{AA'} = {}^t\Phi \mathbf{X} \bar{\Phi} = \left\langle \Phi \, , \ \overline{\mathbf{X}} \Phi \right\rangle_{\mathbb{C}^2} ,$$

where $< .,. >_{\mathbb{C}^2}$ denotes the standard scalar product on $\mathbb{C}^2$. In particular, we see that for the vector

$$Z^a := (l^a + n^a) ,$$

the matrix $Z^{\mathbf{AA'}}$ is the identity and therefore

$$\phi_A \bar{\phi}_{A'} Z^{AA'} = |\phi_0|^2 + |\phi_1|^2 \, . \tag{2.47}$$

The conserved charge (2.6) outside the black hole involves the quadratic form $\phi_A \bar{\phi}_{A'} T^{AA'}$ associated with the normal vector T^a. In the Newman-Penrose tetrad $\mathbf{l}^a$, $\mathbf{n}^a$, $\mathbf{m}^a$, $\bar{\mathbf{m}}^a$, the vector T^a is the sum of the two real frame vectors, whence the above quadratic form becomes simply $|\phi_0|^2 + |\phi_1|^2$. It follows that, with respect to this new tetrad, the conserved charged is exactly the L^2 norm of the vector Φ representing the spinor in the associated spin-frame.

2.5.2 The new expressions of Weyl's equation and the conserved quantity

Having found a Newman-Penrose tetrad meeting our requirements, we now wish to re-calculate Weyl's equation using this new tetrad. We have the possibility of computing the new values of the spin-coefficients using (2.10)-(2.13) or (2.14)-(2.17). This is excessively long and tedious and we prefer to follow a somewhat shorter path. Given any two normalized Newman-Penrose tetrads, there is a unique Lorentz transformation changing the first into the second. To this Lorentz transformation corresponds a unique (modulo sign) spin-transformation. All we have to do here is to calculate the Lorentz transformation L_a^b which transforms the tetrad (2.24)-(2.27) into (2.43)-(2.46), infer the spin transformation S_A^B such that $L_a^b = S_A^B \bar{S}_{A'}^{B'}$, then modify the components of the spinor $\tilde{\phi}_A$ using this spin-transformation. The equation satisfied by the modified components will be the form of Weyl's equation corresponding to the tetrad (2.43)-(2.46) where the unknown is the "density spinor" $\tilde{\phi}_A$ defined by (2.40).

First, in order to obtain the expression of the Lorentz transformation L_a^b, we express the frame-vectors (2.43)-(2.46) in terms of (2.24)-(2.27). We have

$$\begin{aligned}
\mathbf{l}^a &= \frac{\sigma_+}{2\sigma} l^a + \frac{\Delta a^2 \sin^2\theta}{2\sigma\sigma_+} n^a + \frac{\sqrt{\Delta} a \sin\theta}{2\sigma\rho} \left(ip\, m^a - i\bar{p}\, \bar{m}^a \right) , \\
\mathbf{n}^a &= \frac{\Delta a^2 \sin^2\theta}{2\sigma\sigma_+} l^a + \frac{\sigma_+}{2\sigma} n^a + \frac{\sqrt{\Delta} a \sin\theta}{2\sigma\rho} \left(ip\, m^a - i\bar{p}\, \bar{m}^a \right) , \\
\mathbf{m}^a &= -i \frac{\sqrt{\Delta} a \sin\theta}{2\sigma} \left(l^a + n^a \right) + \frac{p\sigma_+}{2\sigma\rho} m^a - \frac{\bar{p} \Delta a^2 \sin^2\theta}{2\sigma\sigma_+\rho} \bar{m}^a ,
\end{aligned}$$

where $\sigma_+ = \sigma + r^2 + a^2$. The matrix of the Lorentz transformation in the basis (2.24)-(2.27) is therefore

$$L_{(a)}^{(b)} = L_a^b \, e_{(a)}{}^a \, e^{(b)}{}_b = \begin{pmatrix} \frac{\sigma_+}{2\sigma} & \frac{\Delta a^2 \sin^2\theta}{2\sigma\sigma_+} & \frac{\sqrt{\Delta} a \sin\theta}{2\sigma\rho} ip & -\frac{\sqrt{\Delta} a \sin\theta}{2\sigma\rho} i\bar{p} \\ \frac{\Delta a^2 \sin^2\theta}{2\sigma\sigma_+} & \frac{\sigma_+}{2\sigma} & \frac{\sqrt{\Delta} a \sin\theta}{2\sigma\rho} ip & -\frac{\sqrt{\Delta} a \sin\theta}{2\sigma\rho} i\bar{p} \\ -i\frac{\sqrt{\Delta} a \sin\theta}{2\sigma} & -i\frac{\sqrt{\Delta} a \sin\theta}{2\sigma} & \frac{p\sigma_+}{2\sigma\rho} & -\frac{\bar{p}\Delta a^2 \sin^2\theta}{2\sigma\sigma_+\rho} \\ i\frac{\sqrt{\Delta} a \sin\theta}{2\sigma} & i\frac{\sqrt{\Delta} a \sin\theta}{2\sigma} & -\frac{p\Delta a^2 \sin^2\theta}{2\sigma\sigma_+\rho} & \frac{\bar{p}\sigma_+}{2\sigma\rho} \end{pmatrix} . \tag{2.48}$$

The matrix $S_{\mathbf{A}}^{\mathbf{B}}$ of the spin-transformation S_A^B in the spin-frame $(o^A,\, \iota^A)$ is uniquely determined, modulo sign, by $L_a^b = S_A^B \bar{S}_{A'}^{B'}$ and $\det\left(S_{\mathbf{A}}^{\mathbf{B}}\right) = 1$. The first condition can be expressed in terms of components as

$$L_{(a)}^{(b)} = \begin{pmatrix} \left|S_0^0\right|^2 & \left|S_0^1\right|^2 & S_0^0 \bar{S}_{0'}^{1'} & S_0^1 \bar{S}_{0'}^{0'} \\ \left|S_1^0\right|^2 & \left|S_1^1\right|^2 & S_1^0 \bar{S}_{1'}^{1'} & S_1^1 \bar{S}_{1'}^{0'} \\ S_0^0 \bar{S}_{1'}^{0'} & S_0^1 \bar{S}_{1'}^{1'} & S_0^0 \bar{S}_{1'}^{1'} & S_0^1 \bar{S}_{1'}^{0'} \\ S_1^0 \bar{S}_{0'}^{0'} & S_1^1 \bar{S}_{0'}^{1'} & S_1^0 \bar{S}_{0'}^{1'} & S_1^1 \bar{S}_{0'}^{0'} \end{pmatrix} . \tag{2.49}$$

Identifying (2.48) and (2.49) and imposing $\det\left(S_{\mathbf{A}}^{\mathbf{B}}\right) = 1$, we obtain

$$S_{\mathbf{A}}^{\mathbf{B}} = \begin{pmatrix} S_0^0 & S_0^1 \\ S_1^0 & S_1^1 \end{pmatrix} = \sqrt{\frac{p}{2\sigma\rho}} \begin{pmatrix} \sqrt{\sigma_+} & -\frac{\bar{p}}{\rho} \frac{ia \sin\theta \sqrt{\Delta}}{\sqrt{\sigma_+}} \\ \frac{ia \sin\theta \sqrt{\Delta}}{\sqrt{\sigma_+}} & \frac{\bar{p}}{\rho} \sqrt{\sigma_+} \end{pmatrix} =: \mathbf{U} , \tag{2.50}$$

where the square root of p is calculated using any given determination of the square root on the complex plane. The spin-transformation (2.50) transforms the spin-frame $(o^A,\, \iota^A)$ into a new spin-frame ($\mathbf{o}^A = S^A_B o^B$, $\boldsymbol{\iota}^A = S^A_B \iota^B$) such that $\mathbf{l}^a = \mathbf{o}^A \bar{\mathbf{o}}^{A'}$, $\mathbf{n}^a = \boldsymbol{\iota}^A \bar{\boldsymbol{\iota}}^{A'}$, $\mathbf{m}^a = \mathbf{o}^A \bar{\boldsymbol{\iota}}^{A'}$, $\bar{\mathbf{m}}^a = \boldsymbol{\iota}^A \bar{\mathbf{o}}^{A'}$. The components of the spinor $\tilde{\phi}_A$ in this spin-frame are given by

$$\psi_0 = \tilde{\phi}_A \mathbf{o}^A = \tilde{\phi}_A S^A_B o^B = S^A_0 \tilde{\phi}_A \,, \;\; \psi_1 = \tilde{\phi}_A \boldsymbol{\iota}^A = \tilde{\phi}_A S^A_B \iota^B = S^A_1 \tilde{\phi}_A \,,$$

that is to say

$$\Psi := \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \mathbf{U}\tilde{\Phi} \,, \;\; \tilde{\Phi} = \begin{pmatrix} \tilde{\phi}_0 \\ \tilde{\phi}_1 \end{pmatrix} = \left(\frac{\Delta\,\sigma^2 \rho^2}{(r^2+a^2)^2} \right)^{\frac{1}{4}} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} . \tag{2.51}$$

The equation satisfied by Ψ is

$$\tilde{M}_t \partial_t \Psi + \tilde{M}_r \partial_r \Psi + \tilde{M}_\theta \left(\partial_\theta + \frac{1}{2} \cot\theta \right) \Psi + \tilde{M}_\varphi \frac{1}{\sin\theta} \partial_\varphi \Psi + \tilde{P}\Psi = 0 \,, \tag{2.52}$$

where

$$\tilde{M}_t = \mathbf{U} M_t \mathbf{U}^{-1} = \begin{pmatrix} \frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} & -\frac{ia\sin\theta}{\sqrt{2\rho^2}} \\ \frac{ia\sin\theta}{\sqrt{2\rho^2}} & \frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} \end{pmatrix} ,$$

$$\tilde{M}_r = \mathbf{U} M_r \mathbf{U}^{-1} = \sqrt{\frac{\Delta}{2\rho^2}} \begin{pmatrix} -\frac{r^2+a^2}{\sigma} & -\frac{ia\sin\theta\sqrt{\Delta}}{\sigma} \\ -\frac{ia\sin\theta\sqrt{\Delta}}{\sigma} & \frac{r^2+a^2}{\sigma} \end{pmatrix} ,$$

$$\tilde{M}_\theta = \mathbf{U} M_\theta \mathbf{U}^{-1} = \frac{-1}{\sigma\sqrt{2}} \begin{pmatrix} \frac{-ia\sin\theta\sqrt{\Delta}}{\rho} & \frac{\sigma\sigma_+ + \Delta a^2 \sin^2\theta}{\rho\sigma_+} \\ \frac{\sigma\sigma_+ + \Delta a^2 \sin^2\theta}{\rho\sigma_+} & \frac{ia\sin\theta\sqrt{\Delta}}{\rho} \end{pmatrix} ,$$

$$\tilde{M}_\varphi = \mathbf{U} M_\varphi \mathbf{U}^{-1} = \begin{pmatrix} \frac{a\sin\theta}{\sqrt{2\Delta\rho^2}} & \frac{-i}{\sqrt{2}\rho} \\ \frac{i}{\sqrt{2}\rho} & \frac{a\sin\theta}{\sqrt{2\Delta\rho^2}} \end{pmatrix} ,$$

$$\tilde{P} = \mathbf{U} P \mathbf{U}^{-1} + \mathbf{U} M_\theta \left[\frac{\partial}{\partial\theta}, \mathbf{U}^{-1} \right] + \mathbf{U} M_r \left[\frac{\partial}{\partial r}, \mathbf{U}^{-1} \right]$$

$$\mathbf{U}^{-1} = \sqrt{\frac{\rho}{2\sigma p}} \begin{pmatrix} \sqrt{\sigma_+} & \frac{ia\sin\theta\sqrt{\Delta}}{\sqrt{\sigma_+}} \\ -\frac{\rho}{\bar{p}} \frac{ia\sin\theta\sqrt{\Delta}}{\sqrt{\sigma_+}} & \frac{\rho}{\bar{p}} \sqrt{\sigma_+} \end{pmatrix} ,$$

and the commutators $[\partial_\theta, \mathbf{U}^{-1}]$, $[\partial_r, \mathbf{U}^{-1}]$ are simply the partial derivatives of $\mathbf{U}^{-1}$ with respect to θ and r. Left-multiplying equation (2.52) by

$$\tilde{M}_t^{-1} = \frac{2\Delta\rho^2}{\sigma^2} \begin{pmatrix} \frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} & \frac{ia\sin\theta}{\sqrt{2\rho^2}} \\ \frac{-ia\sin\theta}{\sqrt{2\rho^2}} & \frac{r^2+a^2}{\sqrt{2\Delta\rho^2}} \end{pmatrix} ,$$

we get

$$\partial_t \Psi + \mathbf{M}_r \partial_r \Psi + \mathbf{M}_\theta \left(\partial_\theta + \frac{1}{2} \cot\theta \right) \Psi + \mathbf{M}_\varphi \frac{1}{\sin\theta} \partial_\varphi \Psi + \tilde{M}_t^{-1} \tilde{P} \Psi = 0 \,, \tag{2.53}$$

where

$$\mathbf{M}_r = \tilde{M}_t^{-1} \tilde{M}_r = \begin{pmatrix} \frac{-\Delta}{\sigma} & 0 \\ 0 & \frac{\Delta}{\sigma} \end{pmatrix} = \frac{\Delta}{\sigma} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\mathbf{M}_\theta = \tilde{M}_t^{-1} \tilde{M}_\theta = \begin{pmatrix} 0 & \frac{-\sqrt{\Delta}}{\sigma} \\ \frac{-\sqrt{\Delta}}{\sigma} & 0 \end{pmatrix} ,$$

$$\mathbf{M}_\varphi = \tilde{M}_t^{-1}\tilde{M}_\varphi = \begin{pmatrix} \frac{2Mra\sin\theta}{\sigma^2} & \frac{-i\sqrt{\Delta}\rho^2}{\sigma^2} \\ \frac{i\sqrt{\Delta}\rho^2}{\sigma^2} & \frac{2Mra\sin\theta}{\sigma^2} \end{pmatrix}.$$

We then modify equation (2.53) by isolating the Dirac operator $\not{D}_{S^2}$ on the 2-sphere S^2 from the rest of the angular terms :

$$\partial_t\Psi + A_r\partial_r\Psi + A_{S^2}i\not{D}_{S^2}\Psi + A_\varphi\partial_\varphi\Psi + B\Psi = 0\,, \tag{2.54}$$

$$A_r = \mathbf{M}_r = \frac{\Delta}{\sigma}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ A_{S^2} = \frac{-\sqrt{\Delta}}{\sigma}\mathrm{Id}_2\,,$$

$$A_\varphi = \begin{pmatrix} \frac{2Mra}{\sigma^2} & \frac{-i\sqrt{\Delta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma}-1\right) \\ \frac{i\sqrt{\Delta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma}-1\right) & \frac{2Mra}{\sigma^2} \end{pmatrix},\ B = \tilde{M}_t^{-1}\tilde{P}\,.$$

Remark 2.3. *The matrix A_{S^2} is now diagonal and furthermore $A_{S^2}\not{D}_{S^2}$ is a short-range perturbation of $A^0_{S^2}\not{D}_{S^2}$, where*

$$A^0_{S^2} = \frac{-\sqrt{\Delta}}{r^2+a^2}\mathrm{Id}_2\,.$$

Remark 2.4. *As was remarked at the end of the previous subsection, the conserved quantity takes a considerably simplified form with respect to the new tetrad, namely*

$$\begin{aligned}\int_{\Sigma_t} T^{AA'}\phi_A\bar{\phi}_{A'}\mathrm{dVol} &= \int_{\Sigma_t} T^{AA'}\phi_A\bar{\phi}_{A'}\sqrt{\frac{\sigma^2\rho^2}{\Delta}}\,\mathrm{d}r\mathrm{d}\omega \\ &= \int_{\Sigma_t} T^{AA'}\phi_A\bar{\phi}_{A'}\sqrt{\frac{\Delta\sigma^2\rho^2}{(r^2+a^2)^2}}\,\mathrm{d}r_*\mathrm{d}\omega \\ &= \int_{\Sigma_t} T^{AA'}\tilde{\phi}_A\bar{\tilde{\phi}}_{A'}\,\mathrm{d}r_*\mathrm{d}\omega \\ &= \int_{\Sigma_t} \langle\Psi,\Psi\rangle_{\mathbb{C}^2}\,\mathrm{d}r_*\mathrm{d}\omega\,. \end{aligned} \tag{2.55}$$

In the tetrad l^a, n^a, m^a, $\bar{m}^a$, the explicit expression of the quantity $T^{AA'}\phi_A\bar{\phi}_{A'}$ involves the matrix $\mathbf{T}$ of $T^{AA'}$ in the associated spin-frame, given by

$$\mathbf{T} = \overline{\mathbf{U}^*\mathbf{U}} = \begin{pmatrix} \frac{r^2+a^2}{\sqrt{\sigma^2}} & \frac{ia\sin\theta\sqrt{\Delta\rho^2}}{\bar{p}\sqrt{\sigma^2}} \\ \frac{-ia\sin\theta\sqrt{\Delta\rho^2}}{p\sqrt{\sigma^2}} & \frac{r^2+a^2}{\sqrt{\sigma^2}} \end{pmatrix}.$$

2.6 The main theorems for the Kerr framework

We start by re-expressing the form (2.54) of Weyl's equation in a manner which makes explicit the existence of two asymptotic regions : one corresponding to the horizon, the other to infinity. This is done by using the Regge-Wheeler-type coordinate r_*, defined in (2.38), instead of r. This coordinate r_*, as was remarked earlier, is chosen so that the principal null geodesics have radial speed ± 1. The consequence is that the horizon is now described as the asymptotic region $r_* \to -\infty$, sometimes referred to as "negative infinity". Equation (2.54) takes the new form

$$\partial_t\Psi = i\not{D}_K\Psi\,, \tag{2.56}$$

where $\not{D}_K$, the Hamiltonian for the Weyl equation on the Kerr metric, is given by

$$\not{D}_K = \frac{r^2+a^2}{\sigma}\gamma D_{r_*} + \frac{\sqrt{\Delta}}{\sigma}\not{D}_{S^2} - A_\varphi D_\varphi + iB\,,$$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,\ D_{r_*} = \frac{1}{i}\frac{\partial}{\partial r_*}\,,\ D_\varphi = \frac{1}{i}\frac{\partial}{\partial \varphi}\,.$$

This expression allows us to define asymptotic dynamics near the horizon and in the neighbourhood of infinity, corresponding to approximations of $\not{D}_K$ in these asymptotic regions. For our first construction of wave operators, we make the simplest choice of asymptotic dynamics : asymptotic profiles. In addition to being simple and intuitive, this has the major advantage of allowing an almost immediate geometrical interpretation of the scattering theory, as providing the solution to a non-trivial Goursat problem on the Penrose compactified block I. The details of this interpretation[1] are given in section 8.

As $r_* \to -\infty$, $\not{D}_K$ approaches

$$\mathbb{D}_H = \gamma D_{r_*} - \frac{2Mr_+a}{\left(r_+^2+a^2\right)^2}\,D_\varphi = \gamma D_{r_*} - \frac{a}{r_+^2+a^2}\,D_\varphi\,, \tag{2.57}$$

whereas in the neighbourhood of infinity, $\not{D}_K$ is close to

$$\mathbb{D}_\infty = \gamma D_{r_*}\,. \tag{2.58}$$

The asymptotic Hamiltonians are both self-adjoint on

$$\mathcal{H} = L^2\left(\left(\mathbb{R}\times S^2\,;\ \mathrm{d}r_*\mathrm{d}\omega\right)\,;\ \mathbb{C}^2\right) \tag{2.59}$$

and for $\Psi = {}^t(\psi_0,\psi_1) \in \mathcal{H}$,

$$\left(e^{it\mathbb{D}_H}\Psi\right)(r_*\,,\theta\,,\varphi) = \begin{pmatrix} \psi_0\left(r_*+t\,,\theta\,,\varphi-\frac{a}{r_+^2+a^2}\,t\right) \\ \psi_1\left(r_*-t\,,\theta\,,\varphi-\frac{a}{r_+^2+a^2}\,t\right)\end{pmatrix}\,,$$

$$\left(e^{it\mathbb{D}_\infty}\Psi\right)(r_*\,,\theta\,,\varphi) = \begin{pmatrix} \psi_0\left(r_*+t\,,\theta\,,\varphi\right) \\ \psi_1\left(r_*-t\,,\theta\,,\varphi\right)\end{pmatrix}\,.$$

The dynamics generated by $\mathbb{D}_H$ operates a radial translation at speed 1 with respect to r_* (towards $-\infty$ for the first component of Ψ and towards $+\infty$ for the second) as well as a rotation of fixed angular velocity $a/(r_+^2+a^2)$, i.e. the rotation speed of the horizon as perceived by an observer static at infinity. The operator $\mathbb{D}_\infty$ induces the same radial translation as $\mathbb{D}_H$ without the rotation. Both Hamiltonians have the same spaces of incoming (resp. outgoing) data :

$$\mathcal{H}^- = \{\Psi=(\psi_0,\psi_1)\in\mathcal{H}\,;\ \psi_1=0\}\ \left(\text{resp. } \mathcal{H}^+ = \{\Psi=(\psi_0,\psi_1)\in\mathcal{H}\,;\ \psi_0=0\}\right)\,.$$

Although the geometric interpretation is less relevant, it is also interesting to use Dirac-type operators, involving the full $\not{D}_{S^2}$ in their angular part, as comparison dynamics. We introduce

$$\not{D}_H = \gamma \mathrm{D}_{r_*} + e^{-\kappa_+|r_*|\theta_0(r_*)}\not{D}_S^2 - \frac{a}{r_+^2+a^2}\mathrm{D}_\varphi\,,\ \not{D}_\infty = \gamma\mathrm{D}_{r_*} + \frac{\theta_1(r_*)}{|r_*|}\not{D}_S^2\,, \tag{2.60}$$

[1] The constructions of section 8 will indeed be based on asymptotic profiles, but they will be slightly different from the ones used here, so as to make their geometric significance more obvious. The scattering results using the profiles of section 8 and the ones described in this section are equivalent

where κ_+, the surface gravity at the outer horizon, is given by

$$\kappa_+ = \frac{r_+ - r_-}{2\left(r_+^2 + a^2\right)} = \frac{\sqrt{M^2 - a^2}}{r_+^2 + a^2}\,, \tag{2.61}$$

$\theta_0, \theta_1 \in \mathcal{C}^\infty(\mathbb{R})$, θ_0 is zero in the neighbourhood of 0 and 1 far from the origin, and $\theta_1(x) = \mathbf{1}_{\mathbb{R}^+}(x)\theta_0(x)$. The choice of $\not{D}_H$ is related to an adequate choice of constant R_0 in the definition (2.39) of r_* (see remark 7.2). These two Hamiltonians are self-adjoint on $\mathcal{H}$.

Our first theorem establishes the existence of asymptotic velocities for all Hamiltonians $\not{D}_K$, $\mathbb{D}_H$, $\mathbb{D}_\infty$, $\not{D}_H$ and $\not{D}_\infty$. Then we give a first construction of wave operators using asymptotic profiles as comparison dynamics in theorem 2 and another construction in theorem 3 using $\not{D}_H$ and $\not{D}_\infty$ instead. We denote by the letters $\mathfrak{W}$ and W the wave operators associated with asymptotic profiles ; we use the letter Ω, in accordance with the notations of section 6, for the wave operators associated with $\not{D}_H$ and $\not{D}_\infty$. All these wave operators are defined using projections onto the positive and negative spectra of our asymptotic velocities.

Theorem 1 (Asymptotic velocities)**.** *1. The three Hamiltonians $\not{D}_H$, $\not{D}_\infty$ and $\not{D}_K$ are self-adjoint on $\mathcal{H}$ and their spectra are purely absolutely continuous ; in particular, their point spectra are empty.*

2. There exist bounded self-adjoint operators $P^\pm$, $P^\pm_H$, $P^\pm_\infty$ such that, for all $J \in \mathcal{C}_\infty(\mathbb{R})$:

$$J(P^\pm) = s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} J\left(\frac{r_*}{t}\right) e^{it\not{D}_K}\,, \tag{2.62}$$

$$J(P^\pm_H) = s - \lim_{t\to\pm\infty} e^{-it\not{D}_H} J\left(\frac{r_*}{t}\right) e^{it\not{D}_H}\,, \tag{2.63}$$

$$J(P^\pm_\infty) = s - \lim_{t\to\pm\infty} e^{-it\not{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\not{D}_\infty}\,, \tag{2.64}$$

$$J(\mp\gamma) = s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_H} = s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}\,. \tag{2.65}$$

In addition, we have $P^- = -P^+$, $P^-_H = -P^+_H$, $P^-_\infty = -P^+_\infty$,

$$\sigma(P^+) = \sigma(P^+_H) = \sigma(P^+_\infty) = \{-1, 1\}\,.$$

Remark 2.5. *Note that*

$$\mathbf{1}_{\mathbb{R}^\pm}(-\gamma) = P_{\mathcal{H}^\pm}\,,$$

where $P_{\mathcal{H}^\pm}$ is the projector from $\mathcal{H}$ onto $\mathcal{H}^\pm$.

Theorem 2 (Asymptotic profiles)**.** *1. The classical wave operators defined by the strong limits*

$$\mathfrak{W}^\pm_H := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_H} P_{\mathcal{H}^\mp}\,, \tag{2.66}$$

$$\mathfrak{W}^\pm_\infty := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_\infty} P_{\mathcal{H}^\pm}\,, \tag{2.67}$$

$$\tilde{\mathfrak{W}}^\pm_H := s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^\pm)\,, \tag{2.68}$$

$$\tilde{\mathfrak{W}}^\pm_\infty := s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^\pm)\,, \tag{2.69}$$

exist and satisfy

$$\tilde{\mathfrak{W}}^\pm_H = \left(\mathfrak{W}^\pm_H\right)^*\,,\ \tilde{\mathfrak{W}}^\pm_\infty = \left(\mathfrak{W}^\pm_\infty\right)^*\,,$$

$$\tilde{\mathfrak{W}}^\pm_H \mathfrak{W}^\pm_H + \tilde{\mathfrak{W}}^\pm_\infty \mathfrak{W}^\pm_\infty = \mathfrak{W}^\pm_H \tilde{\mathfrak{W}}^\pm_H + \mathfrak{W}^\pm_\infty \tilde{\mathfrak{W}}^\pm_\infty = \mathrm{Id}_{\mathcal{H}}\,,$$

$$\ker\left(\mathfrak{W}^\pm_H\right) = \mathcal{H}^\pm\,,\ \ker\left(\mathfrak{W}^\pm_\infty\right) = \mathcal{H}^\mp\,,\ \mathrm{ran}\left(\tilde{\mathfrak{W}}^\pm_H\right) = \mathcal{H}^\mp\,,\ \mathrm{ran}\left(\tilde{\mathfrak{W}}^\pm_\infty\right) = \mathcal{H}^\pm\,.$$

2. *The scattering can be described in a more synthetic manner by defining global wave operators involving both asymptotic dynamics :*

$$\begin{aligned} W^{+}: \quad & \mathcal{H}^{-} \oplus \mathcal{H}^{+} \longrightarrow \mathcal{H} \,, \\ & \left((\psi_0, 0) \,, (0, \psi_1) \right) \longmapsto \mathfrak{W}_H^{+} (\psi_0, 0) + \mathfrak{W}_\infty^{+} (0, \psi_1) \,, \qquad (2.70) \\ W^{-}: \quad & \mathcal{H}^{+} \oplus \mathcal{H}^{-} \longrightarrow \mathcal{H} \\ & \left((0, \psi_1) \,, (\psi_0, 0) \right) \longmapsto \mathfrak{W}_H^{-} (0, \psi_1) + \mathfrak{W}_\infty^{-} (\psi_0, 0) \,. \qquad (2.71) \end{aligned}$$

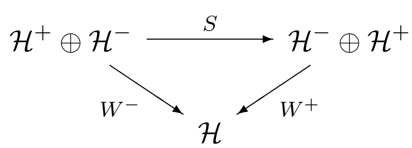
$$\tilde{W}^{+} : \mathcal{H} \longrightarrow \mathcal{H}^{-} \oplus \mathcal{H}^{+} \,, \; \tilde{W}^{+}\Psi = \left(\tilde{\mathfrak{W}}_H^{+}\Psi \,, \tilde{\mathfrak{W}}_\infty^{+}\Psi \right) \,, \qquad (2.72)$$

$$\tilde{W}^{-} : \mathcal{H} \longrightarrow \mathcal{H}^{+} \oplus \mathcal{H}^{-} \,, \; \tilde{W}^{-}\Psi = \left(\tilde{\mathfrak{W}}_H^{-}\Psi \,, \tilde{\mathfrak{W}}_\infty^{-}\Psi \right) \,. \qquad (2.73)$$

The operators $W^{\pm}$ are isometries and satisfy

$$\tilde{W}^{+}W^{+} = \mathrm{Id}_{\mathcal{H}^{-}\oplus\mathcal{H}^{+}} \,, \; \tilde{W}^{-}W^{-} = \mathrm{Id}_{\mathcal{H}^{+}\oplus\mathcal{H}^{-}} \,, \; W^{+}\tilde{W}^{+} = W^{-}\tilde{W}^{-} = \mathrm{Id}_{\mathcal{H}} \,.$$

The scattering operator S is the isometry defined by the commutative diagram :



Theorem 3 (Dirac-type comparison dynamics)**.** *The classical wave operators defined by the strong limits*

$$\Omega_H^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\not{D}_H} \mathbf{1}_{\mathbb{R}^-}(P_H^{\pm}) \,, \qquad (2.74)$$

$$\Omega_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\not{D}_\infty} \mathbf{1}_{\mathbb{R}^+}(P_\infty^{\pm}) \,, \qquad (2.75)$$

$$\tilde{\Omega}_H^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_H} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^{\pm}) \,, \qquad (2.76)$$

$$\tilde{\Omega}_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_\infty} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^{\pm}) \,, \qquad (2.77)$$

exist and satisfy

$$\tilde{\Omega}_H^{\pm} = \left(\Omega_H^{\pm}\right)^* \,, \; \tilde{\Omega}_\infty^{\pm} = \left(\Omega_\infty^{\pm}\right)^* \,,$$
$$\tilde{\Omega}_H^{\pm}\Omega_H^{\pm} + \tilde{\Omega}_\infty^{\pm}\Omega_\infty^{\pm} = \Omega_H^{\pm}\tilde{\Omega}_H^{\pm} + \Omega_\infty^{\pm}\tilde{\Omega}_\infty^{\pm} = \mathrm{Id}_{\mathcal{H}} \,.$$

Remark 2.6. *Theorems 2 and 3 describe the scattering properties of the solutions of (2.56). The scattering properties of the vector Φ (describing the physical Weyl field ϕ_A in the spin-frame (o^A, ι^A)), are obtained from the results of these theorems via the identifying operator*

$$\mathcal{J} \;:\; L^2\left((\Sigma; \mathrm{dVol}); \mathbb{C}^2\right) \longrightarrow \mathcal{H} \,, \; \mathcal{J}\Phi := \left(\frac{\Delta\sigma^2\rho^2}{\left(r^2+a^2\right)^2} \right)^{1/4} \mathbf{U}\Phi \,.$$

More precisely, to a given wave operator $\mathfrak{W}$, W or Ω, for the solution Ψ of (2.56), corresponds the wave operator $\mathcal{J}^{-1}\mathfrak{W}\mathcal{J}$, $\mathcal{J}^{-1}\Omega\mathcal{J}$, or $\mathcal{J}^{-1}W\mathcal{J}$, for the vector Φ.

Remark 2.7. *The theorems above show that the solutions of equation (2.56) satisfy asymptotically the same L^2 properties as the solutions propagated by the simpler comparison dynamics. In particular, the L^2 norm in a compact set tends to zero as $t \to \pm\infty$. Remark 2.6 entails that these properties are also satisfied by the physical field Φ.*

The next four sections describe a complete scattering theory based on a Mourre estimate for a general analytic framework. In section 7, the form (2.56) of Weyl's equation outside a slow Kerr black hole is understood as a special case of this general framework ; theorems 1, 2 and 3 are then deduced from the results of section 6. In section 8, we shall describe the scattering properties of Dirac fields outside a Kerr black hole in a more geometrical manner. A new form of theorem 2 will be derived, using the flows of outgoing and incoming principal null geodesics as comparison dynamics. This form is the most natural geometrically and enables us to interpret the scattering theory as the solution of a singular Goursat problem on the Penrose compactification of block I.

3 Abstract analytic framework

In this section, we describe generic Dirac-type operators on the manifold $\Sigma = \mathbb{R} \times S^2$, endowed with the C^∞ density $d\mu = dr d\omega$. We use the notation r for the "radial" variable, for simplicity ; it is to be understood as corresponding to the variable r_*, and not r, in the Kerr case. We shall often denote f' the derivative of f with respect to r, even for functions depending also on ω. We define several operators : first the reference Dirac operator $\not{D}_0$ then a perturbed and some asymptotic Dirac operators. The perturbed operator is a generalization of the Hamiltonian of equation (2.56). The choice of the others is guided by the wish to compare the full evolution with both asymptotic profiles and the Dirac propagator on simplified Lorentzian manifolds.

3.1 Symbol classes

Let $\eta > 0$. We define the following symbol classes as subsets of $C^\infty(\Sigma)$:

$$\begin{array}{rcl} f & \in & \mathbf{S}^{m,n} \quad \text{iff} \\ \forall \alpha \in \mathbb{N}, \beta \in \mathbb{N}^2 \quad \partial_r^\alpha \partial_\omega^\beta f & \in & \left\{ \begin{array}{ll} O(< r >^{m-\alpha}) & r \to +\infty, \\ O(e^{n\eta |r|}) & r \to -\infty. \end{array} \right. \\ f & \in & \mathbf{S}^m \quad \text{iff} \\ \forall \alpha \in \mathbb{N}, \beta \in \mathbb{N}^2 \quad \partial_r^\alpha \partial_\omega^\beta f & \in & O(< r >^{m-\alpha}). \end{array}$$

Recall that for $f \in C^\infty(\mathbb{R})$, we have :

$$\begin{array}{rcl} f & \in & S^m \quad \text{iff} \\ \forall \alpha \in \mathbb{N} \quad \partial_r^\alpha f & \in & O(< r >^{m-\alpha}). \end{array}$$

We shall understand S^m as the subset of spherically symmetric elements of $\mathbf{S}^m$.

3.2 Technical results

We consider the operator

$$\not{D}_T = \gamma D_r + p(r) \not{D}_{S^2} + c_1$$

with

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \; i\not{D}_{S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\partial_\theta + \frac{1}{2}\cot\theta) + \frac{1}{\sin\theta} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \partial_\varphi \, , \; c_1 \in \mathbb{R}$$

and $p \in C^\infty(\mathbb{R})$, not necessarily bounded. We consider $\not{D}_T$ as an operator acting on the Hilbert-space $\mathcal{H}$ defined earlier in (2.59) :

$$\mathcal{H} = L^2\left(\left(\mathbb{R} \times S^2 ; \; \mathrm{d}r\mathrm{d}\omega \right) \; ; \; \mathbb{C}^2 \right) .$$

In order to describe the domain of $\not{D}_T$ we will introduce spin weighted harmonics Y^l_{sn} (for a complete definition, see for example [45]). For each spinorial weight s, $2s \in \mathbb{Z}$, the family $\{Y^l_{sn} = e^{in\varphi}u^l_{sn} \, ; \; l - |s| \in \mathbb{N} \, , \; l - |n| \in \mathbb{N}\}$ forms a Hilbert basis of $L^2(S^2_\omega, d\omega)$ and we have the following relations

$$\begin{aligned} \frac{du^l_{sn}}{d\theta} - \frac{n - s\cos\theta}{\sin\theta} u^l_{sn} &= -i \left[(l+s)(l-s+1)\right]^{1/2} u^l_{s-1,n} \, , \\ \frac{du^l_{sn}}{d\theta} + \frac{n - s\cos\theta}{\sin\theta} u^l_{sn} &= -i \left[(l+s+1)(l-s)\right]^{1/2} u^l_{s+1,n} \, . \end{aligned}$$

We define $\otimes_2$ as the following operation between two vectors of $\mathbb{C}^2$

$$\forall v = (v_1, v_2), \, u = (u_1, u_2), \quad v \otimes_2 u = (u_1 v_1, u_2 v_2).$$

Since the families

$$\{Y^l_{\frac{1}{2},n}; (n,l) \in \mathcal{I}\}, \quad \{Y^l_{-\frac{1}{2},n}; (n,l) \in \mathcal{I}\}, \quad \mathcal{I} = \{(n,l)/l - \frac{1}{2} \in \mathbb{N}, l - |n| \in \mathbb{N}\}$$

form a Hilbert basis of $L^2(S^2_\omega, d\omega)$, we express $\mathcal{H}$ as a direct sum

$$\mathcal{H} = \oplus_{(n,l)\in\mathcal{I}} \mathcal{H}_{nl}, \quad \mathcal{H}_{nl} = L^2\left((\mathbb{R}; dr) ; \mathbb{C}^2 \right) \otimes_2 Y_{nl}, \; Y_{nl} = (Y^l_{-\frac{1}{2},n}, Y^l_{\frac{1}{2},n}).$$

We shall henceforth identify $\mathcal{H}_{nl}$ and $L^2\left((\mathbb{R}; dr) ; \mathbb{C}^2 \right)$ as well as $\psi_{nl} \otimes_2 Y_{nl}$ and ψ_{nl}. We see that

$$\not{D}_T = \oplus_{nl} \not{D}^{nl}_T \quad \text{with} \quad \not{D}^{nl}_T := \gamma D_r + p(r)\tau(l + \tfrac{1}{2}) + c_1, \; \tau := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

In what follows, we put $q := l + \frac{1}{2}$ and we assume :

$$\exists C > 0, \quad \forall r \in \mathbb{R} \quad |p'(r)| \leq C|p(r)|.$$

We put

$$\begin{aligned} D(\not{D}^{nl}_T) &= \{u \in \mathcal{H}_{nl}; \not{D}^{nl}_T u \in \mathcal{H}_{nl}\}, \\ D(\not{D}_T) &= \{u = \sum_{nl} u_{nl}; \, u_{nl} \in D(\not{D}^{nl}_T), \sum_{nl} ||u_{nl}||^2 + ||\not{D}^{nl}_T u_{nl}||^2 < \infty\}. \end{aligned}$$

Our aim is to show that $(\not{D}_T, D(\not{D}_T))$ is selfadjoint. We will need several lemmata.

Lemma 3.1. *$(C_0^\infty(\mathbb{R}))^2$ is dense in $D(\not{D}^{nl}_T)$ equipped with the graph norm.*

Proof.

For $\phi \in C_0^\infty(\mathbb{R})$ and $f = (f_1, f_2) \in \mathcal{H}_{nl}$ we put $\phi * f := (\phi * f_1, \phi * f_2)$, "$*$" denoting the convolution. Let $f \in D(\not{D}_T^{nl})$. We use a standard approximation procedure. Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \geq 0, \int \phi = 1, \phi_\delta(x) := \delta^{-1}\phi(\frac{x}{\delta}), \chi \in C_0^\infty\{|x| < 1\}$, $\chi \equiv 1$ in a neighbourhood of 0, $||\chi^{(\alpha)}|| \leq 1$, $\alpha = 0, 1$, $\chi_m(x) := \chi(\frac{x}{m})$. We put :

$$f_{\delta,m}(x) := \phi_\delta * (\chi_m f),\ f_m := \chi_m f.$$

We write

$$||\not{D}_T^{nl} f - \not{D}_T^{nl} f_{\delta,m}|| \leq ||\not{D}_T^{nl} f - \not{D}_T^{nl} f_m|| + ||\not{D}_T^{nl} f_m - \not{D}_T^{nl} f_{\delta,m}||.$$

Let us first consider

$$\not{D}_T^{nl} f - \not{D}_T^{nl} f_m = (1 - \chi_m)\not{D}_T^{nl} f + i\gamma(\chi_m)' f\ .$$

We obtain :

$$\forall \epsilon > 0, \exists M; \forall m \geq M \quad ||\not{D}_T^{nl} f - \not{D}_T^{nl} f_m|| < \frac{\epsilon}{2}.$$

For given $\epsilon > 0$ we fix $m \geq M$. Note that $f_m \in (H^1_{\text{comp}})^2$. Indeed :

$$\not{D}_T^{nl} f_m = \gamma D_r f_m + p\tau q f_m$$

and $\not{D}_T^{nl} f_m, p\tau q f_m \in \mathcal{H}_{nl}$. We have :

$$||\not{D}_T^{nl} f_m - \not{D}_T^{nl} f_{\delta,m}|| \quad \leq \quad ||\gamma D_r f_m - \gamma D_r f_{\delta,m}|| + ||p\tau q f_m - p\tau q f_{\delta,m}||. \tag{3.2}$$

It is well known that

$$||\gamma D_r f_m - \gamma D_r f_{\delta,m}|| \to 0 \quad (\delta \to 0).$$

Let us consider the second term in (3.2).

We have $\text{supp} f_m \subset B(0, m)$, $\quad \text{supp}\phi_\delta \subset B(0, 1)$. It follows

$$\text{supp}\phi_\delta * f_m \subset \overline{B(0,1) + B(0,m)} =: K.$$

We estimate $(|f|^2 = |f_1|^2 + |f_2|^2)$:

$$\begin{aligned} \int |p(f_{\delta,m} - f_m)|^2 dx \quad &\leq \quad \sup_{x \in K} |p(x)|^2 \int_K |f_{\delta,m} - f_m|^2 dx \\ &\leq \quad C \int_K |f_{\delta,m} - f_m|^2 dx \to 0 \quad (\delta \to 0). \end{aligned}$$

This concludes the proof of the lemma. □

Lemma 3.2. *We have:*

$$\forall u \in D(\not{D}_T) \quad ||\gamma D_r u|| \quad \leq \quad C(||\not{D}_T u|| + ||u||), \tag{3.3}$$

$$\forall u \in D(\not{D}_T) \quad ||p(r)\not{D}_{S^2} u|| \quad \leq \quad C(||\not{D}_T u|| + ||u||), \tag{3.4}$$

$$\forall u \in D(\not{D}_T^{nl}) \quad ||\gamma D_r u|| \quad \leq \quad C(||\not{D}_T^{nl} u|| + ||u||), \tag{3.5}$$

$$\forall u \in D(\not{D}_T^{nl}) \quad ||p(r)\tau q u|| \quad \leq \quad C(||\not{D}_T^{nl} u|| + ||u||). \tag{3.6}$$

This implies :

$$D(\not{D}_T^{nl}) = (H^1(\mathbb{R}))^2 \cap D(p), \quad \textit{where} \quad D(p) = \{u \in \mathcal{H}_{nl}; pu \in \mathcal{H}_{nl}\}.$$

Proof.
(3.5), (3.6) follow from (3.3), (3.4). In the sense of quadratic forms on $D(\not{D}_T)$ we have:

$$\not{D}_T^2 = D_r^2 + p^2(r)\not{D}_{S^2}^2 + \frac{\gamma}{i}p'(r)\not{D}_{S^2} \geq \frac{1}{2}(D_r^2 + p^2(r)\not{D}_{S^2}^2) - C\,,$$

which proves the lemma. □

Corollary 3.1. *We have :*

- *(i)* $D(\not{D}_T^{nl}) \subset (H^1(\mathbb{R}))^2$,
- *(ii)* *If* $f = (f_{ij})$ *and* $f_{ij}, g \in C_\infty(\mathbb{R})$, *then* $f(r)g(\not{D}_T^{nl})$ *is compact.*

Lemma 3.3. $(\not{D}_T^{nl}, D(\not{D}_T^{nl}))$ *is selfadjoint.*

Proof.
By a classical result due to Thaller[2] ([55, theorem 4.3]), we know that $(\not{D}_T^{nl}, (C_0^\infty(\mathbb{R}))^2)$ is essentially selfadjoint. Let us denote by $D_T(\not{D}_T^{nl})$ the domain of its selfadjoint extension. We have to show that $D_T(\not{D}_T^{nl}) = D(\not{D}_T^{nl})$.

If u belongs to $D_T(\not{D}_T^{nl})$ then, by definition, there exists a sequence $u_m \in (C_0^\infty(\mathbb{R}))^2$ such that $u_m \to u$, $\not{D}_T^{nl} u_m \to v =: \not{D}_T^{nl} u$ in $\mathcal{H}_{nl}$. Besides $\not{D}_T^{nl} u_m \to \not{D}_T^{nl} u$ in the sense of distributions and we find that $\not{D}_T^{nl} u$, defined in the sense of distributions, belongs to $\mathcal{H}_{nl}$, i.e. $u \in D(\not{D}_T^{nl})$.

Let now $u \in D(\not{D}_T^{nl})$. As $(C_0^\infty(\mathbb{R}))^2$ is dense in $D(\not{D}_T^{nl})$ by lemma 3.1, there exists a sequence $u_m \in (C_0^\infty(\mathbb{R}))^2$ s.t. $u_m \to u$ in $\mathcal{H}_{nl}$, $\not{D}_T^{nl} u_m \to \not{D}_T^{nl} u$ in $\mathcal{H}_{nl}$, i.e. $u \in D_T(\not{D}_T^{nl})$. □

Lemma 3.4. $(C_0^\infty(\Sigma))^2$ *is dense in* $D(\not{D}_T)$ *equipped with the graph norm.*

Proof.
Recall that

$$D(\not{D}_T) = \{u = \sum_{nl} u_{nl} \in \mathcal{H};\, u_{nl} \in \mathcal{H}_{nl}^1,\, \sum_{nl} ||\not{D}_T^{nl} u_{nl}||^2 < \infty\}.$$

Let $u = \sum_{nl} u_{nl} \in D(\not{D}_T)$. For $\epsilon > 0$ we choose $N > 0$ s.t.

$$\sum_{|(n,l)| \geq N} ||\not{D}_T^{nl} u_{nl}||^2 + ||u_{nl}||^2 < \frac{\epsilon}{2}.$$

$(C_0^\infty(\mathbb{R}))^2$ being dense in $D(\not{D}_T^{nl})$ we can choose $\phi_{nl}^N \in (C_0^\infty(\mathbb{R}))^2$ s.t.

$$\forall |(n,l)| \leq N \quad ||\not{D}_T^{nl}(u_{nl} - \phi_{nl}^N)||^2 + ||(u_{nl} - \phi_{nl}^N)||^2 < \frac{\epsilon}{2N^2}.$$

We put :

$$\phi_N := \sum_{|(n,l)| \leq N} \phi_{nl}^N \in (C_0^\infty(\Sigma))^2.$$

We have :

$$\begin{aligned} ||\not{D}_T(u - \phi_N)||^2 + ||u - \phi_N||^2 &= \sum_{|(n,l)| \leq N} ||\not{D}_T^{nl}(u_{nl} - \phi_{nl}^N)||^2 + ||(u_{nl} - \phi_{nl}^N)||^2 \\ &+ \sum_{|(n,l)| \geq N} ||\not{D}_T^{nl} u_{nl}||^2 + ||u_{nl}||^2 < \epsilon. \end{aligned}$$ □

We find :

[2] Thaller's result is proved in dimension 3 but the proof is independent of the dimension.

Lemma 3.5. *The operator* $\not{D}_T$ *with domain*

$$D(\not{D}_T) = \{u \in \mathcal{H};\, \not{D}_T u \in \mathcal{H}\} = \{u = \sum_{nl} u_{nl}\,;\, u_{nl} \in D(\not{D}_T^{nl}),\, \sum_{nl} ||\not{D}_T^{nl} u_{nl}||^2 < \infty\}\,,$$

is selfadjoint.

Proof.
Let us first show that

$$D(\not{D}_T) = \{u \in \mathcal{H};\, \not{D}_T u \in \mathcal{H}\} = \{u = \sum_{nl} u_{nl};\, u_{nl} \in D(\not{D}_T^{nl}),\, \sum_{nl} ||\not{D}_T^{nl} u_{nl}||^2 < \infty\}. \tag{3.7}$$

Let $u = \sum_{nl} u_{nl} \in \mathcal{H}$. As $\not{D}_T : \mathcal{H} \to \mathcal{D}'$ is continuous, it follows that

$$\not{D}_T u = \sum_{nl} \not{D}_T^{nl} u_{nl}$$

in the sense of distributions. The equality (3.7) then follows from the fact that

$$\sum_{nl} \not{D}_T^{nl} u_{nl} \in \mathcal{H} \Leftrightarrow \forall n, l\ \not{D}_T^{nl} u_{nl} \in \mathcal{H}_{nl},\ \sum_{nl} ||\not{D}_T^{nl} u_{nl}||^2 < \infty.$$

We now have to show:

1. $(\not{D}_T, D(\not{D}_T))$ is closed,
2. $\mathrm{ran}(\not{D}_T \pm i) = \mathcal{H}$.

We will start with 1. Let $u_m \in D(\not{D}_T)$, $u_m \to u \quad \not{D}_T u_m \to v$. We must show that $u \in D(\not{D}_T)$ and $\not{D}_T u = v$. Let

$$u_m = \sum_{nl} u_{nl}^m,\ u = \sum u_{nl},\ v = \sum v_{nl}.$$

Clearly $u_{nl}^m \to u_{nl}$, $\not{D}_T^{nl} u_{nl}^m \to v_{nl}$. As $(\not{D}_T^{nl}, D(\not{D}_T^{nl}))$ is closed, $u_{nl} \in D(\not{D}_T^{nl})$ and $\not{D}_T^{nl} u_{nl} = v_{nl}$. We have :

$$\not{D}_T u = \sum \not{D}_T^{nl} u_{nl} = \sum v_{nl} = v.$$

But $v \in \mathcal{H}$, i.e.

$$\sum_{nl} ||\not{D}_T^{nl} u_{nl}||^2 = \sum ||v_{nl}||^2 < \infty \,\text{and}\, u \in D(\not{D}_T).$$

Let us now show 2. Let $v = \sum v_{nl} \in \mathcal{H}$. We have to find $u \in D(\not{D}_T)$ s.t. $(\not{D}_T \pm i)u = v$. As $(\not{D}_T^{nl}, D(\not{D}_T^{nl}))$ is selfadjoint, for each n, l we find $u_{nl} \in D(\not{D}_T^{nl})$ s.t. $(\not{D}_T \pm i)u_{nl} = v_{nl}$. We put $u := \sum u_{nl}$ and check

$$\not{D}_T u = \sum_{nl} \not{D}_T^{nl} u_{nl} = \mp iu + v \in \mathcal{H}\,,\ \text{i.e.}\, u \in D(\not{D}_T)\,.$$

This concludes the proof of the lemma. □

Remark 3.1. *If we suppose that* p *is bounded our results follow immediately from the Kato-Rellich theorem.*

For technical reasons, we shall need to consider the case $p(r) = c_0 e^{\eta r}$ for some constant $c_0 \geq 0$. We put

$$\not{D}_e = \gamma D_r + c_0 e^{\eta r} \not{D}_{S^2} + c_1\,.$$

3.3 The reference Dirac operator

We consider on $(C_0^\infty(\Sigma))^2$ the operator

$$\not{D}_0 \;:=\; \gamma D_r + g(r)\not{D}_{S^2} + f(r), \quad f, g \in C_b^\infty(\mathbb{R}),\; g > 0.$$

We assume $g \in \mathbf{S}^{-1,-1}$, $f' \in \mathbf{S}^{-3}$, the existence of some constants $c_0 \geq 0$ and $c_1 \in \mathbb{R}$ such that

$$(g(r) - c_0 e^{\eta r})^{(i)} = O\left(e^{(\eta+\varepsilon)r}\right) \text{ as } r \to -\infty\,,\ \varepsilon > 0\,,\ i = 0, 1\,, \tag{3.8}$$

$$f(r) - c_1 \in O(< r >^{-2}), \quad r \to -\infty, \tag{3.9}$$

$$\left(g(r) - \frac{1}{r}\right)^{(i)} \in O\left(< r >^{-1-i-\varepsilon}\right) \text{ as } r \to +\infty\,,\ \varepsilon > 0\,,\ i = 0, 1\,, \tag{3.10}$$

$$f(r) \in O(< r >^{-2}), \quad r \to \infty. \tag{3.11}$$

Remark 3.2. *Properties (3.8) and (3.10) imply the existence of $R_0 > 0$ and $c_2 > 0$ such that*

$$\forall r \geq R_0\,,\ g(r) \geq \frac{c_2}{r} \text{ and } \forall r \leq -R_0\,,\ g(r) \geq c_2 e^{\eta r}\,. \tag{3.12}$$

Remark 3.3. *Note that the reference Dirac operator has the same principal terms as the Dirac operator associated with the Riemannian metric*

$$g_0 = \mathrm{d}r^2 + g^{-2}(r)\mathrm{d}\omega^2$$

on $\mathbb{R}_r \times S^2$. The Riemannian manifold $(\mathbb{R}_r \times S^2, g_0)$ has two asymptotic ends : the end corresponding to $r \to +\infty$ is asymptotically flat and that corresponding to $r \to -\infty$ is asymptotically hyperbolic, in other words exponentially large (the size of the 2-sphere grows exponentially as $r \to -\infty$).

We have :

$$\not{D}_0 = \oplus_{(n,l)\in\mathcal{I}} \not{D}_0^{nl}\ ,\quad \not{D}_0^{nl} = \gamma D_r + g(r) q\tau + f(r)\,.$$

$\not{D}_0^{nl}$ is selfadjoint with domain $D(\not{D}_0^{nl}) = (H^1(\mathbb{R}))^2$. By lemma 3.5 $\not{D}_0$ is selfadjoint with domain :

$$D(\not{D}_0) = \{\Psi = \sum_{(n,l)\in\mathcal{I}} \psi_{nl};\ \psi_{nl} \in D(\not{D}_0^{nl}) \sum_{(n,l)\in\mathcal{I}} ||\not{D}_0^{nl}\psi_{nl}||^2_{(L^2(\mathbb{R}))^2} < \infty\}.$$

3.4 The perturbed Dirac operator

We consider on $C_0^\infty(\Sigma)$ a Dirac-type operator of the form :

$$\not{D} = h\not{D}_0 h + V;\ V = (V_{ij}), h \geq 0. \tag{3.13}$$

We suppose that $V_{ij}, h \in C_b^\infty(\Sigma)$ are some real functions satisfying the following conditions :

$$\exists 0 < \alpha < 1 \quad |h^2 - 1| \leq \alpha\,, \tag{3.14}$$

$$\partial_\omega h \in \mathbf{S}^{-2}\,, \tag{3.15}$$

$$h - 1 \in \mathbf{S}^{-2}\,, \tag{3.16}$$

$$V_{ij} \in \mathbf{S}^{-2}\,. \tag{3.17}$$

We define $\not{D}_1 := \not{D} - \not{D}_0$.

3.5 Asymptotic dynamics

We have two asymptotic regions ($r \to \pm\infty$) and to each we associate an asymptotic operator. Let $\theta_0 \in C_b^\infty(\mathbb{R})$ s.t. $\theta_0 = 0$ in a neighbourhood of 0 and for all $|x| \geq 1$, $\theta_0(x) = 1$ and $\theta_1(x) := \mathbf{1}_{\mathbb{R}^+}(x)\theta_0(x)$. We first consider negative infinity. We put

$$\not{D}_- := \gamma D_r + ce^{-\eta\theta_0(r)|r|}\not{D}_{S^2} + c_1 \quad (c \geq 0)$$

and we define $\not{D}_-^{nl}, D(\not{D}_-^{nl}) = (H^1(\mathbb{R}))^2$ in the same way as for $\not{D}_0$. Clearly $(\not{D}_-^{nl}, D(\not{D}_-^{nl}))$ is selfadjoint and $\not{D}_-$ is selfadjoint with domain

$$D(\not{D}_-) = \{\psi = \sum_{nl} \psi_{nl};\ \psi_{nl} \in D(\not{D}_-^{nl}), \sum_{nl} ||\not{D}_-^{nl}\psi_{nl}||^2 < \infty\}.$$

For positive infinity, we put

$$\not{D}_+ = \gamma D_r + \theta_1(r)\frac{1}{|r|}\not{D}_{S^2}.$$

$\not{D}_+^{nl}, D(\not{D}_+^{nl})$ are defined as for $\not{D}_0$ and $\not{D}_-$, $(\not{D}_+^{nl}, D(\not{D}_+^{nl}))$ is selfadjoint and $\not{D}_+$ is selfadjoint with domain

$$D(\not{D}_+) = \{\psi = \sum \psi_{nl};\ \psi_{nl} \in D(\not{D}_+^{nl}), \sum_{nl} ||\not{D}_+^{nl}\psi_{nl}||^2 < \infty\}.$$

The constant c in $\not{D}_-$ will be taken equal to 0 for a comparison with asymptotic profiles and to c_0 for a Dirac-type asymptotic operator (see introduction to this section). We denote in what follows :

$$\mathcal{N} := \{0, \pm\},\ g_0 := g,\ g_- := ce^{-\eta\theta_0(r)|r|},\ g_+(r) := \frac{1}{|r|}\theta_1(r),\ f_0 := f,\ f_+ := 0,\ f_- := c_1.$$

4 Some fundamental properties of our Dirac-type Hamiltonians

This section is mostly devoted to the proof of technical results that will be important later on. Many results stated here concern functions of selfadjoint operators. Their proof requires to use the Helffer-Sjöstrand formula (see e.g. [15]). Let $\chi \in C_0^\infty(\mathbb{R})$, H a selfadjoint operator, there exists an almost analytic extension $\tilde{\chi}$ of χ s.t.

$$\tilde{\chi}|_{\mathbb{R}} = \chi, \quad |\frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)| \leq C|\mathrm{Im}z|^N,\ \forall N \in \mathbb{N},$$
$$\chi(H) = \frac{1}{2\pi i}\int \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(z-H)^{-1}dz \wedge d\bar{z}.$$

4.1 Description of the domains

Let us first note that the operator $\not{D}$ is selfadjoint with the same domain as $\not{D}_0$:

Lemma 4.1.

$$(\not{D}, D(\not{D}_0)) \quad \textit{is selfadjoint.}$$

Proof.
As $h : D(\not{D}_0) \to D(\not{D}_0)$, $(\not{D}, D(\not{D}_0))$ is well-defined and symmetric. We have

$$\not{D} = h^2 \not{D}_0 + \tilde{V} \tag{4.1}$$

with $\tilde{V} = (\tilde{V}_{ij})$, $\tilde{V}_{ij} \in \mathbf{S}^{-2}$. The selfadjointness of $(\not{D}, D(\not{D}_0))$ follows from (3.14) and the Kato-Rellich theorem. □

We put $D(\not{D}) := D(\not{D}_0)$. By (3.13), (3.14) it can easily be checked that for $u \in \mathcal{H}$, the properties $\not{D}u \in \mathcal{H}$ and $\not{D}_0 u \in \mathcal{H}$ are equivalent. So we obtain :

$$D(\not{D}) = \{u \in \mathcal{H};\ \not{D}u \in \mathcal{H}\}.$$

We denote in what follows $\mathcal{H}^1 := D(\not{D}) = D(\not{D}_0)$, $\mathcal{H}^1_{nl} := D(\not{D}_0^{nl}) = (H^1(\mathbb{R}))^2 \otimes_2 Y_{nl}$. Recall from lemma 3.4 that $(C_0^\infty(\Sigma))^2$ is dense in $D(\not{D}_0) = D(\not{D})$. Let $||u||_{\mathcal{H}^1} = ||u|| + ||\not{D}_0 u||$ be the graph norm of $\not{D}_0$ and $\mathcal{V}^1$ the closure of $(C_0^\infty(\Sigma))^2$ in this norm.

Lemma 4.2.

$$\mathcal{H}^1 = \mathcal{V}^1 .$$

Proof.

- Let us first show that $\mathcal{V}^1 \subset \mathcal{H}^1$. Let $u \in \mathcal{V}^1$, $u_m \in (C_0^\infty(\Sigma))^2$ s.t. $u_m \to u$. $\not{D}_0 u_m$ is a Cauchy sequence, so $\not{D}_0 u_m \to v \in \mathcal{H}$. Besides, $\not{D}_0 u_m \to \not{D}_0 u$ in the sense of distributions, so $\not{D}_0 u = v \in \mathcal{H}$, i.e. $u \in \mathcal{H}^1$.
- We now show $\mathcal{H}^1 \subset \mathcal{V}^1$. Let $u \in \mathcal{H}^1$. By lemma 3.4 there exists a sequence $u_m \in (C_0^\infty(\Sigma))^2$ s.t. $||u_m - u||_{\mathcal{H}^1} \to 0$, it follows $u \in \mathcal{V}^1$. □

We consider the quadratic forms associated to $\not{D}_0^2$ and

$$H := D_r^2 + g^2(r)\not{D}_{S^2}^2$$

which we denote by Q_0 and Q_H, for example $Q_0(u,u) = (\not{D}_0^2 u, u) + ||u||^2$. We also denote $H^{nl} := D_r^2 + g^2(r)q^2$. Let $D(Q_i)$, $i \in \{0, H\}$, be the closure of $(C_0^\infty(\Sigma))^2$ in the norm $Q_i(u,u)$.

Lemma 4.3. *The norms $Q_0(u,u)$ and $Q_H(u,u)$ are equivalent.*

Proof.
Let us first note :

$$\exists C > 0 \quad \forall r \in \mathbb{R} \quad |g'(r)| \le C g(r).$$

In the sense of quadratic forms on $(C_0^\infty(\Sigma))^2$ we have :

$$\not{D}_0^2 = D_r^2 + g^2(r)\not{D}_{S^2}^2 + \frac{\gamma}{i} g'(r)\not{D}_{S^2} + \gamma D_r f(r) + f(r)\gamma D_r + 2f(r)g(r)\not{D}_{S^2} ,$$

whence,

$$\frac{1}{2}(D_r^2 + g^2(r)\not{D}_{S^2}^2) - C \le \not{D}_0^2 \le 2(D_r^2 + g^2(r)\not{D}_{S^2}^2) + C .$$

This establishes the equivalence of the norms Q_0 and Q_H. □

Corollary 4.1.

$$D(Q_0) \subset (H^1_{\mathrm{loc}}(\Sigma))^2.$$

Proof.
$D(Q_0) = D(Q_H)$ by the previous lemma. But $D(Q_H) \subset (H^1_{\text{loc}}(\Sigma))^2$ by the local ellipticity of the operator H. □

Corollary 4.1 and lemma 4.2 together give the

Lemma 4.4.

$$\mathcal{H}^1 \subset (H^1_{\text{loc}}(\Sigma))^2.$$

Corollary 4.2. *If $f_{ij}, g \in C_\infty(\mathbb{R})$, $f(r) = (f_{ij}(r))_{ij}$, then $f(r)g(\not{D})$ is compact on $\mathcal{H}$.*

Proof.
It is sufficient to suppose $f, g \in C_0^\infty(\mathbb{R})$. So

$$f(r)g(\not{D}) : \mathcal{H} \to (H^1(\Omega))^2 \hookrightarrow \mathcal{H}$$

where Ω is some bounded set and the above embedding is compact by the Rellich theorem. □

Lemma 4.5. *Let $\chi \in C_0^\infty(\mathbb{R})$. Then the operator $\chi(\not{D}) - \chi(\not{D}_0)$ is compact.*

Proof. Using the Helffer-Sjöstrand formula it is sufficient to show :

$$||(z - \not{D}_0)^{-1}(\not{D} - \not{D}_0)(z - \not{D})^{-1}|| \le C|\text{Imz}|^{-2}, \tag{4.2}$$

$$(z - \not{D}_0)^{-1}(\not{D} - \not{D}_0)(z - \not{D})^{-1} \text{ is compact for all } z \in \mathbb{C} \setminus (\sigma(\not{D}_0) \cup \sigma(\not{D})). \tag{4.3}$$

(4.2) is clear, let us show (4.3). We have :

$$\not{D} - \not{D}_0 = (h-1)\not{D}_0 h + \not{D}_0(h-1) + V.$$

It follows :

$$\begin{aligned}(z - \not{D}_0)^{-1}(\not{D} - \not{D}_0)(z - \not{D})^{-1} &= (z - \not{D}_0)^{-1}(h-1)\not{D}_0 h(z - \not{D})^{-1} \\ &+ (z - \not{D}_0)^{-1}\not{D}_0(h-1)(z - \not{D})^{-1} \\ &+ (z - \not{D}_0)^{-1}V(z - \not{D})^{-1}.\end{aligned}$$

(4.3) now follows from the fact that $h : D(\not{D}) \to D(\not{D})$, (3.16), (3.17) and corollary 4.2. □

We will also need the following

Lemma 4.6.

$$D(\not{D}^2) = D(\not{D}_0^2) = D(H)\,.$$

Proof.
We have :

$$\begin{aligned}\not{D} &= h\not{D}_0 h + V, \\ \not{D}^2 &= h\not{D}_0 hV + Vh\not{D}_0 h + h\not{D}_0 h^2\not{D}_0 h + V^2, \\ D(\not{D}_0^2) &= \{u \in D(\not{D}_0);\, \not{D}_0 u \in D(\not{D}_0)\}, \\ D(\not{D}^2) &= \{u \in D(\not{D});\, \not{D}u \in D(\not{D})\}, \\ D(\not{D}) &= D(\not{D}_0).\end{aligned} \tag{4.4}$$

Let $u \in D(\not{D}_0^2)$. We have to show that $\not{D}^2 u \in \mathcal{H}$. This follows from (4.4) and the fact that $h, V : D(\not{D}_0) \to D(\not{D}_0), D(\not{D}_0^2) \to D(\not{D}_0^2)$. The proof for $D(\not{D}_0^2) \subset D(\not{D}^2)$ is analogous using the fact that h is non vanishing (see (3.14)). The following estimates give $D(H) = D(\not{D}_0^2)$:

$$\begin{aligned} ||Hu||^2 &\leq ||\not{D}_0^2 u||^2 + ||\frac{\gamma}{i} g'(r) \not{D}_{S^2} u||^2 \\ &\leq C(||\not{D}_0^2 u||^2 + ||u||^2), \\ ||\not{D}_0^2 u||^2 &\leq C(||Hu||^2 + ||\frac{\gamma}{i} g'(r) \not{D}_{S^2} u||^2) \\ &\leq C(||Hu||^2 + ||u||^2). \quad \Box \end{aligned}$$

We shall henceforth denote $\mathcal{H}^2 := D(\not{D}^2) = D(\not{D}_0^2) = D(H)$.

4.2 Resolvent estimates

Lemma 4.7. *We have for all $u \in D(\not{D}_0)$:*

$$||g(r) \not{D}_{S^2} u|| \leq C(||\not{D}_0 u|| + ||u||), \tag{4.5}$$

$$||\gamma D_r u|| \leq C(||\not{D}_0 u|| + ||u||). \tag{4.6}$$

Proof.
The lemma follows from the equivalence of the norms Q_0 and Q_H and the fact that

$$||g(r) \not{D}_{S^2} u||^2 + ||\gamma D_r u||^2 = (Hu, u). \quad \Box$$

Lemma 4.8. *For $u \in D(\not{D}_0)$ we have :*

$$|| < r >^2 \not{D}_1 u|| \leq C(||\not{D}_0 u|| + ||u||), \tag{4.7}$$

$$|| < r > \not{D}_1 < r > u|| \leq C(||\not{D}_0 u|| + ||u||), \tag{4.8}$$

$$||\not{D}_1 u < r >^2 || \leq C(||\not{D}_0 u|| + ||u||). \tag{4.9}$$

Proof.
We will only show (4.7), the proof for the other estimates is analogous. We have, using (3.14)-(3.17) :

$$\not{D}_1 = (h^2 - 1)\not{D}_0 + \tilde{V} \quad \text{with } \tilde{V} = (\tilde{V}_{ij}), \, \tilde{V}_{ij} \in \mathbf{S}^{-2}.$$

which gives (4.7). $\Box$

4.3 Absence of eigenvalues for $\not{D}_\nu$, $\nu \in \mathcal{N}$

The following lemma is analogous to [6, lemma VI.1].

Lemma 4.9. *$\not{D}_\nu$ has no eigenvalues for all $\nu \in \mathcal{N}$. Similarly, $\not{D}_e$ has no eigenvalues.*

Proof.
We prove the lemma only for $\not{D}_0$, the other cases are analogous. It is sufficient to show that $\not{D}_0^{nl} - c_1 = \gamma D_r + g(r)\tau q + f(r) - c_1$ has no eigenvalues. We put $\hat{V}(r) = g(r)\tau q + f(r) - c_1$. If $u \in (L^2(\mathbb{R}))^2$ is an eigenvector of $\not{D}_0^{nl} - c_1$ with eigenvalue λ, then $w(r) = e^{-i\lambda\gamma r} u(r)$ satisfies :

$$w'(r) - i\gamma e^{-i\lambda\gamma r} \hat{V}(r) e^{i\lambda\gamma r} w(r) = 0. \tag{4.10}$$

Each solution of (4.10) is in H^1 and therefore $\lim_{r \to -\infty} w(r) = 0$. As

$$\int_{-\infty}^0 |\hat{V}| dr < \infty,$$

we conclude by Gronwall's lemma that $w = 0$. $\Box$

5 The Mourre estimate

5.1 Preliminary remarks

The Mourre estimate is a positive commutator estimate between the Hamiltonian and another selfadjoint operator, called the conjugate operator. The conjugate operator thus represents an observable that increases along the evolution. For Schrödinger or Dirac equations in flat space-time, the situation has been thoroughly studied and we can take the generator of dilations as conjugate operator.

In our case we have two asymptotic regions. The space-time is asymptotically flat at positive infinity and we can use the generator of dilations as conjugate operator there. Near the black-hole horizon the problem is much more complicated. Let us consider a toy model of this situation :

$$\not{D} = \gamma D_r + e^{\eta r} \not{D}_{S^2} \text{ on } \mathbb{R}_- \times S^2.$$

The Dirac operators that we consider are short range perturbations of an operator of this kind. For such a Hamiltonian, if we try to use the generator of dilations

$$A := \frac{1}{2} (r D_r + D_r r)$$

as conjugate operator, we find :

$$[i\not{D}, A] = \gamma D_r - \eta r e^{\eta r} \not{D}_{S^2} .$$

For $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi(\not{D})\,[i\not{D}, A]\,\chi(\not{D})$ generically has no sign. Moreover, this commutator is not controlled by $\not{D}$.

In this spherically symmetric setting, we can use spin weighted harmonics and write :

$$\not{D}^{nl} = \gamma D_r + e^{\eta r} \tau q\,.$$

The angular part is replaced by $e^{\eta r}\tau q$ ($q = l + 1/2$), a mere potential. Therefore, after diagonalization, we can use the generator of dilations. If the metric is not spherically symmetric, we cannot procede in this manner. We consider instead the unitary transformation $U = e^{\eta^{-1} i D_r \ln |\not{D}_{S^2}|}$. We obtain :

$$\hat{\not{D}} = U^* \not{D} U = \gamma D_r + e^{\eta r} \frac{\not{D}_{S^2}}{|\not{D}_{S^2}|}.$$

On each spherical harmonics $\hat{\not{D}}$ reduces to the operator

$$\hat{\not{D}}^{nl} = \gamma D_r + e^{\eta r} \tau.$$

If now we use the generator of dilations as conjugate operator, all the necessary estimates are uniform in q simply because no term involves q ! In particular, $e^{\eta r}\tau \chi(\hat{\not{D}}^{nl})$ and $\eta r e^{\eta r}\tau\chi(\hat{\not{D}}^{nl})$ will be compact and thus small if the support of χ is sufficiently small. This "smallness-result" is uniform in q. If we apply our unitary transformation to the generator of dilations, we find an operator similar to the one introduced by Froese and Hislop (see [24]). The argument is however different. In the case of the Laplacian we can show that the commutator between the angular part of the Laplacian and the Froese-Hislop conjugate operator is positive. In our case we cannot find such a conjugate operator because the angular part has no sign. One might think better to use $\not{D}^2$ rather than $\not{D}$ to get a Mourre estimate and then apply known results about the Mourre estimate for the square root of an operator (see [14], [32]). Let us first remark that the angular part of $\not{D}^2$ also has no sign :

$$\not{D}^2 = D_r^2 + e^{2\eta r}\not{D}_{S^2}^2 + e^{\eta r}\frac{\gamma}{i}\not{D}_{S^2}.$$

Note also that the connection term $e^{\eta r}\frac{\gamma}{i}\not{D}_{S^2}$ is not a perturbation (not even a long range one) of the Laplacian. This is typical for exponentially large ends. It is however reasonable to expect that the connection term is a perturbation of the Laplacian for a large class of asymptotic ends. For manifolds with such ends, a Mourre theory for the Laplacian implies directly a Mourre theory for the Dirac operator.

5.2 The abstract setting of Mourre theory

In this section we recall the technical hypotheses for the Mourre estimate. We consider the commutator $[H, iA]$ between the Hamiltonian H and another selfadjoint operator A, called the conjugate operator. We say that the pair (H, A) satisfies a Mourre estimate on some energy interval Δ, if

$$\mathbf{1}_\Delta(H)[iH, A]\mathbf{1}_\Delta(H) \geq \delta\mathbf{1}_\Delta(H);\ \delta > 0.$$

As both operators H and A are unbounded we have to be careful to define correctly the commutator. We say that the pair (H, A) satisfies the Mourre conditions (see [42]) iff

(M1') $D(A) \cap D(H)$ is dense in $D(H)$,

(M2') e^{isA} preserves $D(H), \sup_{|s|\leq 1} ||He^{isA}u|| < \infty, \quad \forall u \in D(H)$,

(M3') $[iH, A]$ which is defined as a quadratic form on $D(H) \cap D(A)$ is semibounded, closable and can be extended to a bounded operator from $D(H)$ to $\mathcal{H}$:

$$|[iH, A](u, v)| \leq C||Hu||||v||, \quad \forall u, v \in D(H) \cap D(A).$$

It has been remarked in [25] that the Virial theorem remains valid under the following conditions :

(M1) e^{isA} preserves $D(H)$,

(M2) $[iH, A]$ defined as a quadratic form on $D(H) \cap D(A)$ can be extended to a bounded operator from $D(H)$ to $\mathcal{H}$:

$$|[iH, A](u, v)| \leq C||Hu||||v||, \quad \forall u, v \in D(H) \cap D(A).$$

(M1')+(M2') is even equivalent to (M1) (see [1] proposition 3.2.5). Note also that even in Mourre's original work [42], the assumption that $[iH, A]$ is semi-bounded is not necessary.

In our opinion the simplest and most useful condition for the Mourre estimate is the following (see [1]) :

A bounded operator C is of class $C^k(A; \mathcal{H})$ iff

$$\mathbb{R} \ni s \mapsto e^{isA}Ce^{-isA} \quad \text{is} \quad C^k \quad \text{for the strong topology of} \quad \mathcal{B}(\mathcal{H}).$$

$H \in C^k(A)$ if there exists $z \in \mathbb{C} \setminus \sigma(H)$ such that $(z - H)^{-1} \in C^k(A; \mathcal{H})$. (M1)-(M2) implies $H \in C^1(A)$ and the Virial theorem is valid under the only condition $H \in C^1(A)$ (see [1]).

The condition $H \in C^1(A)$ has been caracterized in [1, theorem 6.2.10] by the following property of the commutator $[H, iA]$:

Proposition 5.1. *The operator H is of class $C^1(A)$ iff the following two conditions are satisfied :*

(i) *There exists $C < \infty$ s.t.*

$$|(Au, Hu) - (Hu, Au)| \leq C||(H+i)u||^2, \quad \forall u \in D(H) \cap D(A),$$

(ii) *There exists $z \in \mathbb{C} \setminus \sigma(H)$ s.t.*

$$\{u \in D(A) \quad | \quad (z-H)^{-1}u \in D(A), (\bar{z}-H)^{-1}u \in D(A)\}$$

is a core for A.

In general it is not easy to check the conditions $(i), (ii)$ if the domains of H and A are not explicitly known. In such a case, a possibility for checking the condition $H \in C^1(A)$ consists in searching first a common core for H and A. This is described in [26]. We start with an extension of the Nelson theorem (see [26, lemma 1.2.5]) :

Lemma 5.1. *Let $\mathcal{H}$ be a Hilbert space, $N \geq 1$ a selfadjoint operator on $\mathcal{H}$, A a symmetric operator on $\mathcal{H}$ such that $D(N) \subset D(A)$ and*

$$\begin{aligned} &(i) \quad ||Au|| \leq C||Nu||, \quad u \in D(N), \\ &(ii) \quad |(Au, Nu) - (Nu, Au)| \leq C||N^{\frac{1}{2}}u||^2, \quad u \in D(N). \end{aligned}$$

Then A is essentially selfadjoint on $D(N)$. If $u \in D(\bar{A})$, then $(1 + i\epsilon N)^{-1}u$ converges to u in the graph topology of $D(\bar{A})$ when $\epsilon \to 0$.

The operator N is called a *comparison operator*. In this situation, it is sufficient to calculate the commutator on $D(N)$, more precisely, we have the following lemma (see [26, lemma 3.2.2]) :

Lemma 5.2. *Let H, H_0, N be selfadjoint operators on a Hilbert space $\mathcal{H}$ such that $N \geq 1$, $D(H) = D(H_0)$ as Banach spaces, and $(z-H)^{-1}$ sends $D(N)$ into itself. Let A be a symmetric operator with domain $D(N)$. Suppose that H_0 and A satisfy the assumptions of lemma 5.1 with comparison operator N and denote still A the unique selfadjoint extension of A. Suppose furthermore :*

$$|(Au, Hu) - (Hu, Au)| \leq C(||Hu||^2 + ||u||^2), \quad \forall u \in D(N).$$

Then :
(i) $D(N)$ is dense in $D(A) \cap D(H)$ equipped with the norm $||Hu|| + ||Au|| + ||u||$,
(ii) the quadratic form $[H, iA]$ on $D(A) \cap D(H)$ is the unique extension of $[H, iA]$ on $D(N)$,
(iii) H is of class $C^1(A)$.

We will also use the following lemma (see [25, lemma 2]) :

Lemma 5.3. *Let $H \in C^1(A)$ and suppose that the commutator $[iH, A]$ can be extended to a bounded operator from $D(H)$ to $\mathcal{H}$. Then e^{isA} preserves $D(H)$.*

5.3 Technical results

We now define the comparison operator by

$$\begin{aligned} N &:= H + r^2 + 1 \quad (\text{acting on } \mathcal{H}), \\ N^{nl} &:= H^{nl} + r^2 + 1 \quad (\text{acting on } \mathcal{H}_{nl}). \end{aligned}$$

We put :

$$\begin{aligned} D(N^{nl}) &:= \{u \in \mathcal{H}_{nl};\, N^{nl}u \in \mathcal{H}_{nl}\}, \\ D(N) &:= \{u \in \mathcal{H};\, Nu \in \mathcal{H}\}, \\ &= \{u = \sum_{nl} u_{nl}; u_{nl} \in D(N^{nl}), \sum ||N^{nl}u_{nl}||^2 < \infty\}. \end{aligned}$$

We recall (a slightly weaker version for the first one) [32, lemmata 4.1.1 and 5.1.1] :

Lemma 5.4. *$(C_0^\infty(\mathbb{R}))^2$ is dense in $D(N^{nl})$ and $(C_0^\infty(\Sigma))^2$ is dense in $D(N)$.*

Lemma 5.5. *We have for all $u \in D(N)$:*

$$\begin{aligned} ||r^2u|| &\leq ||Nu||^2 + ||u||^2, \\ ||Hu||^2 &\leq ||Nu||^2 + ||u||^2. \end{aligned}$$

Therefore we can caracterize the domains of N^{nl} and N in the following way :

$$\begin{aligned} D(N^{nl}) &= D(H^{nl}) \cap D(r^2) = D((\not{D}_0^{nl})^2) \cap D(r^2), \\ D(N) &= D(H) \cap D(r^2) = D(\not{D}^2) \cap D(r^2) \\ &= \{u = \sum u_{nl} \in \mathcal{H}; u_{nl} \in D(N^{nl}), \quad \sum_{nl} ||H^{nl}u_{nl}||^2 + ||r^2u_{nl}||^2 < \infty\}, \end{aligned}$$

where we have used lemma 4.6.

Lemma 5.6. *Let $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \sigma(\not{D})$, we have :*

$$\begin{aligned} (i) \quad & (z - \not{D})^{-1} : D(<r>^n) \to D(<r>^n), \\ (ii) \quad & (z - \not{D})^{-1} : D(N) \to D(N)\,. \end{aligned}$$

Proof.
We have clearly $(z - \not{D})^{-1} : D(\not{D}^2) \to D(\not{D}^2)$ and (ii) follows from (i) because $D(N) = D(\not{D}^2) \cap D(<r>^2)$. Let us first show that $(z - \not{D})^{-1} : D(r) \to D(r)$. This is equivalent to

$$\sup_{|s|\leq 1} ||\frac{e^{isr} - 1}{s}(z - \not{D})^{-1}u|| < \infty, \quad \forall u \in D(r). \tag{5.1}$$

We have :

$$\frac{e^{isr}-1}{s}(z-\not{D})^{-1} = \frac{e^{isr}(z-\not{D})^{-1}e^{-isr} - (z-\not{D})^{-1}}{s} + \frac{e^{isr}(z-\not{D})^{-1}(1-e^{-isr})}{s}.$$

Clearly :

$$\sup_{|s|\leq 1} ||e^{isr}(z-\not{D})^{-1}\frac{1-e^{-isr}}{s}u|| < \infty, \quad \forall u \in D(r)\,.$$

Moreover

$$\frac{e^{isr}(z-\not{D})^{-1}e^{-isr} - (z-\not{D})^{-1}}{s} = (z-\not{D}_s)^{-1}\frac{\not{D}_s - \not{D}}{s}(z-\not{D})^{-1}$$

with

$$\not{D}_s = e^{isr}\not{D}e^{-isr}, \quad \not{D}_s - \not{D} = -s\gamma h^2.$$

Using $(z-\not{D}_s)^{-1} = e^{isr}(z-\not{D})^{-1}e^{-isr}$, this gives (5.1).

Let us now suppose :

$$(z - \not{D})^{-1} \ : D(<r>^n) \to D(<r>^n)$$

and show that

$$(z - \not{D})^{-1} \ : D(<r>^{n+1}) \to D(<r>^{n+1})\,. \tag{5.2}$$

If $u \in D(<r>^{n+1})$, then $<r>u \in D(<r>^n)$ and

$$<r>^{n+1}(z - \not{D})^{-1} = <r>^n (<r>(z - \not{D})^{-1} <r>^{-1}) <r> u\,.$$

In order to prove (5.2), it is therefore sufficient to show :

$$<r>(z - \not{D})^{-1} <r>^{-1} \ : D(<r>^n) \to D(<r>^n)\,.$$

We have $<r>(z - \not{D})^{-1} <r>^{-1} = (z - <r>\not{D}<r>^{-1})^{-1}$ and $<r>\not{D}<r>^{-1}$ can be treated in exactly the same way as $\not{D}$. It follows

$$(z - <r>\not{D}<r>^{-1})^{-1} \ : D(<r>^n) \to D(<r>^n)\,. \quad \square$$

Lemma 5.7. *We have $\not{D} \in C^1(<r>)$ and the commutator $[i\not{D}, <r>]$ is bounded.*

Proof.
We use proposition 5.1. By lemma 5.6 we have for all $z \in \mathbb{C} \setminus \sigma(\not{D})$

$$(z - \not{D})^{-1} : D(<r>) \to D(<r>)\,.$$

Furthermore $[i\not{D}, <r>] = h\gamma <r>^{-1} rh$ and this is a bounded operator. $\square$

5.4 Conjugate operator for $\not{D}$

Let $F \in C^\infty(\mathbb{R})$ with $F(x) = 0$ for $x \geq 1$ and $F(x) = 1$ for $x \leq \frac{1}{2}$. Let $\eta > 0$ be the constant of section 3.1. We define

$$F_S \ := F\left(\frac{\eta r + \ln|\not{D}_{S^2}|}{S}\right).$$

Note that F_S is well defined because $0 \notin \sigma(\not{D}_{S^2})$. Let $j_\pm \in C^\infty(\mathbb{R})$, $j_\pm \geq 0$ with $j_-(x) = 1$ for $x \leq 0$, $j_-(x) = 0$ for $x \geq 1$, $j_+(x) = 1$ for $x \geq 1$, $j_+(x) = 0$ for $x \leq 0$ and $j_-^2 + j_+^2 = 1$. Let $j_{\pm,R}(.) = j_\pm(\frac{.}{R})$. We put :

$$\begin{aligned}
K_S &:= (\eta r + \ln|\not{D}_{S^2}|)F_S^2, \quad D(K_S) = \{u \in \mathcal{H}, K_S u \in \mathcal{H}\}\,,\\
X_-(r, \not{D}_{S^2}) &:= j_{-,R}^2(r) K_S\,,\\
X_+(r) &:= r j_{+,R}^2(r)\,,\\
Z &:= X_- + X_+\,.
\end{aligned}$$

We obtain from [32, corollary 5.2.2] :

Lemma 5.8.

1. $|X_-(r,q)| \leq C <r>$ *uniformly in* q, R *for all* S,

   $|X_-^{(i)}(r,q)| \leq C$ *uniformly in* q, R, *for all* $i \geq 1$ *and for all* S,

2. $j^2_{-,R}K_S$ *is bounded from* $D(N)$ *to* $D(D_r)$,
$j^2_{-,R}D_r$ *is bounded from* $D(N)$ *to* $D(K_S)$.

We put:

$$A_- := \frac{1}{2}(X_-(r,\not{D}_{S^2})D_r + hc) + c_1\gamma X_-(r,\not{D}_{S^2}),$$
$$A_+ := \frac{1}{2}(X_+(r)D_r + hc), \quad A := A_- + A_+.$$

Remark 5.1. *In [39], a term of type* $c_1\eta r F\left(\frac{r}{S}\right)\gamma$ *was introduced to treat an electromagnetic scalar potential, constant on the horizon of the Reissner-Nordstrøm black hole. Near the horizon, the effects of rotation in the case of Dirac's equation outside a slow Kerr black hole are similar to the effects of charge on a Reissner-Nordstrøm background. We therefore use the same extra-term as in [39], but conjugated by the unitary transformation introduced in section 5.1. After a cut-off near the horizon, this gives the term* $c_1\gamma X_-(r,\not{D}_{S^2})$ *in* A_-.

By lemma 5.8 the operators $A_\pm, A$ are well defined on $D(N)$.

Remark 5.2. *From now on we will consider systematically all commutators between two of the operators* $\not{D}_0, A_\pm, A, N$ *as quadratic forms on* $D(N)$. *All these operators preserve* $\mathcal{H}_{nl}$, *hence it is sufficient to calculate the commutators on* $D(N^{nl})$ *; in fact we can even do these calculations on* $(C_0^\infty(\mathbb{R}))^2$ *using the density of* $(C_0^\infty(\mathbb{R}))^2$ *in* $D(N^{nl})$. *This justifies in particular the application of the Leibniz rule. In order to extend the commutators on larger spaces, we need to obtain estimates that are uniform in* n, l.

Lemma 5.9. *The pairs* $(\not{D}_0, N)$ *and* (A, N) *satisfy the hypotheses of lemma 5.1.*

Proof.
Let us start with $(\not{D}_0, N)$:

$$D(N) \subset D(\not{D}_0^2) \subset D(\not{D}_0),$$
$$||\not{D}_0 u||^2 \leq C(||\not{D}_0^2 u|| + ||u||^2) \leq C||Nu||^2, \ \forall u \in D(N).$$

For $u \in D(N)$, we have :

$$|[i\not{D}_0, N](u,u)| \leq |(2\gamma r u, u)| + 2|(f'(r)u, D_r u)| + 2|(g'(r)\not{D}_{S^2}u, D_r u)| \leq C(Nu, u).$$

The proof for (A, N) is similar to the proof of [32, lemme 5.2.4]. We have one extra term which is

$$(c_1\gamma X'_- D_r + hc)(u,u) \leq C(Nu, u).$$

We omit the details. □

Lemma 5.10. *We have* $\not{D} \in C^1(A)$ *and the commutator* $[i\not{D}, A]$ *can be extended to a bounded operator from* $D(\not{D})$ *to* $\mathcal{H}$, *that we denote* $[i\not{D}, A]_0$.

Proof.
We use lemma 5.2. We will show

$$\forall u \in D(N) \quad |(Au, \not{D}u) - (\not{D}u, Au)| \leq C(||\not{D}u||||u|| + ||u||^2). \tag{5.3}$$

1st step : We will estimate $|[i\not{D}_0, A](u,u)|$.

1.1 Let us first estimate $|[i\not{D}_0, A_-](u,u)|$. We have as a quadratic form on $D(N^{nl})$:

$$[i\not{D}_0^{nl}, A_-^{nl}] = \gamma X_-' D_r - X_-(g'(r)\tau q + f'(r)) + c_1 X_-' + [ig(r)\tau q, c_1\gamma X_-]\ ,$$

i.e.

$$\begin{aligned} [i\not{D}_0^{nl}, A_-^{nl}](u_{nl}, u_{nl}) &= (\gamma X_-' D_r u_{nl}, u_{nl}) - (X_-(g'(r)\tau q + f'(r))u_{nl}, u_{nl}) \\ &\quad + (c_1\gamma X_-' u_{nl}, u_{nl}) + [ig(r)\tau q, c_1\gamma X_-]\,(u_{nl}, u_{nl})\,. \end{aligned}$$

As X_-' is uniformly bounded in n, l the first term can be estimated by

$$(\gamma X_-' D_r u_{nl}, u_{nl}) \le C(||\not{D}_0^{nl} u_{nl}||||u_{nl}|| + ||u_{nl}||^2),$$

where we have also used lemma 4.7. The second term is in fact bounded:

$$\begin{aligned} |X_- g'(r)\tau q| &\le C|X_- e^{\eta r + \ln q}| \le C, \\ |X_- f'(r)| &\le C \end{aligned}$$

where we have used $g \in \mathbf{S}^{-1,-1}$, $f' \in \mathbf{S}^{-3}$ and lemma 5.8. The third term is bounded by lemma 5.8 again and the last is bounded uniformly in q :

$$|g(r) q X_- c_1| \le C \left| X_- e^{\eta r + \ln q} \right| \le C\,.$$

1.2 Let us now estimate $|[i\not{D}_0, A_+](u,u)|$. As a quadratic form on $D(N^{nl})$ we find :

$$[i\not{D}_0^{nl}, A_+^{nl}] \;=\; \gamma X_+' D_r - X_+(g'(r)\tau q + f'(r))\,.$$

We obtain the estimate for the first term using the fact that X_+' is bounded. In order to estimate the second term we use lemma 4.7 and

$$|X_+ g'(r)| \le C|g(r)|, \quad |X_+ f'(r)| \le C.$$

2nd step : We have now to estimate $[i\not{D}, A]$. We have

$$[i\not{D}, A] = h[\not{D}_0, A]h + h\not{D}_0[h, A] + [h, A]\not{D}_0 h + [V, A].$$

We have $[V, A] = -ZV'$, $[h, A] = -Zh'$. The estimate now follows from the fact that

$$h, Zh', ZV' : D(\not{D}_0) \to D(\not{D}_0)\,. \quad \square$$

Using lemma 5.3 we get:

Corollary 5.1. *The pair $(\not{D}, A)$ satisfies the Mourre conditions (M1), (M2).*

We obtain from [32, lemme A.2.1 and lemme A.2.2] :

Lemma 5.11. *We have $|Z^{(i)} Z^{(k)}| \le C$ and $|Z^{(i)} X_-^{(k)}| \le C$ uniformly in q if $i + k \ge 2$.*

Lemma 5.12. *Let $i, j, k \in \mathbb{N}$. We have uniformly in q :*

$$|g^{(i)} q X_-^{(j)}| \le C, \quad |X_-^{(k)} g^{(i)} q X_-^{(j)}| \le C\,.$$

If in addition, $3 \ge i \ge 1$ and $i + k \ge 2$, then we have uniformly in q

$$|g^{(i)} q X_+^{(j)}| \le C g(r) q, \quad |X_+^{(j)} g^{(i)} q X_+^{(k)}| \le C g(r) q,$$
$$|X_-^{(j)} g^{(i)} q X_+^{(k)}| \le C, \quad |X_+^{(j)} g^{(i)} q X_-^{(k)}| \le C.$$

Lemma 5.13. *The double commutator $[[i\not{D}, A]_0, A]$ defined as a quadratic form on $D(N)$ can be extended to a bounded operator from $\mathcal{H}^1$ to $\mathcal{H}$.*

Proof.
Recall from the proof of lemma 5.10 that

$$[i\not{D}_0^{nl}, A^{nl}] = \gamma Z' D_r - Z(g'(r)\tau q + f'(r)) + c_1 X_-' + [ig(r)\tau q, c_1 \gamma X_-]\,.$$

So we get:

$$\begin{aligned}
[[i\not{D}_0^{nl}, A^{nl}], iA^{nl}] &= \gamma (Z')^2 D_r - \gamma Z Z'' D_r + c_1 Z' X_-' + Z(Z(g'(r)\tau q + f'(r)))' \\
&- \left[Z\left(g'(r)\tau q + f'(r)\right), ic_1 \gamma X_-\right] - c_1 Z X_-'' \\
&- Z\left([ig(r)\tau q, c_1 \gamma X_-]\right)' + \left[[ig(r)\tau q, c_1 \gamma X_-], ic_1 \gamma X_-\right]
\end{aligned}$$

and the lemma follows with $\not{D}$ replaced by $\not{D}_0$ using lemmata 4.7, 5.8, 5.11 and 5.12. Recall now that $\not{D} = h\not{D}_0 h + V$. So we have (see the proof of lemma 5.10) :

$$\begin{aligned}
[i\not{D}, A] &= -h\not{D}_0 h' Z - h' Z \not{D}_0 h + h[i\not{D}_0, A]h - ZV', \\
[[i\not{D}, A], iA] &= h' Z \not{D}_0 h' Z + h\not{D}_0 (h'Z)' Z + (h'Z)' Z \not{D}_0 h + h' Z \not{D}_0 h' Z \\
&- h[i\not{D}_0, A]h' Z - h' Z [i\not{D}_0, A]h - h' Z[i\not{D}_0, A]h - h[i\not{D}_0, A]h' Z \\
&+ h[[i\not{D}_0, A], iA]h + (ZV')' Z.
\end{aligned}$$

Using $h' \in \mathbf{S}^{-2}, V_{ij} \in \mathbf{S}^{-2}$ we observe that

$$h'Z, (h'Z)'Z, (ZV')'Z : D(\not{D}_0) \to D(\not{D}_0),$$

so the double commutator is bounded from $\mathcal{H}^1 \to \mathcal{H}$. □

5.5 The Mourre estimate for $\not{D}$

Let us start with some technical lemmata.

Lemma 5.14. *Let $\chi \in C_0^\infty(\mathbb{R})$. Then*

$$j_{-,R}(\chi(\not{D}_0) - \chi(\not{D}_e)) \textit{ is compact.}$$

Proof.
Using the Helffer-Sjöstrand formula, it is sufficient to show :

$$j_{-,R}(z - \not{D}_0)^{-1}(\not{D}_0 - \not{D}_e)(z - \not{D}_e)^{-1} \text{ is compact for all } z \in \mathbb{C} \setminus (\sigma(\not{D}_0) \cup \sigma(\not{D}_e)), \tag{5.4}$$
$$||j_{-,R}(z - \not{D}_0)^{-1}(\not{D}_0 - \not{D}_e)(z - \not{D}_e)^{-1}|| \le C|\mathrm{Imz}|^{-2}. \tag{5.5}$$

(5.5) is clear, let us show (5.4). We have:

$$j_{-,R}(z-\not{D}_0)^{-1}(\not{D}_0-\not{D}_e)(z-\not{D}_e)^{-1} = \begin{array}{ll} - & (z-\not{D}_0)^{-1}\gamma j'_{-,R}(z-\not{D}_0)^{-1}(\not{D}_0-\not{D}_e)(z-\not{D}_e)^{-1} \\ + & (z-\not{D}_0)^{-1}j_{-,R}(g(r)-c_0e^{\eta r})\not{D}_{S^2}(z-\not{D}_e)^{-1}. \end{array}$$

Both terms are compact by (3.8), corollary 3.1 and lemma 3.2. □

Let us put $T^{nl}=\ln(q)D_r$, $D(T^{nl})=\mathcal{H}^1_{nl}$. $(T^{nl},D(T^{nl}))$ is clearly selfadjoint and the operator $T:=\ln|\not{D}_{S^2}|D_r$ is selfadjoint with domain :

$$D(T)=\{u=\sum_{nl}u_{nl}\,;\,u_{nl}\in D(T^{nl}),\,\sum_{nl}||(T^{nl}+i)u_{nl}||^2<\infty\}.$$

Lemma 5.15. *Let $f,\chi\in C_\infty(\mathbb{R})$. Then*

$$f(\eta r+\ln q)\chi(\not{D}_e^{nl})\text{ is compact on }\mathcal{H}_{nl}.$$

Proof.

It is sufficient to show that

$$e^{-\frac{1}{\eta}iD_r\ln q}f(\eta r+\ln q)\chi(\not{D}_e^{nl})e^{\frac{1}{\eta}iD_r\ln q}=f(\eta r)\chi(\gamma D_r+c_1+c_0e^{\eta r}\tau)$$

is compact and this follows from corollary 3.1. □

Lemma 5.16. *Let $f\in C_\infty(\mathbb{R}),\lambda\in\mathbb{R}$.*

$$\forall\epsilon>0,\,\exists\delta>0\quad||f(\eta r+\ln|\not{D}_{S^2}|)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_e)||<\epsilon.$$

Proof.

We have:

$$\begin{aligned}||f(\eta r+\ln|\not{D}_{S^2}|)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_e)|| &= ||e^{-\frac{i}{\eta}T}f(\eta r+\ln|\not{D}_{S^2}|)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_e)e^{\frac{i}{\eta}T}||\\ &= ||f(\eta r)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\gamma D_r+c_0e^{\eta r}\frac{\not{D}_{S^2}}{|\not{D}_{S^2}|}+c_1)||\end{aligned}$$

and it is sufficient to show:

$$||f(\eta r)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\gamma D_r+c_0e^{\eta r}\tau+c_1)||<\epsilon\,,$$

uniformly in q. The operator

$$f(\eta r)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\gamma D_r+c_0e^{\eta r}\tau+c_1)$$

is compact by corollary 3.1. So for any given $\epsilon>0$, we can find $\delta>0$ s.t.

$$||f(\eta r)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\gamma D_r+c_0e^{\eta r}\tau+c_1)||<\varepsilon\,.$$

This concludes the proof of lemma 5.16. □

The following corollary estimates a remainder term in the Mourre estimate.

Corollary 5.2. *For all $S,R>0$, $\epsilon>0$, $\lambda\in\mathbb{R}$, there exists $\delta>0$ such that :*

$$\begin{aligned}&\Big\|\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)\left(-\eta g(r)\not{D}_{S^2}F_S^2j_{-,R}^2+j_{-,R}^2(\eta r+ln|\not{D}_{S^2}|)(F_S^2)'\gamma D_r-X_-g'(r)\not{D}_{S^2}\right.\\ &\left.+c_1j_{-,R}^2(\eta r+\ln|\not{D}_{S^2}|)\left(F_S^2\right)'+ig(r)\not{D}_{S^2}X_-\gamma c_1-i\gamma c_1X_-g(r)\not{D}_{S^2}\right)\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)\Big\|<\epsilon\,.\end{aligned}$$

Proof.
Let us treat

$$\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)\eta g(r)\not{D}_{S^2}F_S^2 j_{-,R}^2 \mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0).$$

Using that

$$\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)(g(r)-c_0e^{\eta r})\not{D}_{S^2}F_S^2 j_{-,R}^2 \text{ and } (\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)-\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_e))j_{-,R}^2$$

are compact and that $e^{\eta r}\not{D}_{S^2}F_S^2$ is bounded, it is sufficient to treat

$$\mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_e)c_0e^{\eta r}\not{D}_{S^2}F_S^2$$

and this term can be estimated using lemma 5.16. The estimation of the remaining terms is analogous. □

Lemma 5.17. *We have*

(i) *$j_{\pm,R}$ preserves $D(\not{D}^2)=D(\not{D}_0^2)$, its norm in $\mathcal{L}\left(D(\not{D}^2)\right)$ is bounded uniformly in R,*

(ii) *F_S preserves $D(\not{D}^2)=D(\not{D}_0^2)$, its norm in $\mathcal{L}\left(D(\not{D}^2)\right)$ is bounded uniformly in S,*

(iii) *$W:=j_{-,R}(1-F_S)$ preserves $D(\not{D}^2)=D(\not{D}_0^2)$, its norm in $\mathcal{L}(D(\not{D}^2))$ is bounded uniformly in R,S.*

We have an analogous statement if we replace $\not{D}^2$ (resp. $\not{D}_0^2$) by $\not{D}$ (resp. $\not{D}_0$).

Proof.
We have :

$$[(z-\not{D}_0^2)^{-1}, ij_{\pm,R}] = (z-\not{D}_0^2)^{-1}(2f(r)\gamma j'_{\pm,R}+j'_{\pm,R}D_r+D_rj'_{\pm,R})\frac{1}{R}(z-\not{D}_0^2)^{-1},$$

which gives

$$[(z-\not{D}_0^2)^{-1}, j_{\pm,R}] \in O(R^{-1})|\mathrm{Im} z|^{-2} \text{ for } z\in K\subset\subset\mathbb{C}. \tag{5.6}$$

In the same manner we calculate :

$$[(z-\not{D}_0^2)^{-1},F_S] = (z-\not{D}_0^2)^{-1}[\not{D}_0^2,F_S](z-\not{D}_0^2)^{-1}\,.$$

Therefore,

$$[(z-\not{D}_0^2)^{-1},F_S]\in O(S^{-1})|\mathrm{Im} z|^{-2} \tag{5.7}$$

and F_S preserves $D(\not{D}_0^2)$. Finally (5.6) and (5.7) give *(iii)*. □

Lemma 5.18. *If* $\mathrm{supp}\chi\subset]0,\infty[$, *then*

$$\lim_{S\to\infty}||\chi(\not{D}_0)W|| = 0 \textit{ uniformly in } R \textit{ large.}$$

Proof.
Let $\hat{\chi} \in C_0^\infty(]0,\infty[)$ with $\hat{\chi}\chi = \chi$. As supp$\chi \subset]0,\infty[$,

$$\chi(\not{D}_0) = \hat{\chi}(\not{D}_0)\chi(|\not{D}_0|) = \hat{\chi}(\not{D}_0)\tilde{\chi}(\not{D}_0^2)$$

where $\tilde{\chi}(x) = \chi(\sqrt{x})$. We have on supp j_{-R} (R sufficiently large) :

$$\begin{aligned} \not{D}_0^2 &= D_r^2 + g^2(r)\not{D}_{S^2}^2 + \frac{\gamma}{i}g'(r)\not{D}_{S^2} + \gamma D_r f(r) + f(r)\gamma D_r + f^2(r) + 2f(r)g(r)\not{D}_{S^2} \\ &\geq g^2(r)\not{D}_{S^2}^2 - Cg(r)|\not{D}_{S^2}| - C \\ &\geq C_1 e^{2\eta r}\not{D}_{S^2}^2 - C_2 e^{\eta r}|\not{D}_{S^2}| - C_3 \,. \end{aligned} \tag{5.8}$$

On supp$(1-F_S)j_{-,R}^2$ we have :

$$\ln(e^{\eta r}|\not{D}_{S^2}|) \geq \frac{S}{2}, \quad e^{\eta r}|\not{D}_{S^2}| \geq e^{S/2}.$$

Using (5.8) we get for S large enough :

$$\not{D}_0^2 \geq Ce^{S/2}\,,\ \text{i.e.}\,,\ \forall M > 1\,,\ \exists S_0\,;\ \forall S \geq S_0\,,\ W\not{D}_0^2 W \geq MW^2 \tag{5.9}$$

in the sense of quadratic forms on $D(\not{D}_0)$. Using (5.9), we get :

$$\begin{aligned} (z-\not{D}_0^2)^{-1}W^2(\bar{z}-\not{D}_0^2)^{-1} &\leq \frac{1}{M}(z-\not{D}_0^2)^{-1}W\not{D}_0^2 W(\bar{z}-\not{D}_0^2)^{-1} \\ &= \frac{1}{M}W(z-\not{D}_0^2)^{-1}\not{D}_0^2(\bar{z}-\not{D}_0^2)^{-1}W + \frac{1}{M}O(R^{-1},S^{-1})|\mathrm{Im}z|^{-3} \end{aligned}$$

for $z \in K \subset\subset \mathbb{C}$. This follows from (5.6) and (5.7). We have:

$$(z-\not{D}_0^2)^{-1}\not{D}_0^2(z-\not{D}_0^2)^{-1} \leq |\mathrm{Im}z|^{-2}, z \in K \subset\subset \mathbb{C}.$$

It follows :

$$\begin{aligned} (z-\not{D}_0^2)^{-1}W^2(\bar{z}-\not{D}_0^2)^{-1} &\leq \frac{C}{M}|\mathrm{Im}z|^{-2} + \frac{1}{M}O(R^{-1},S^{-1})|\mathrm{Im}z|^{-3}, \\ ||(z-\not{D}_0^2)^{-1}W|| &\leq \frac{C}{\sqrt{M}}\left(|\mathrm{Im}z|^{-1} + O(R^{-1},S^{-1})|\mathrm{Im}z|^{-\frac{3}{2}}\right),\ z \in K. \end{aligned}$$

Using the Helffer-Sjöstrand formula we obtain:

$$||\tilde{\chi}(\not{D}_0^2)W|| \leq \frac{C}{\sqrt{M}}. \quad \square$$

Lemma 5.19. *We have for R,S large enough: for all $\lambda_0 > 0$, there exists an interval I, neighbourhood of λ_0, and $\mu > 0$ s.t. :*

$$\mathbf{1}_I(\not{D})[i\not{D},A]\mathbf{1}_I(\not{D}) \geq \mu\mathbf{1}_I(\not{D}) + K,\ K \text{ compact}.$$

For $\lambda_0 < 0$, we take $-A$ instead of A and obtain a similar estimate.

Proof.
We work with $\lambda_0 > 0$, the case $\lambda_0 < 0$ is proved similarly.

1st step : We first calculate the commutator $[i\not{D}_0, A]$. We have :

$$[i\not{D}_0^{nl}, A_-^{nl}] = \gamma X'_- D_r - X_-(g'(r)\tau q + f'(r)) + c_1 X'_- + [ig(r)\tau q, c_1\gamma X_-] \,.$$

We have for $\chi \in C_0^\infty(]0,\infty[)$, $0 \le \chi \le 1$:

$$\chi(\not{D}_0^{nl})\gamma X'_- D_r \chi(\not{D}_0^{nl}) = \eta\chi(\not{D}_0^{nl}) j_{-,R} F_S \gamma D_r F_S j_{-,R} \chi(\not{D}_0^{nl}) + O(R^{-1}, S^{-1}) + T_1^{nl} + \tilde{K}_1^{nl} \,,$$

uniformly in n, l, where

$$\begin{aligned} T_1^{nl} &:= \chi(\not{D}_0^{nl}) j_{-,R}^2 (\eta r + \ln q)(F_S^2)' \gamma D_r \chi(\not{D}_0^{nl}) \,, \\ \tilde{K}_1^{nl} &:= \chi(\not{D}_0^{nl}) (j_{-,R}^2)' (\eta r + \ln q) F_S^2 \gamma D_r \chi(\not{D}_0^{nl}) \,. \end{aligned}$$

We put

$$T_1 := \bigoplus_{n,l} T_1^{nl} \ , \quad \tilde{K}_1 := \bigoplus_{n,l} \tilde{K}_1^{nl} \,.$$

The operator $\tilde{K}_1$ is compact by lemma 5.8 and corollary 4.2. Using lemma 5.17, we obtain:

$$\chi(\not{D}_0)[i\not{D}_0, A_-]\chi(\not{D}_0) = \eta\chi(\not{D}_0) j_{-,R} F_S \not{D}_0 F_S j_{-,R} \chi(\not{D}_0) + O(R^{-1}, S^{-1}) + \tilde{K}_1 + R_1 + T \,, \quad (5.10)$$

with

$$\begin{aligned} R_1 &= -\chi(\not{D}_0) \left(X_- f'(r) - c_1 (j_{-,R}^2(\eta r + \ln|\not{D}_{S^2}|))' F_S^2 + \eta j_{-,R} F_S f(r) F_S j_{-,R} \right) \chi(\not{D}_0) \,, \\ T &= -\chi(\not{D}_0) \Big(X_- g'(r) \not{D}_{S^2} + \eta j_{-,R} F_S g(r) \not{D}_{S^2} F_S j_{-,R} \\ &\quad - c_1 j_{-,R}^2 \left(\eta r + \ln|\not{D}_{S^2}|\right) \left(F_S^2\right)' - ig(r)\not{D}_{S^2} c_1 \gamma X_- + i c_1 \gamma X_- g(r) \not{D}_{S^2} \Big) \chi(\not{D}_0) + T_1 \,, \end{aligned}$$

as an identity between quadratic forms on $D(\not{D}_0)$. We first estimate R_1. We have :

$$\chi(\not{D}_0)\eta j_{-,R}^2 F_S^2 f(r) \chi(\not{D}_0) = \eta\chi(\not{D}_0)(F_S^2 - 1) j_{-,R}^2 f(r) \chi(\not{D}_0) + \eta\chi(\not{D}_0) j_{-,R}^2 f(r) \chi(\not{D}_0)$$

and $\lim_{S\to\infty} ||\chi(\not{D}_0)(F_S^2 - 1) j_{-,R}^2 f(r) \chi(\not{D}_0)|| = 0$ using lemma 5.18. We put

$$\hat{R}_1 := R_1 + \eta\chi(\not{D}_0)(F_S^2 - 1) j_{-,R}^2 f(r) \chi(\not{D}_0).$$

Let us now consider the term $\chi(\not{D}_0) c_1 (j_{-,R}^2(\eta r + \ln|\not{D}_{S^2}|))' F_S^2 \chi(\not{D}_0)$. We have

$$\begin{aligned} \chi(\not{D}_0) c_1 \left(j_{-,R}^2(\eta r + \ln|\not{D}_{S^2}|)\right)' F_S^2 \chi(\not{D}_0) &= \chi(\not{D}_0) c_1 \left(j_{-,R}^2\right)' \left(\eta r + \ln|\not{D}_{S^2}|\right) F_S^2 \chi(\not{D}_0) \\ &\quad + \chi(\not{D}_0) c_1 j_{-,R}^2 \eta F_S^2 \chi(\not{D}_0) \,. \end{aligned} \quad (5.11)$$

We use lemma 5.18 to obtain

$$\lim_{S\to\infty} \left\| \chi(\not{D}_0) c_1 j_{-,R}^2 \left(F_S^2 - 1\right) \chi(\not{D}_0) \right\| = 0 \,.$$

We put

$$\hat{K}_1 := \hat{R}_1 - \chi(\not{D}_0) c_1 j_{-,R}^2 \left(F_S^2 - 1\right) \chi(\not{D}_0) \,.$$

Let us show that $\hat{K}_1$ is compact. We first note that the first term in (5.11) is compact by corollary 4.2 and lemma 5.8. Furthermore, $(\eta(c_1 - f(r)) j_{-,R}^2) \mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)$ is compact by (3.9) and corollary 4.2. Besides,

$$X_- f'(r) \mathbf{1}_{[\lambda-\delta,\lambda+\delta]}(\not{D}_0)$$

is compact by corollary 4.2 and lemma 5.8. We introduce the compact operator

$$K_1 := \tilde{K}_1 + \hat{K}_1 .$$

Let us now treat the first term in (5.10). We have :

$$\begin{aligned} &\chi(\not{D}_0) j_{-,R} F_S \not{D}_0 F_S j_{-,R} \chi(\not{D}_0) \\ &= \chi(\not{D}_0) j_{-,R} \not{D}_0 j_{-,R} \chi(\not{D}_0) + \chi(\not{D}_0) j_{-,R} \not{D}_0 j_{-,R} (F_S - 1) \chi(\not{D}_0) \\ &+ \chi(\not{D}_0) j_{-,R} (F_S - 1) \not{D}_0 j_{-,R} F_S \chi(\not{D}_0) . \end{aligned} \tag{5.12}$$

We know by lemma 5.17 that $||\chi(\not{D}_0) j_{-,R} \not{D}_0||$ is bounded uniformly in R and

$$||j_{-,R}(F_S - 1)\chi(\not{D}_0)|| \to 0 \text{ as } S \to \infty .$$

This estimates the second term in (5.12). The last term can be treated in the same manner. We obtain :

$$\chi(\not{D}_0)[i\not{D}_0, A_-]\chi(\not{D}_0) \geq \eta \chi(\not{D}_0) j_{-,R} \not{D}_0 j_{-,R} \chi(\not{D}_0) - \epsilon \chi^2(\not{D}_0) + T + K_1$$

if R, S large enough.

Let us now estimate $[i\not{D}_0, A_+]$. We obtain :

$$\begin{aligned} &\chi(\not{D}_0)[i\not{D}_0, A_+]\chi(\not{D}_0) \\ &= \chi(\not{D}_0) j_{+,R} \gamma D_r j_{+,R} \chi(\not{D}_0) - \chi(\not{D}_0) X_+ (g'(r)\not{D}_{S^2} + f'(r))\chi(\not{D}_0) + k, \end{aligned}$$

where k is a compact operator. We now use the fact that the operator

$$\begin{aligned} &X_+(g'(r)\not{D}_{S^2} + f'(r))\chi(\not{D}_0) + (g(r)\not{D}_{S^2} + f(r)) j_{+,R}^2 \chi(\not{D}_0) \\ &= ((X_+ g'(r) + j_{+,R}^2 g(r))\not{D}_{S^2} + (X_+ f'(r) + j_{+,R}^2 f(r)))\chi(\not{D}_0) \\ &= \left(j_{+,R}^2 (r g'(r) + g(r))\not{D}_{S^2} + j_{+,R}^2 (r f'(r) + f(r)) \right) \chi(\not{D}_0) \end{aligned}$$

is compact, by (3.10), (3.11) and corollary 4.2, to obtain :

$$\chi(\not{D}_0)[i\not{D}_0, A_+]\chi(\not{D}_0) = \chi(\not{D}_0) j_{+,R} \not{D}_0 j_{+,R} \chi(\not{D}_0) + K_2$$

with some compact operator K_2. Putting everything together, we find for R, S large enough :

$$\begin{aligned} &\chi(\not{D}_0)[i\not{D}_0, A]\chi(\not{D}_0) \\ &\geq \eta \chi(\not{D}_0) j_{-,R} \not{D}_0 j_{-,R} \chi(\not{D}_0) + \chi(\not{D}_0) j_{+,R} \not{D}_0 j_{+,R} \chi(\not{D}_0) - \epsilon \chi^2(\not{D}_0) + T + K_1 + K_2 . \end{aligned}$$

Using the compactness of $\chi(\not{D}_0) j'_{\pm,R}$, we obtain :

$$\chi(\not{D}_0)[i\not{D}_0, A]\chi(\not{D}_0) \geq \tilde{\mu} \chi^2(\not{D}_0) + K + T \quad \text{with a compact operator } K \text{ and } \tilde{\mu} > 0.$$

We fix now R, S large enough. We can apply corollary 5.2 to obtain:

$$\chi(\not{D}_0)[i\not{D}_0, A]\chi(\not{D}_0) \geq \mu \chi^2(\not{D}_0) \ , \quad \mu > 0 \, ,$$

if the support of χ is sufficiently small.

2$^{\text{nd}}$ step : We now estimate $[i\not{D}_1, A]$. The operator $\chi(\not{D})[i\not{D}_1, A]\chi(\not{D})$ is in fact compact. Let us write :

$$\begin{aligned}
&\chi(\not{D})\not{D}_1 A\chi(\not{D}) \\
&= \ (\chi(\not{D})(\not{D}_0 + i) < r >^{-1}) \times ((\not{D}_0 + i)^{-1} < r > \not{D}_1 < r >< r >^{-1} A\chi(\not{D})) \\
&+ \ \chi(\not{D})(\not{D}_0 + i) < r >^{-1} (\not{D}_0 + i)^{-1}[\gamma D_r, < r >] \\
&\times \ (\not{D}_0 + i)^{-1}\not{D}_1 < r >< r >^{-1} A\chi(\not{D}).
\end{aligned}$$

The first factor of each term is compact, the others are bounded. This concludes the proof of the lemma using that $\chi(\not{D}) - \chi(\not{D}_0)$ is compact. □

Using [42] we obtain the following consequence of the limiting absorption principle :

Theorem 4. *For all* $\chi \in C_0^\infty(\mathbb{R} \setminus (\{0\} \cup \sigma_{pp}(\not{D})))$, $\mu > \frac{1}{2}$, $\psi \in \mathcal{H}$, *we have*

$$\int_0^\infty || < A >^{-\mu} e^{it\not{D}}\chi(\not{D})\psi||^2 dt \leq C||\psi||^2.$$

The operator $\not{D}$ *has no singular continuous spectrum and the pure point spectrum is locally finite in* $\mathbb{R} \setminus \{0\}$.

5.6 The Mourre estimate for $\not{D}_\nu$, $\nu \in \mathcal{N}$.

Let us first remark that we cannot apply directly the results of the previous sections to $\not{D}_\pm$. The situation for $\not{D}_\nu$, $\nu \in \mathcal{N}$ is however much simpler as we can restrict our attention to the subspaces of spherical harmonics. So we shall work in what follows with the spaces $\mathcal{H}_{nl}$ and the operators $\not{D}_\nu^{nl}$. We will drop the indices n, l. We define :

$$B_\nu \ := \ \frac{1}{2}(rD_r + D_r r) + \gamma c_1^\nu r\,,\ c_1^\nu = \begin{cases} c_1 & \text{if } \nu = -\,, \\ c_1 j_-^2 & \text{if } \nu = 0\,, \\ 0 & \text{if } \nu = +\,. \end{cases}$$

Let $N := D_r^2 + r^2 + 1$, $D(N) = \{\psi \in \mathcal{H},\ N\psi \in \mathcal{H}\}$. N is selfadjoint with this domain and we have also :

$$D(N) = (H^2(\mathbb{R}))^2 \cap D(r^2).$$

All commutators in this subsection are defined as quadratic forms on $D(N)$. As $(C_0^\infty(\mathbb{R}))^2$ is dense in $D(N)$, it is sufficient to calculate them on $(C_0^\infty(\mathbb{R}))^2$.

Lemma 5.20. *The pairs* (B_ν, N) *and* $(\not{D}_\nu, N)$, $\nu \in \mathcal{N}$ *satisfy the hypotheses of lemma 5.1.*

Proof.
The proof for (B_ν, N) is contained in the proof of [39, lemma 4.5]. Let us treat $(\not{D}_\nu, N)$:

$$\begin{aligned}
D(N) \ &\subset \ D(\not{D}_\nu^2) = (H^2(\mathbb{R}))^2 \text{ and} \\
||\not{D}_\nu u||^2 \ &\leq \ C(||\not{D}_\nu^2 u|| + ||u||^2) \\
&\leq \ C||Nu||^2,\ \forall u \in D(N).
\end{aligned}$$

We calculate :

$$[i\not{D}_\nu, N] = 2\gamma r + D_r g'\tau q + g'\tau q D_r + D_r f_\nu' + f_\nu' D_r\,.$$

This implies:

$$|[i\not{D}_\nu, N](u, u)| \leq C(Nu, u). \quad \square$$

Lemma 5.21. *We have:*

$$\forall \nu \in \mathcal{N}, \forall z \in \mathbb{C} \setminus \sigma(\not{\mathfrak{D}}_\nu) \quad (z - \not{\mathfrak{D}}_\nu)^{-1} \ : D(N) \to D(N).$$

The argument for the proof is the same as in the proof of lemma 5.6, we omit the details.

Lemma 5.22. *We have $\mathcal{D}_\nu \in C^1(B_\nu)$ for all $\nu \in \mathcal{N}$ and the commutator $[i\not{\mathfrak{D}}_\nu, B_\nu]$ can be extended to a bounded operator from $D(\not{\mathfrak{D}}_\nu)$ to $\mathcal{H}$, that we denote by $[i\not{\mathfrak{D}}_\nu, B_\nu]_0$.*

Proof.

We use lemma 5.2. We show :

$$|(B_\nu u, \not{\mathfrak{D}}_\nu u) - (\not{\mathfrak{D}}_\nu u, B_\nu u)| \leq C(||\not{\mathfrak{D}}_\nu u|| ||u|| + ||u||^2). \tag{5.13}$$

We have

$$[i\not{\mathfrak{D}}_\nu, B_\nu] = \gamma D_r - r g_\nu' \tau q - r f_\nu' + (c_1^\nu r)' + [i g_\nu \tau q, \gamma c_1^\nu r] \,.$$

This gives the desired estimate by lemma 4.7 and (3.8)-(3.11). □

Using lemma 5.3 we obtain :

Corollary 5.3. *The pair $(\not{\mathfrak{D}}_\nu, B_\nu)$ satisfies the Mourre conditions (M1), (M2) for all $\nu \in \mathcal{N}$.*

Lemma 5.23. *For all $\nu \in \mathcal{N}$ the double commutator $[iB_\nu, [i\not{\mathfrak{D}}_\nu, B_\nu]_0]$, defined as a quadratic form on $D(N)$, can be extended to a bounded operator from $D(\not{\mathfrak{D}}_\nu)$ to $\mathcal{H}$.*

Proof.

We have :

$$\begin{aligned}[][[i\not{\mathfrak{D}}_\nu, B_\nu], iB_\nu] \;=\;& \gamma D_r + r(r g_\nu')' \tau q + r(r f_\nu)' - r(c_1^\nu r)'' \\ & - r\left([i g_\nu \tau q, \gamma c_1^\nu r]\right)' + (c_1^\nu r)' + [[i g_\nu \tau q, \gamma c_1^\nu r], i\gamma c_1^\nu r]\end{aligned}$$

and the estimate for the double commutator follows in the same way as for the commutator. □

Lemma 5.24. *Let $\nu \in \mathcal{N}$. For all $\lambda_0 > 0$ there exists a neighbourhood I of λ_0 and $\mu > 0$ s.t.*

$$\mathbf{1}_I(\not{\mathfrak{D}}_\nu)[i\not{\mathfrak{D}}_\nu, B_\nu]\mathbf{1}_I(\not{\mathfrak{D}}_\nu) \geq \mu \mathbf{1}_I(\not{\mathfrak{D}}_\nu) + k_\nu$$

where k_ν is a compact operator. For $\lambda_0 < 0$, taking $-B_\nu$ instead of B_ν, we obtain a similar estimate.

Proof.

We work in the case $\lambda_0 > 0$, the proof is similar for $\lambda_0 < 0$. By the proof of lemma 5.22 we have :

$$[i\not{\mathfrak{D}}_\nu, B_\nu] = \not{\mathfrak{D}}_\nu + (c_1^\nu r)' - f_\nu - r f_\nu' - r g_\nu' \tau q - g_\nu \tau q + [i g_\nu \tau q, \gamma c_1^\nu r] \,.$$

Using corollary 3.1 and (3.8)-(3.11), we see that for $\chi \in \mathcal{C}_0^\infty(]0, +\infty[)$,

$$\chi(\not{\mathfrak{D}}_\nu)(r f_\nu' + r g_\nu' \tau q + g_\nu \tau q - (c_1^\nu r)' + f_\nu - [i g_\nu \tau q, \gamma c_1^\nu r])\chi(\not{\mathfrak{D}}_\nu)$$

is compact. Putting everything together we find :

$$\chi(\not{\mathfrak{D}}_\nu)[i\not{\mathfrak{D}}_\nu, B_\nu]\chi(\not{\mathfrak{D}}_\nu) \geq \mu \chi^2(\not{\mathfrak{D}}_\nu) + k_\nu$$

where k_ν is compact. □

Using [42] we obtain the following consequence of the limiting absorption principle :

Theorem 5. *For all $\nu \in \mathcal{N}$, $\chi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, $\mu > \frac{1}{2}$ and $\psi \in \mathcal{H}$, we have*

$$\int_0^\infty || < B_\nu >^{-\mu} e^{it\not{\mathfrak{D}}_\nu} \chi(\not{\mathfrak{D}}_\nu)\psi||^2 dt \leq C||\psi||^2.$$

The operators $\not{\mathfrak{D}}_\nu$ have no singular continuous spectrum.

6 Asymptotic completeness

Let $j_\pm \in C_b^\infty(\mathbb{R})$ be a partition of unity as follows : there exists $R > 0$ such that

$$j_-(r) = 0\,\forall r \geq R\,,\; j_-(r) = 1\,\forall r \leq -R\,,$$
$$j_+(r) = 0\,\forall r \leq -R\,,\; j_+(r) = 1\,\forall r \geq R\,,$$
$$j_+^2 + j_-^2 = 1\,.$$

6.1 Technical results

Lemma 6.1. *Let $\chi \in C_0^\infty(\mathbb{R})$. Then the operator $< A > \chi(\not{D}) < r >^{-1}$ is bounded.*

Proof.
We first show that $[< A >, \chi(\not{D})]$ is bounded. Using the Helffer-Sjöstrand formula, it is sufficient to show that $(z - \not{D})^{-1}[< A >, \not{D}](z - \not{D})^{-1}$ is bounded and that

$$\|(z - \not{D})^{-1}[< A >, \not{D}](z - \not{D})^{-1}\| \leq C|\mathrm{Im}z|^{-2}.$$

This follows from the commutator and double commutator estimates and from [15, lemma C.3.2]. It remains to show that $\chi(\not{D}) < A >< r >^{-1}$ is bounded. This follows from lemma 5.8. □

Lemma 6.2. *Let $\chi \in C_0^\infty(\mathbb{R})$. Then the operator $< A > \chi(\not{D})(\not{D} - \not{D}_0)\chi(\not{D}_0) < A >$ is bounded.*

Proof.
Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\chi}\chi = \chi$. We have :

$$\begin{aligned} &< A > \chi(\not{D})(\not{D} - \not{D}_0)\chi(\not{D}_0) < A > \\ &= \;< A > \chi(\not{D}) < r >^{-1}< r > \tilde{\chi}(\not{D})(\not{D} - \not{D}_0)\tilde{\chi}(\not{D}_0) < r >< r >^{-1} \chi(\not{D}_0) < A > \end{aligned}$$

and it is sufficient to show that

$$< r > \tilde{\chi}(\not{D})(\not{D} - \not{D}_0)\tilde{\chi}(\not{D}_0) < r > \text{ is bounded.}$$

By lemma 5.7 $[< r >, \tilde{\chi}(\not{D})]$ is bounded from $\mathcal{H}$ to $\mathcal{H}^1$, so it remains to show that

$$\tilde{\chi}(\not{D}) < r > (\not{D} - \not{D}_0) < r > \tilde{\chi}(\not{D}_0) \text{ is bounded,}$$

which is a consequence of lemma 4.8. □

6.2 Comparison with the intermediate dynamics

Theorem 6. *The limits*

$$s - \lim_{t\to\infty} e^{-it\not{D}} e^{it\not{D}_0}\,, \tag{6.1}$$
$$s - \lim_{t\to\infty} e^{-it\not{D}_0} e^{it\not{D}} \mathbf{1}^{\mathrm{c}}(\not{D}) \tag{6.2}$$

exist ($\mathbf{1}^{\mathrm{c}}(\not{D})$ is the projector onto the continuous subspace of $\not{D}$). If we denote (6.1) by Ω^+, then (6.2) equals $(\Omega^+)^*$ and we have

$$(\Omega^+)^*\Omega^+ = \mathbf{1}, \quad \Omega^+(\Omega^+)^* = \mathbf{1}^{\mathrm{c}}(\not{D})\,.$$

Proof.

By a density argument and using that $\sigma_{\rm pp}(\not{D})$ has no accumulation point except possibly 0, as well as the fact that $\not{D}_0$ does not have any eigenvalue, it is sufficient to show the existence of

$$s-\lim_{t\to\infty} e^{-it\not{D}}e^{it\not{D}_0}\tilde{\chi}^2(\not{D}_0), \tag{6.3}$$

$$s-\lim_{t\to\infty} e^{-it\not{D}_0}e^{it\not{D}}\chi^2(\not{D}) \tag{6.4}$$

with $\chi,\tilde{\chi}\in C_0^\infty(\mathbb{R})$ and $\text{supp}\chi\subset\mathbb{R}\setminus(\{0\}\cup\sigma_{\rm pp}(\not{D})), \text{supp}\tilde{\chi}\subset\mathbb{R}\setminus\{0\}$. We have :

$$e^{-it\not{D}_0}e^{it\not{D}}\chi^2(\not{D}) \;=\; e^{-it\not{D}_0}(\chi(\not{D})-\chi(\not{D}_0))e^{it\not{D}}\chi(\not{D})+e^{-it\not{D}_0}\chi(\not{D}_0)e^{it\not{D}}\chi(\not{D}). \tag{6.5}$$

By lemma 4.5, $\chi(\not{D}_0)-\chi(\not{D})$ is compact. As $\text{supp}\chi\cap\sigma_{\rm pp}(\not{D})=\emptyset$ and $\sigma_{\rm sc}(\not{D})=\emptyset$, $e^{it\not{D}}\chi(\not{D})\to 0$ weakly, so $(\chi(\not{D}_0)-\chi(\not{D}))e^{it\not{D}}\chi(\not{D})\to 0$ strongly, and the first term in (6.5) tends strongly to zero. We have :

$$\frac{d}{dt}\chi(\not{D}_0)e^{-it\not{D}_0}e^{it\not{D}}\chi(\not{D})=\chi(\not{D}_0)e^{-it\not{D}_0}i(\not{D}-\not{D}_0)e^{it\not{D}}\chi(\not{D}).$$

Let $\hat{\chi}\in C_0^\infty(\mathbb{R}\setminus(\{0\}\cup\sigma_{\rm pp}(\not{D}))$ with $\chi\hat{\chi}=\chi$. Then :

$$\begin{aligned}&\frac{d}{dt}\chi(\not{D}_0)e^{-it\not{D}_0}e^{it\not{D}}\chi(\not{D})\\ &=\;(e^{-it\not{D}_0}\chi(\not{D}_0)<A>^{-1})(<A>\hat{\chi}(\not{D}_0)i(\not{D}-\not{D}_0)\hat{\chi}(\not{D})<A>)(<A>^{-1}\chi(\not{D})e^{it\not{D}}).\end{aligned}$$

The second operator is bounded by lemma 6.2. The first and third operators, using theorem 4 and a duality argument, give the integrability of the whole expression. This shows the existence of (6.2). The proof of the existence of (6.1) is analogous. □

6.3 Technical results concerning the separable problem

In this subsection we drop the indices n,l. Recall that $D(\not{D}_\nu)=(H^1(\mathbb{R}))^2$, $\forall\nu\in\mathbb{N}$.

Lemma 6.3. *Let $\psi\in C_b^\infty(\mathbb{R})$ s.t. $\psi'\in C_0^\infty(\mathbb{R})$, $\chi\in C_0^\infty(\mathbb{R})$. Then $[\psi,\chi(\not{D}_\nu)]$ is compact.*

Proof.

By the Helffer-Sjöstrand formula it is sufficient to show :

$$\forall z\in\mathbb{C}\setminus\sigma(\not{D}_\nu)\;(z-\not{D}_\nu)^{-1}[\psi,\not{D}_\nu](z-\not{D}_\nu)^{-1}\text{ is compact}, \tag{6.6}$$

$$||(z-\not{D}_\nu)^{-1}[\psi,\not{D}_\nu](z-\not{D}_\nu)^{-1}||\le C|\text{Im}z|^{-2}. \tag{6.7}$$

(6.7) is clear and (6.6) follows from corollary 3.1 because $[i\not{D}_\nu,\psi]=\gamma\psi'$. □

Lemma 6.4.

The operator $j_\pm(\chi(\not{D}_\pm)-\chi(\not{D}_0))$ is compact.

Proof.

Using the Helffer-Sjöstrand formula it is sufficient to show :

$$||j_\pm(z-\not{D}_0)^{-1}(\not{D}_0-\not{D}_\pm)(z-\not{D}_\pm)^{-1}||\le C|\text{Im}z|^{-2}, \tag{6.8}$$

$$\forall z\in\mathbb{C}\setminus(\sigma(\not{D}_0)\cup\sigma(\not{D}_\pm))\;j_\pm(z-\not{D}_0)^{-1}(\not{D}_0-\not{D}_\pm)(z-\not{D}_\pm)^{-1}\text{ is compact}. \tag{6.9}$$

(6.8) is clear, let us show (6.9). We have :

$$\begin{aligned} & j_\pm(z-\not{D}_0)^{-1}(\not{D}_0-\not{D}_\pm)(z-\not{D}_\pm)^{-1} \\ = \;& (z-\not{D}_0)^{-1}[j_\pm,\not{D}_0](z-\not{D}_0)^{-1}(\not{D}_0-\not{D}_\pm)(z-\not{D}_\pm)^{-1} \\ + \;& (z-\not{D}_0)^{-1}j_\pm(\not{D}_0-\not{D}_\pm)(z-\not{D}_\pm)^{-1}. \end{aligned} \tag{6.10}$$

The first term is compact by lemma 6.3. We have

$$j_\pm(\not{D}_0-\not{D}_\pm) = j_\pm(g_0(r)-g_\pm(r))\tau q + j_\pm(f(r)-f_\pm).$$

Both terms are functions which tend to zero as $|r|\to\infty$, so the last term in (6.10) is compact by corollary 3.1. □

Lemma 6.5. *For all $\chi \in C_0^\infty(\mathbb{R})$*

$$< B_\nu > \chi(\not{D}_0)(\not{D}_0 j_\pm - j_\pm \not{D}_\pm)\chi(\not{D}_\pm) < B_\nu > \textit{ is bounded.}$$

Proof.
Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\chi\tilde{\chi} = \chi$. We have :

$$\begin{aligned} & < B_\nu > \chi(\not{D}_0)(\not{D}_0 j_\pm - j_\pm \not{D}_\pm)\chi(\not{D}_\pm) < B_\nu > \\ = \;& < B_\nu > \chi(\not{D}_0) < r >^{-1} < r > \tilde{\chi}(\not{D}_0)(\not{D}_0 j_\pm - j_\pm \not{D}_\pm)\tilde{\chi}(\not{D}_\pm) < r >< r >^{-1} \chi(\not{D}_\pm) < B_\nu > . \end{aligned}$$

We first show that

$$< B_\nu > \chi(\not{D}_\nu) < r >^{-1} \text{ is bounded for all } \nu \in \mathcal{N}. \tag{6.11}$$

By [15, lemma C.3.2], the Helffer-Sjöstrand formula and the commutator estimates included in the proofs of lemmata 5.22 and 5.23, $[< B_\nu >, \chi(\not{D}_\nu)]$ is bounded. As $\chi(\not{D}_\nu) < B_\nu >< r >^{-1}$ is also bounded, (6.11) follows. It remains to show that

$$< r > \tilde{\chi}(\not{D}_0)(\not{D}_0 j_\pm - j_\pm \not{D}_\pm)\tilde{\chi}(\not{D}_\pm) < r > \text{ is bounded.} \tag{6.12}$$

Clearly, the commutator $[< r >, \not{D}_\nu]$ is bounded and by the Helffer-Sjöstrand formula, we find :

$$[< r >, \tilde{\chi}(\not{D}_\nu)] \text{ is bounded for all } \nu \text{ from } \mathcal{H} \text{ to } \mathcal{H}^1. \tag{6.13}$$

It remains to show that :

$$\tilde{\chi}(\not{D}_0) < r > (\not{D}_0 j_\pm - j_\pm \not{D}_\pm) < r > \tilde{\chi}(\not{D}_\pm) \text{ is bounded.} \tag{6.14}$$

We have

$$\not{D}_0 j_\pm - j_\pm \not{D}_\pm = \frac{1}{i} j'_\pm \gamma + j_\pm(g(r)-g_\pm(r))\tau q + j_\pm(f(r)-f_\pm(r)) .$$

It follows that

$$< r > \left(\not{D}_0 j_\pm - j_\pm \not{D}_\pm\right) < r >$$

is a uniformly bounded function thanks to (3.8)-(3.11). □

6.4 Comparison with the asymptotic dynamics

Theorem 7. *The limits*

$$s-\lim_{t\to\infty} e^{-it\not{D}_0} j_\pm e^{it\not{D}_\pm}, \tag{6.15}$$

$$s-\lim_{t\to\infty} e^{-it\not{D}_\pm} j_\pm e^{it\not{D}_0} \tag{6.16}$$

exist. If we denote (6.15) by $\Omega_{0,\pm}^{+}$, then (6.16) equals $(\Omega_{0,\pm}^{+})^*$ and we have

$$\Omega_{0,+}^{+}(\Omega_{0,+}^{+})^* + \Omega_{0,-}^{+}(\Omega_{0,-}^{+})^* = (\Omega_{0,+}^{+})^*\Omega_{0,+}^{+} + (\Omega_{0,-}^{+})^*\Omega_{0,-}^{+} = \mathbf{1}.$$

$\Omega_{0,\pm}^{+}, (\Omega_{0,\pm}^{+})^*$ are independent of the choice of the partition of unity.

Proof.

We start by proving the existence of $\Omega_{0,\pm}$. It is sufficient to show for all n,l the existence of :

$$s-\lim_{t\to\infty} e^{-it\not{D}_0^{nl}} j_\pm e^{it\not{D}_\pm^{nl}} \chi^2(\not{D}_\pm^{nl})$$

for $\chi \in C_0^\infty(\mathbb{R}\setminus\{0\})$. From now on we omit the indices n,l. We have :

$$\begin{aligned} e^{-it\not{D}_0} j_\pm e^{it\not{D}_\pm}\chi^2(\not{D}_\pm) &= e^{-it\not{D}_0} j_\pm(\chi(\not{D}_\pm)-\chi(\not{D}_0)) e^{it\not{D}_\pm}\chi(\not{D}_\pm) \\ &+ e^{-it\not{D}_0}[j_\pm,\chi(\not{D}_0)] e^{it\not{D}_\pm}\chi(\not{D}_\pm) \\ &+ e^{-it\not{D}_0}\chi(\not{D}_0) j_\pm e^{it\not{D}_\pm}\chi(\not{D}_\pm). \end{aligned} \tag{6.17}$$

Using that $[j_\pm,\chi(\not{D}_0)]$ is compact by lemma 6.3 and that $j_\pm(\chi(\not{D}_\pm)-\chi(\not{D}_0))$ is compact by lemma 6.4, it is sufficient to show that the last term in (6.17) has a limit. Let $\hat{\chi} \in C_0^\infty(\mathbb{R}\setminus\{0\})$ s.t. $\hat{\chi}\chi = \chi$. We have :

$$\begin{aligned} &\frac{d}{dt}\chi(\not{D}_0) e^{-it\not{D}_0} j_\pm e^{it\not{D}_\pm}\chi(\not{D}_\pm) \\ &= -ie^{-it\not{D}_0}\chi(\not{D}_0)(\not{D}_0 j_\pm - j_\pm \not{D}_\pm) e^{it\not{D}_\pm}\chi(\not{D}_\pm) \\ &= -i(e^{-it\not{D}_0}\chi(\not{D}_0) < B_\nu >^{-1}) \\ &\quad \times(< B_\nu > \hat{\chi}(\not{D}_0)(\not{D}_0 j_\pm - j_\pm\not{D}_\pm)\hat{\chi}(\not{D}_\pm) < B_\nu >)(< B_\nu >^{-1} \chi(\not{D}_\pm) e^{it\not{D}_\pm}). \end{aligned}$$

We conclude as in the proof of theorem 6 using theorem 5 and lemma 6.5. The proof of the existence of (6.16) is analogous. In order to prove the last statement, it is sufficient to show that

$$s-\lim_{t\to\infty} e^{-it\not{D}_\pm}\psi e^{it\not{D}_0} = 0$$

for all $\psi \in C_0^\infty(\mathbb{R})$. This will follow from

$$s-\lim_{t\to\infty} e^{-it\not{D}_\pm}\psi e^{it\not{D}_0}\chi(\not{D}_0) = 0$$

for all $\chi \in C_0^\infty(\mathbb{R}\setminus\{0\})$, which is true because $\psi\chi(\not{D}_0)$ is compact. □

Theorem 8. *The limits*

$$s-\lim_{t\to\infty} e^{-it\not{D}} j_\pm e^{it\not{D}_\pm}, \tag{6.18}$$

$$s-\lim_{t\to\infty} e^{-it\not{D}_\pm} j_\pm e^{it\not{D}}\mathbf{1}^c(\not{D}) \tag{6.19}$$

exist. If we denote (6.18) by $\Omega_\pm^+$, then (6.19) equals $(\Omega_\pm^+)^*$ and we have :

$$\Omega_+^+(\Omega_+^+)^* + \Omega_-^+(\Omega_-^+)^* = \mathbf{1}^c(\not{D}), \quad (\Omega_+^+)^*\Omega_+^+ + (\Omega_-^+)^*\Omega_-^+ = \mathbf{1}.$$

$\Omega_\pm^+, (\Omega_\pm^+)^*$ are independent of the choice of the partition of unity.

Proof.
This follows from theorems 6, 7 and the chain rule. □

7 Proof of the theorems of section 2.6

7.1 Absence of point spectrum

The separability of Weyl's equation in Boyer-Lindquist coordinates was proved independently by W.G. Unruh [57] and S.A. Teukolski [56] ; S. Chandrasekhar then extended the result to the full Dirac equation (see [11] and [12]). All these proofs rely on the Newman-Penrose formalism and adopt Kinnersley's null tetrad (for its definition, see equations (2.20)-(2.23) or Kinnersley's original work [36]). The absence of stationary solutions to the charged massive Dirac equation outside a non extreme Kerr-Newman black-hole is the object of a work by F. Finster, N. Kamran, J. Smoller and S.-T. Yau [22] ; the class of solutions considered there is specified by means of so-called matching conditions across the horizon. In fact, if we simply impose the physical requirement that solutions should have finite total charge, the absence of stationary solutions becomes an immediate consequence of the separability of the equations. We prove this for the massless Dirac equation on the Kerr metric. The same technique should be valid on Kerr-Newman backgrounds for charged and massive fields. In the extreme case however, the method fails because of the lack of integrability at the horizon.

Proposition 7.1. *There are no stationary finite charge Weyl fields outside a Kerr black hole, i.e. there are no non-zero solutions*

$$\phi_A \in \mathcal{C}\left(\mathbb{R}_t\,;\ L^2\left((\Sigma\,;\ \mathrm{dVol})\ ;\ \mathbb{S}_A\right)\right)$$

of (2.5), of the form

$$\phi(t,r,\theta,\varphi) = e^{-i\alpha t}\chi(r,\theta,\varphi)\,,\ \ \alpha \in \mathbb{R}\,.$$

In other words, the Hamiltonian $\not{D}_K$ has empty point spectrum on $\mathcal{H}$.

Proof.
Let us consider

$$\alpha \in \mathbb{R} \text{ and } \chi_A \in L^2\left((\Sigma\,;\ \mathrm{dVol})\ ;\ \mathbb{S}_A\right)$$

such that $\phi_A(t,r,\theta,\varphi) = e^{-i\alpha t}\chi_A(r,\theta,\varphi)$ satisfies (2.5). We denote f_0 and f_1 the components of χ_A in the spin-frame $(O^A,\, I^A)$ associated with Kinnersley's null tetrad L^a, N^a, m^a, $\bar{m}^a$, i.e.

$$L^a = O^A\bar{O}^{A'}\,,\ N^a = I^A\bar{I}^{A'}\,,\ m^a = O^A\bar{I}^{A'}\,,\ \bar{m}^a = I^A\bar{O}^{A'}\,,$$
$$f_0 = \chi_A O^A\,,\ f_1 = \chi_A I^A\,.$$

A simple calculation shows that $\chi_A \in L^2\left((\Sigma\,;\ \mathrm{dVol})\ ;\ \mathbb{S}_A\right)$ if and only if

$$f_0 \in L^2\left(\Sigma\,;\ \Delta\mathrm{d}r_*\mathrm{d}\omega\right)\ ,\ \ f_1 \in L^2\left(\Sigma\,;\ \rho^2\mathrm{d}r_*\mathrm{d}\omega\right)\,. \tag{7.1}$$

We know from [11] and [56] that for $s = \pm 1/2$ and for each value of α, there exist orthonormal bases $\left\{Y_{l,n}^{s,\alpha}(\theta,\varphi) = S_{l,n}^{s,\alpha}(\theta)e^{in\varphi}\right\}_{l,n}$ of $L^2\left(S^2\,;\ \mathrm{d}\omega\right)$, such that f_0 (resp. f_1) can be decomposed into a series, convergent in $L^2(\Sigma;\ \Delta \mathrm{d}r_* \mathrm{d}\omega)$ (resp. in $L^2\left(\Sigma;\ \rho^2 \mathrm{d}r_* \mathrm{d}\omega\right)$) of functions of the form[3] :

$$f_0^{l,n}(t,r,\theta,\varphi) = e^{in\varphi} R_{1/2}(r) S_{1/2}(\theta)\,,\ f_1^{l,n}(t,r,\theta,\varphi) = \frac{1}{\bar{p}} e^{in\varphi} R_{-1/2}(r) S_{-1/2}(\theta)$$

(where we omit in R and S the indices l, n and α for simplicity of notation), the functions $e^{in\varphi} S_{\pm 1/2}(\theta)$ are smooth functions on the sphere and $R_{\pm 1/2}$ satisfy :

$$\begin{gathered}\left(\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial r_*} + \frac{iK}{\Delta}\right) R_{-1/2} = \lambda R_{1/2}\,,\\ \left(\left(r^2+a^2\right)\frac{\partial}{\partial r_*} - iK + (r-M)\right) R_{1/2} = 2\lambda R_{-1/2}\,,\\ K = (r^2+a^2)\alpha + an\,,\end{gathered}$$

λ being a separation constant depending on the discrete parameters l and n. The condition (7.1) is equivalent to (since $|\bar{p}|^2 = \rho^2$)

$$R_{1/2} \in L^2(\mathbb{R};\ \Delta \mathrm{d}r_*)\ ,\ \ R_{-1/2} \in L^2(\mathbb{R};\ \mathrm{d}r_*)\ . \tag{7.2}$$

Putting

$$U = R_{-1/2}\,,\ V = \sqrt{\Delta} R_{1/2}\,,$$

we obtain

$$\begin{gathered}\left(\frac{\partial}{\partial r_*} + \frac{iK}{r^2+a^2}\right) U = \lambda\sqrt{\Delta} V\,,\\ \left(\frac{\partial}{\partial r_*} - \frac{iK}{r^2+a^2}\right) V = 2\lambda\sqrt{\Delta} U\,.\end{gathered}$$

We can multiply U and V by phase factors in order to get rid of the terms involving iK :

$$\begin{aligned}\tilde{U} &= \exp\left(\int_0^{r_*} \frac{iK}{(r(s))^2+a^2}\mathrm{d}s\right) U =: \beta U\,,\\ \tilde{V} &= \exp\left(-\int_0^{r_*} \frac{iK}{(r(s))^2+a^2}\mathrm{d}s\right) V = \bar{\beta} V\,,\end{aligned}$$

where $s \mapsto r(s)$ is the reciprocal function of $r \mapsto r_*$. Now $\tilde{U}$ and $\tilde{V}$ satisfy the differential system

$$\begin{gathered}\tilde{U}' = \lambda\beta^2\sqrt{\Delta}\tilde{V}\,,\\ \tilde{V}' = 2\lambda\bar{\beta}^2\sqrt{\Delta}\tilde{U}\,,\end{gathered}$$

i.e.

$$\left(\begin{array}{c}\tilde{U}\\ \tilde{V}\end{array}\right)' = B\left(\begin{array}{c}\tilde{U}\\ \tilde{V}\end{array}\right)\ ,\ \ B = \left(\begin{array}{cc} 0 & \lambda\beta^2\sqrt{\Delta}\\ 2\lambda\bar{\beta}^2\sqrt{\Delta} & 0\end{array}\right) \tag{7.3}$$

and (7.2) is equivalent to

$$\tilde{U}\,,\ \tilde{V} \in L^2(\mathbb{R};\ \mathrm{d}r_*)\ .$$

The factor $\lambda\beta^2\sqrt{\Delta}$ falls off exponentially fast as $r_* \to -\infty$ and is therefore integrable at $-\infty$. Hence, there exists a solution ${}^t(\tilde{U},\tilde{V})$ of (7.3) that tends to ${}^t(c_1, c_2)$ at $-\infty$ for any $c_1, c_2 \in \mathbb{C}$. The space of solutions of (7.3) being of complex dimension 2, it follows that non zero solutions of (7.3) do not belong to $L^2(\mathbb{R}; \mathrm{d}r_*)$. □

[3] Note that Chandrasekhar's unknowns F_1 and F_2 are the components of ϕ^A, and not ϕ_A, with respect to the spin-frame $(O^A,\ I^A)$; the correspondence with our unknowns is therefore $f_0 = -e^{i\alpha t} F_2$, $f_1 = e^{i\alpha t} F_1$.

7.2 Compatibility of the general analytic framework

With the notations of section 3 we put

$$\begin{aligned}
g(r_*) &:= \frac{\sqrt{\Delta}}{r^2+a^2}\,,\ f(r_*) := -\frac{2Mran}{(r^2+a^2)^2}\,,\ h(r_*) := \sqrt{\frac{r^2+a^2}{\sigma}}\,,\\
V^n &:= \frac{i\sqrt{\Delta}}{\sigma\sin\theta}(\frac{\rho^2}{\sigma}-1)\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} + iB + i\sqrt{\frac{r^2+a^2}{\sigma}}\gamma\left(\sqrt{\frac{r^2+a^2}{\sigma}}\right)'\\
&+ \frac{i\Delta^{3/2}a^2\sin\theta\cos\theta}{2\sigma^3}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} + \frac{2Mran}{\sigma}\left(\frac{1}{r^2+a^2}-\frac{1}{\sigma}\right),\\
\not{D}_0^n &:= \gamma D_{r_*} + g(r_*)\not{D}_{S^2} + f(r_*)n\,,\ \not{D}^n := h\not{D}_0^n h + V^n.
\end{aligned}$$

We have :

$$\begin{aligned}
\not{D}^n = h\not{D}_0^n h + V^n &= \frac{r^2+a^2}{\sigma}\gamma D_{r_*} + \frac{\sqrt{\Delta}}{\sigma}\not{D}_{S^2} - \frac{2Mran}{\sigma(r^2+a^2)} - \frac{i\Delta^{3/2}a^2\sin\theta\cos\theta}{2\sigma^3}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\\
&+ \sqrt{\frac{r^2+a^2}{\sigma}}\frac{1}{i}\gamma\left(\sqrt{\frac{r^2+a^2}{\sigma}}\right)' + V^n = \not{D}_K^n,
\end{aligned}$$

where $\not{D}_K^n$ is obtained from $\not{D}_K$ by fixing $D_\varphi = n$ without changing $\not{D}_{S^2}$.

Remark 7.1. *The operators $\not{D}^n$ and $\not{D}_0^n$ are operators acting on $\mathcal{H}$. They coincide with the restrictions of $\not{D}$ and $\not{D}_0$ to the subspace of functions whose dependence in φ is given by $e^{in\varphi}$. For such restrictions, the operator $\not{D}_{S^2}$ is not "simplified" ; it is better to keep the whole $\not{D}_{S^2}$ which is a regular operator acting on this subspace, than to use an explicit expression with coordinate singularities.*

Let us recall from section 3 that ($\eta = \kappa_+$ defined in (2.61)) :

$$\begin{aligned}
f &\in \mathbf{S}^{m,n} \quad \text{iff} \quad \forall\alpha,\beta\in\mathbb{N}\quad \partial_{r_*}^\alpha\partial_\theta^\beta f \in \left\{\begin{array}{ll} O(<r_*>^{m-\alpha}) & r_*\to+\infty\,,\\ O(e^{n\eta|r_*|}) & r_*\to-\infty\,,\end{array}\right.\\
f &\in \mathbf{S}^m \quad \text{iff} \quad \forall\alpha,\beta\in\mathbb{N}\quad \partial_{r_*}^\alpha\partial_\theta^\beta f\in O(<r_*>^{m-\alpha})\ |r_*|\to\infty.
\end{aligned}$$

We check :

$$\mathbf{S}^{m,n}\times\mathbf{S}^{m',n'}\subset\mathbf{S}^{m+m',n+n'}, \tag{7.4}$$
$$\forall\alpha\in\mathbb{N}\,,\quad \partial_{r_*}^\alpha : \mathbf{S}^{m,n}\to\mathbf{S}^{m-\alpha,n}, \tag{7.5}$$
$$\forall\beta\in\mathbb{N}\,,\quad \partial_\theta^\beta : \mathbf{S}^{m,n}\to\mathbf{S}^{m,n}. \tag{7.6}$$

For a function depending on r and θ we define :

$$f(r,\theta)\in\Pi^m \quad \text{iff} \quad \forall\alpha,\beta\in\mathbb{N}\,,\quad \partial_r^\alpha\partial_\theta^\beta f\in\left\{\begin{array}{ll} O(<r>^{m-\alpha}) & r\to\infty\\ O(1) & r\to r_+\end{array}\right.$$

We have (see [32, lemme 9.7.1]) :

Lemma 7.1. ***(i)*** *If $f(r)\in\Pi^m$, then we have for all $\alpha\in\mathbb{N}$ $\quad D_{r_*}^\alpha f(r(r_*))\in\mathbf{S}^{m-\alpha,-2}$.*

(ii) *If $f(r_*)\in\mathbf{S}^{m,n}$ and $g(r)\in\Pi^k$, then $f(r_*)g(r(r_*))\in\mathbf{S}^{m+k,n}$.*

Examples

1) $e_1(r,\theta) = \Delta \in \mathbf{S}^{2,-2}$,

2) $e_2(r,\theta) = \sqrt{\Delta} \in \mathbf{S}^{1,-1}$,

3) $e_3(r,\theta) = \frac{1}{\sigma^p} \in \mathbf{S}^{-2p,0}$,

4) $e_4(r,\theta) = \partial_\theta(\frac{1}{\sigma^p}) \in \mathbf{S}^{-2(p+1),-2}$.

Lemma 7.2. *The functions g, f, h, V satisfy the hypotheses of section 3.3.*

Proof.

We start with $g(r_*) = \frac{\sqrt{\Delta}}{r^2+a^2} \in \mathbf{S}^{-1,-1}$ according to example **2)** and (7.4). We put

$$c_0 := \frac{(r_+ - r_-)^{\frac{1}{2}(\frac{r_-}{r_+}+1)}}{r_+^2 + a^2} e^{-\eta r_+}.$$

We have $\Delta = (r(r_*) - r_-)^{\frac{r_-}{r_+}+1} e^{2\eta(r_* - r(r_*))}$ and therefore :

$$g(r_*) - c_0 e^{\eta r_*} = \tilde{g}(r_*) e^{\eta r_*}$$

with

$$\tilde{g}(r_*) := \frac{(r-r_-)^{\frac{1}{2}(\frac{r_-}{r_+}+1)} e^{-\eta r}}{r^2+a^2} - c_0 = O(r - r_+) = O\left(e^{2\eta r_*}\right) \text{ as } r_* \to -\infty\,.$$

Therefore $g(r_*)$ satisfies condition (3.8). We have

$$g(r) - \frac{1}{r_*} = \frac{\sqrt{\Delta}}{r^2+a^2} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r_*}\,.$$

The term

$$\frac{\sqrt{\Delta}}{r^2+a^2} - \frac{1}{r}$$

is $O(< r_* >^{-2})$ while the remainder, due to the logarithmic terms in r_*, is $O(< r_* >^{-2+\varepsilon})$ for any $\varepsilon > 0$. The condition on the derivative of g is checked in the same manner. Therefore $g(r_*)$ satisfies condition (3.10).

Let us now check the condition on $f(r_*)$. Recall that $f(r_*) = -\frac{2Mran}{(r^2+a^2)^2}$. We have

$$\hat{f}(r) := -\frac{2Mran}{(r^2+a^2)^2} \in \Pi^{-3}$$

and $f' \in \mathbf{S}^{-4,-2}$ by lemma 7.1. We put $c_1 := -(r_+^2 + a^2)^{-1}$ and we obtain :

$$f(r_*) - c_1 = -\left(\frac{2M(r-r_+)}{(r^2+a^2)^2} + \frac{2Mr_+(r_+^2 - r^2)(r_+^2 + r^2 + 2a^2)}{(r^2+a^2)^2(r_+^2+a^2)^2}\right) \in O(e^{2\eta r_*})\,,\ r_* \to -\infty.$$

Clearly $f(r_*) \in O(< r_* >^{-2})$, $r_* \to +\infty$. This proves that (3.9), (3.11) are fulfilled.

Let us now check the conditions on h. Recall that $h(r_*) = \sqrt{\frac{r^2+a^2}{\sigma}}$. We have :

$$0 \le h^2 - 1 = \frac{h^4-1}{h^2+1} \le h^4 - 1 = \frac{(r^2+a^2)^2 - \sigma^2}{\sigma^2} = \frac{\Delta a^2 \sin^2\theta}{\sigma^2} \le \frac{a^2}{r^2} \le \alpha < 1\,,$$

where we have used that $r \geq r_+ \geq \frac{1}{\alpha} M$ $(0 < \alpha < 1)$ if $M > |a|$.

We have :

$$\partial_\theta h = \frac{\sqrt{r^2+a^2}\Delta a^2 \sin\theta\cos\theta}{2\sigma^{5/2}} \in \mathbf{S}^{-2,-2},$$

which shows that (3.15) is fulfilled. We have :

$$h - 1 = \frac{h^2-1}{h+1} = \frac{h^4-1}{(h+1)(h^2+1)} = \frac{\Delta a^2 \sin^2\theta}{\sigma^2(h+1)(h^2+1)} \in \mathbf{S}^{-2,-2}.$$

We have now to check the conditions on (V_{ij}). We put :

$$\begin{aligned}
V_1 &:= \frac{i\sqrt{\Delta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma}-1\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V_2 := i\sqrt{\frac{r^2+a^2}{\sigma}}\gamma\left(\sqrt{\frac{r^2+a^2}{\sigma}}\right)', \\
V_3 &:= \frac{i\Delta^{3/2}a^2\sin\theta\cos\theta}{2\sigma^3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
V_4 &:= \frac{2Mran}{\sigma}\left(\frac{1}{r^2+a^2}-\frac{1}{\sigma}\right), \ V_5 := iB.
\end{aligned}$$

Let us first treat V_1. We have

$$\frac{\sqrt{\Delta}(\rho^2-\sigma)}{\sigma\sin\theta} = \frac{(\rho^4-\sigma^2)\sqrt{\Delta}}{(\rho^2+\sigma)\sigma\sin\theta} = -\frac{(a^2\sin\theta\rho^2+2Mra^2\sin\theta)\sqrt{\Delta}}{(\rho^2+\sigma)\sigma} \in \mathbf{S}^{-2,-1}.$$

An explicit calculation gives :

$$V_2 = \frac{i}{2}\gamma\frac{(r^2+a^2)(r-M)a^2\sin^2\theta - 2ra^2\Delta\sin^2\theta}{\sigma^3}\frac{\Delta}{r^2+a^2} \in \mathbf{S}^{-3,-2}.$$

We have $V_3 \in \mathbf{S}^{-3,-3}$ and

$$V_4 = -\frac{2Mrn\Delta a^3\sin^2\theta}{\sigma^2\sigma_+(r^2+a^2)} \in \mathbf{S}^{-5,-2}.$$

To treat V_5 let us first remark that $\tilde{M}_t = (\tilde{M}_{ij})$ with $\tilde{M}_{ij} \in \mathbf{S}^{0,1}$ and $\tilde{M}_t^{-1} = (\tilde{M}_{ij}^-)$ with $\tilde{M}_{ij}^- \in \mathbf{S}^{0,-1}$. We have $V_5 = \tilde{M}_t^{-1}\tilde{P}$ and it is therefore sufficient to show that $\tilde{P} \in \mathbf{S}^{-2,0}$. We have

$$\tilde{P} = \mathbf{U}P\mathbf{U}^{-1} + \mathbf{U}M_\theta[\partial_\theta, \mathbf{U}^{-1}] + \mathbf{U}M_r[\partial_r, \mathbf{U}^{-1}].$$

We first claim that the components of $UM_\theta[\partial_\theta, \mathbf{U}^{-1}]$ are in $\mathbf{S}^{-2,0}$. Indeed we have $\mathbf{U}^{-1} =: (\mathbf{U}_{ij}^-)$ with $\mathbf{U}_{12}^-, \mathbf{U}_{21}^- \in \mathbf{S}^{-1,-1}$, $\partial_\theta\mathbf{U}_{ii}^- \in \mathbf{S}^{-1,0}$, $M_\theta = (M_\theta^{ij})$ with $M_\theta^{ij} \in \mathbf{S}^{-1,0}$ and $\mathbf{U} = (\mathbf{U}_{ij})$ with $\mathbf{U}_{ij} \in \mathbf{S}^{0,0}$. Using (7.4) we get the desired fall-off. We now claim that the components of $\mathbf{U}M_r[\partial_r, \mathbf{U}^{-1}]$ are in $\mathbf{S}^{-2,0}$. Note that the derivative is with respect to r but the symbol class is defined in reference to r_*. It is sufficient to show that $\partial_r\mathbf{U}_{ij}^- \in \mathbf{S}^{-2,0}$ for all i,j. This is obvious for $\partial_r\mathbf{U}_{12}^-$ and $\partial_r\mathbf{U}_{21}^-$. Let us consider $\partial_r\mathbf{U}_{11}^-$. We have

$$X := \frac{\rho\sigma_+}{\sigma p} = 1 - \frac{a^2\cos^2\theta}{\rho(r+\rho)} - \frac{ia\cos\theta}{\rho} + \frac{r^2+a^2}{\sigma p} = 1 + O(r^{-1}).$$

Consequently,

$$\partial_r\mathbf{U}_{11}^- = \frac{1}{2\sqrt{2}}X^{-1/2}\partial_r X \in S^{-2,0}.$$

The proof is similar for $\partial_r \mathbf{U}^-_{22}$ using in addition that $\rho/\bar{p} = 1 + O(r^{-1})$. We now have to show that $\mathbf{U}P\mathbf{U}^{-1}$ belongs to $\mathbf{S}^{-2,0}$. This is equivalent to $P \in \mathbf{S}^{-2,0}$ which follows from the explicit form of P. ☐

Remark 7.2. *We can choose the arbitrary constant R_0 (see (2.39)) so that $e^{-r_*}\sqrt{\Delta}/\sigma \to 1$ as $r_* \to -\infty$, i.e. the constant c_0 then becomes 1.*

7.3 Proof of theorems 1, 2 and 3

We start by a proof of the scattering theories of theorems 2 and 3 using cut-off functions. Then we use these results to construct the asymptotic velocity. This in turn gives us the more elegant definitions of the wave operators given in the theorems.

7.3.1 Scattering theory in terms of cut-off functions

We first note that, by lemma 3.5, $\not{D}_H$ and $\not{D}_\infty$ are self-adjoint on $\mathcal{H}$ and, by lemma 4.8, their point spectra are empty. Moreover $\not{D}_K$ is selfadjoint on $\mathcal{H}$ by lemma 4.1 (the conservation of the total charge guarantees that $\not{D}_K$ is symmetric). The absence of point spectrum for $\not{D}_K$ is shown in section 7.1.

Proof of the scattering theory of theorem 3.

We consider cut-off functions $j_\pm \in \mathcal{C}_b^\infty(\mathbb{R})$ satisfying : there exists $R > 0$ such that

$$j_+ \equiv 0 \text{ on }]-\infty, -R]\,,\ j_+ \equiv 1 \text{ on } [R, +\infty[\,, \tag{7.7}$$

$$j_- \equiv 1 \text{ on }]-\infty, -R]\,,\ j_- \equiv 0 \text{ on } [R, +\infty[\,, \tag{7.8}$$

$$j_+^2 + j_-^2 \equiv 1 \text{ on } \mathbb{R}\,. \tag{7.9}$$

We prove the existence of the following direct and inverse wave operators, defined by the strong limits :

$$\Omega_H^\pm := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} j_- e^{it\not{D}_H}\ ,\ \Omega_\infty^\pm := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} j_+ e^{it\not{D}_\infty}\ ,$$
$$\tilde{\Omega}_H^\pm := s - \lim_{t\to\pm\infty} e^{-it\not{D}_H} j_- e^{it\not{D}_K}\ ,\ \tilde{\Omega}_\infty^\pm := s - \lim_{t\to\pm\infty} e^{-it\not{D}_\infty} j_+ e^{it\not{D}_K}\ .$$

They are denoted by the same notations as in theorem 3. It will in fact become apparent at the end of the proof that they are the same operators.

Let us decompose $\mathcal{H}$ into the direct sum of the spaces

$$\mathcal{H}^n := \{u = e^{in\varphi} v,\ v \in L^2(\mathbb{R} \times [0,\pi], dr \sin\theta d\theta)\}.$$

The dynamics $e^{it\not{D}_K}, e^{it\not{D}_H}, e^{it\not{D}_\infty}$ as well as the cut-off functions $j_\pm$ preserve the spaces $\mathcal{H}^n$. We have furthermore

$$e^{it\not{D}_K}|_{\mathcal{H}^n} = e^{it\not{D}_K^n}|_{\mathcal{H}^n},\ e^{it\not{D}_H}|_{\mathcal{H}^n} = e^{it\not{D}_H^n}|_{\mathcal{H}^n},\ e^{it\not{D}_\infty}|_{\mathcal{H}^n} = e^{it\not{D}_\infty^n}|_{\mathcal{H}^n},$$

where the operator with index n is obtained from the operator without index by replacing D_φ by n without changing $\not{D}_{S^2}$, e.g.

$$\not{D}_K^n = \frac{r^2 + a^2}{\sigma} \gamma D_{r_*} + \frac{\sqrt{\Delta}}{\sigma} \not{D}_{S^2} - A_\varphi n + iB.$$

Using the absence of pure point spectrum for $\not{D}_K$, it is thus sufficient to show the existence of the limits (on $\mathcal{H}^n$) :

$$\Omega_H^{\pm,n} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K^n} j_- e^{it\not{D}_H^n} \,,\ \text{etc...}$$

These limits exist on $\mathcal{H} \supset \mathcal{H}^n$ by theorem 8 and lemma 7.2 and have the required properties. They are moreover independent of the choice of the cut-off function. □

Proof of the scattering theory of theorem 2.

Theorem 8 also gives the existence of limits :

$$s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} j_- e^{it\mathbb{D}_H} \,,\ s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} j_- e^{it\not{D}_K} \,,$$

with the correct properties. From [44, lemmata 6.1 and 6.2], we also infer the existence of the limits

$$s - \lim_{t\to\pm\infty} e^{-it\not{D}_\infty} e^{it\mathbb{D}_\infty} \,,\ s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} e^{it\not{D}_\infty} \,.$$

The existence of the direct and inverse wave operators

$$\begin{aligned}
\mathfrak{W}_H^{\pm} &:= s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} j_- e^{it\mathbb{D}_H} \,,\ \mathfrak{W}_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} j_+ e^{it\mathbb{D}_\infty} \,,\\
\tilde{\mathfrak{W}}_H^{\pm} &:= s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} j_- e^{it\not{D}_K} \,,\ \tilde{\mathfrak{W}}_\infty^{\pm} := s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} j_+ e^{it\not{D}_K} \,,
\end{aligned}$$

then follows from the chain rule. These operators are independent of the choice of the cut-off functions. □

7.3.2 Asymptotic velocity (proof of theorem 1)

We first establish results for the asymptotic profiles.

Lemma 7.3. *For each $J \in \mathcal{C}_\infty(\mathbb{R})$, we have :*

$$\exists\ s - \lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty} = J(-\gamma) \,, \tag{7.10}$$

$$\exists\ s - \lim_{t\to+\infty} e^{-it\mathbb{D}_H} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_H} = J(-\gamma) \,. \tag{7.11}$$

Proof.

We only prove (7.10), the proof of (7.11) is analogous. Recall that for $\Psi = {}^t(\psi_0,\psi_1)$ in $\left(L^2(\mathbb{R}\times S^2\,;\ \mathrm{d}r_*\mathrm{d}\omega)\right)^2$, the action of $e^{it\mathbb{D}_\infty}$ on Ψ is given by :

$$\left(e^{it\mathbb{D}_\infty}\Psi\right)(r_*,\omega) = \begin{pmatrix} \psi_0(r_*+t,\omega) \\ \psi_1(r_*-t,\omega) \end{pmatrix} .$$

We establish the following properties, fundamental for the proof. Consider $\Psi \in \left(\mathcal{C}_0^\infty(\mathbb{R}\times S^2)\right)^2$ and $J \in \mathcal{C}_\infty(\mathbb{R})$. We have :

1. if $J \equiv 0$ in a neighbourhood of 1, then

$$\lim_{t\to+\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty} \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix} = 0 \,;$$

2. if $J \equiv 0$ in a neighbourhood of -1, then

$$\lim_{t\to+\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty} \left(\begin{array}{c}\psi_0\\0\end{array}\right) = 0\,.$$

We establish the first limit. Let $\varepsilon > 0$ such that $J \equiv 0$ in $[1-\varepsilon, 1+\varepsilon]$,

$$\begin{aligned}&\left\| J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty} \left(\begin{array}{c}0\\\psi_1\end{array}\right)\right\|^2\\ &\le\ C\left\{\int_{-\infty}^{(1-\varepsilon)t} |\psi_1(r_*-t,\omega)|^2\,\mathrm{d}r_*\mathrm{d}\omega + \int_{(1+\varepsilon)t}^{+\infty} |\psi_1(r_*-t,\omega)|^2\,\mathrm{d}r_*\mathrm{d}\omega\right\}\end{aligned}$$

which is zero for t large enough. The proof of the second limit is similar. It follows that :

1. if $J \equiv 0$ in a neighbourhood of $\{-1, 1\}$, then

$$s-\lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty} = 0\,,$$

2. if $J \equiv 1$ in a neighbourhood of 1 and $J \equiv 0$ in a neighbourhood of -1, then

$$\lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}\Psi = \lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}\left(\begin{array}{c}0\\\psi_1\end{array}\right) = \left(\begin{array}{c}0\\\psi_1\end{array}\right)$$

since $J-1 \equiv 0$ in a neighbourhood of 1,

3. if $J \equiv 1$ in a neighbourhood of -1 and $J \equiv 0$ in a neighbourhood of 1, then

$$\lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}\Psi = \lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}\left(\begin{array}{c}\psi_0\\0\end{array}\right) = \left(\begin{array}{c}\psi_0\\0\end{array}\right)$$

since $J-1 \equiv 0$ in a neighbourhood of -1.

Hence the result. □

We now introduce the following wave operators, whose existence is a trivial consequence of our previous results :

$$\begin{aligned}\mathcal{V}_\infty^+ &:= s-\lim_{t\to+\infty} e^{-it\not{D}_\infty} e^{it\mathbb{D}_\infty}\,,\ \tilde{\mathcal{V}}_\infty^+ := s-\lim_{t\to+\infty} e^{-it\mathbb{D}_\infty} e^{it\not{D}_\infty} = \left(\mathcal{V}_\infty^+\right)^*\,,\\ \mathcal{V}_H^+ &:= s-\lim_{t\to+\infty} e^{-it\not{D}_H} e^{it\mathbb{D}_H}\,,\ \tilde{\mathcal{V}}_H^+ := s-\lim_{t\to+\infty} e^{-it\mathbb{D}_H} e^{it\not{D}_H} = \left(\mathcal{V}_H^+\right)^*\,.\end{aligned}$$

The asymptotic velocity constructed for $\mathbb{D}_\infty$ and $\mathbb{D}_H$ in lemma 7.3, together with these operators give us asymptotic velocities for $\not{D}_\infty$ and $\not{D}_H$:

Corollary 7.1. *For all $J \in \mathcal{C}_\infty(\mathbb{R})$,*

$$\exists\ s-\lim_{t\to+\infty} e^{-it\not{D}_H} J\left(\frac{r_*}{t}\right) e^{it\not{D}_H} = \mathcal{V}_H^+ J(-\gamma)\tilde{\mathcal{V}}_H^+\,, \tag{7.12}$$

$$\exists\ s-\lim_{t\to+\infty} e^{-it\not{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\not{D}_\infty} = \mathcal{V}_\infty^+ J(-\gamma)\tilde{\mathcal{V}}_\infty^+\,, \tag{7.13}$$

$$\exists\ s-\lim_{t\to+\infty} e^{-it\not{D}_K} J\left(\frac{r_*}{t}\right) e^{it\not{D}_K} = \mathfrak{W}_H^+ J(-\gamma)\tilde{\mathfrak{W}}_H^+ + \mathfrak{W}_\infty^+ J(-\gamma)\tilde{\mathfrak{W}}_\infty^+\,. \tag{7.14}$$

Proof.

We only establish (7.14) ; the proof of (7.12) and (7.13) is trivial. We consider $j_\pm, j_0 \in \mathcal{C}_b^\infty(\mathbb{R})$ such that

$$\mathrm{supp}\, j_+ \subset]0, +\infty[\,,\ \mathrm{supp}\, j_- \subset]-\infty, 0[\,,\ \mathrm{supp}\, j_0 \subset \left]-\frac{1}{2}, \frac{1}{2}\right[\ ,\ j_-^2 + j_0^2 + j_+^2 \equiv 1\,.$$

We have clearly

$$s - \lim_{t\to+\infty} e^{-it\not{D}_K} j_0(r_*) J\left(\frac{r_*}{t}\right) j_0(r_*) e^{it\not{D}_K} = 0\,.$$

Now we have the following strong convergence as $t \to +\infty$:

$$\begin{aligned} & e^{-it\not{D}_K} j_+(r_*) J\left(\frac{r_*}{t}\right) j_+(r_*) e^{it\not{D}_K} \\ = \;& e^{-it\not{D}_K} j_+(r_*) e^{it\mathbb{D}_\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty} e^{-it\mathbb{D}_\infty} j_+(r_*) e^{it\not{D}_K} \longrightarrow \mathfrak{W}_\infty^+ J(-\gamma) \tilde{\mathfrak{W}}_\infty^+ \end{aligned}$$

and we have an analogous result for j_-. □

Using [15, proposition B.2.1], we obtain the existence of self-adjoint operators P^+, P_H^+ and P_∞^+ such that, for all $J \in \mathcal{C}_\infty$,

$$J(P^+) = s - \lim_{t\to+\infty} e^{-it\not{D}_K} J\left(\frac{r_*}{t}\right) e^{it\not{D}_K}\ ,\ J(P_H^+) = s - \lim_{t\to+\infty} e^{-it\not{D}_H} J\left(\frac{r_*}{t}\right) e^{it\not{D}_H}\ ,$$
$$J(P_\infty^+) = s - \lim_{t\to+\infty} e^{-it\not{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\not{D}_\infty}\ .$$

They are referred to as the asymptotic velocities associated with $\not{D}_K$, $\not{D}_H$ and $\not{D}_\infty$. By lemma 7.3, $-\gamma$ is the asymptotic velocity associated with $\mathbb{D}_H$ and $\mathbb{D}_\infty$. We next calculate the spectra of P^+, P_H^+ and P_∞^+, describing for each Hamiltonian the allowed radial propagation speeds.

Lemma 7.4.

$$\sigma(P^+) = \sigma(P_H^+) = \sigma(P_\infty^+) = \{-1, 1\}\ .$$

Proof.

The result for P_H^+ and P_∞^+ is clear. Using

$$\mathrm{ran}\tilde{\mathfrak{W}}_H^+ = \mathcal{H}^- \text{ and } \mathrm{ran}\tilde{\mathfrak{W}}_\infty^+ = \mathcal{H}^+\,,$$

we find

$$J(P^+) = J(1)\mathfrak{W}_\infty^+\tilde{\mathfrak{W}}_\infty^+ + J(-1)\mathfrak{W}_H^+\tilde{\mathfrak{W}}_H^+\,.$$

Clearly, if $J(1) = J(-1) = 0$, we have $J(P^+) = 0$. In the case where $J(-1) \neq 0$, we choose $\Psi \in \mathcal{H}^-$, $\Psi \neq 0$ and we put $\Phi := \mathfrak{W}_H^+\Psi$. We have $\Phi \neq 0$ since $\|\mathfrak{W}_H^+\Psi\| = \|\Psi\|$. Applying $J(P^+)$ to Φ, we find

$$J(P^+)\Phi = J(-1)\mathfrak{W}_H^+\tilde{\mathfrak{W}}_H^+\Phi = J(-1)\Phi \neq 0\,,$$

since $\mathrm{ran}\mathfrak{W}_H^+ \subset \ker\tilde{\mathfrak{W}}_\infty^+$. A similar construction can be done for $J(1) \neq 0$. □

Remark 7.3. *As a trivial consequence of the previous lemma, we have :*

$$P^+ = \mathfrak{W}_H^+(-\gamma)\tilde{\mathfrak{W}}_H^+ + \mathfrak{W}_\infty^+(-\gamma)\tilde{\mathfrak{W}}_\infty^+\ ,$$
$$P_H^+ = \mathcal{V}_H^+(-\gamma)\tilde{\mathcal{V}}_H^+\ ,\quad P_\infty^+ = \mathcal{V}_\infty^+(-\gamma)\tilde{\mathcal{V}}_\infty^+\ .$$

7.3.3 Proof of theorems 2 and 3

Let $0 < \varepsilon < 1$, $J \in \mathcal{C}_b^\infty(\mathbb{R})$, $\mathrm{supp} J \subset]-\infty, 0[$, such that $J \equiv 1$ on $]-\infty, -\varepsilon[$. Let $j_\pm$, j_0 a partition of unity as in the proof of corollary 7.1. We show that

$$s - \lim_{t\to+\infty} e^{-it\not{D}_H} J\left(\frac{r_*}{t}\right) e^{it\not{D}_K} = s - \lim_{t\to+\infty} e^{-it\not{D}_H} j_-^2(r_*) e^{it\not{D}_K} .$$

We clearly have

$$s - \lim_{t\to+\infty} e^{-it\not{D}_H} j_0^2(r_*) J\left(\frac{r_*}{t}\right) e^{it\not{D}_K} = s - \lim_{t\to+\infty} e^{-it\not{D}_H} j_+^2(r_*) J\left(\frac{r_*}{t}\right) e^{it\not{D}_K} = 0 .$$

Now, for $\Psi \in \mathcal{H}$,

$$\begin{aligned} & e^{-it\not{D}_H} j_-^2(r_*) J\left(\frac{r_*}{t}\right) e^{it\not{D}_K}\Psi \\ = \;& e^{-it\not{D}_H} e^{it\mathbb{D}_H} e^{-it\mathbb{D}_H} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_H} e^{-it\mathbb{D}_H} j_-^2(r_*) e^{it\not{D}_K}\Psi \\ & \longrightarrow \mathcal{V}_H^+ J(-\gamma) \tilde{\mathfrak{W}}_H^+ \Psi = \mathcal{V}_H^+ \tilde{\mathfrak{W}}_H^+ \Psi = \tilde{\Omega}_H^+ \Psi \quad \text{as } t \to +\infty , \end{aligned}$$

since $J(-\gamma) = P_{\mathcal{H}^-}$ and $\mathcal{H}^- = \mathrm{ran} \tilde{\mathfrak{W}}_H^+$. The proof is similar for the other limits. □

8 Geometric interpretation

All the constructions of this section are based on the skeleton of the conformal geometry of block I : the two congruences of principal null geodesics. We start by giving a new version of theorem 2, using the flow of principal null geodesics as comparison dynamics. The new wave operators are denoted as in theorem 2 but with an additional index *pn* for "principal null". Then, we interpret geometrically this scattering theory as providing the solution to a Goursat problem on a singular null hypersurface on the Penrose compactification of block I.

- The first step is to interpret the inverse wave operators $\tilde{\mathfrak{W}}_{H,pn}^\pm$ as representations of trace operators on the future (resp. past) horizon. We describe the standard choices of coordinates used to understand the horizon as the union of two smooth null hypersurfaces at the boundary of block I. This provides an explicit diffeomorphism between the $\{t = 0\}$ hypersurface and the future (resp. past) horizon. Next, we construct a spin-frame that is regular in the neighbourhood of the horizon, and we describe its relation with (o^A, ι^A) defined in section 2. This, together with standard regularity results for hyperbolic equations (essentially Leray's theorem), enables us to define traces on the future and past horizons for the vector Ψ. We then understand the operators $\tilde{\mathfrak{W}}_{H,pn}^\pm$ as the pull-back of the trace operators by the explicit diffeomorphisms.
- The next step is a similar interpretation of the wave operators $\tilde{\mathfrak{W}}_{\infty,pn}^\pm$ as trace operators on future and past null infinities ; this is based on the Penrose compactification of the exterior of the black hole.

Each of the global inverse wave operators $\tilde{W}_{pn}^\pm$ is then understood as a trace operator, on the union of two of the previous null hypersurfaces : the future (resp. past) horizon and future (resp. past) null infinity. This larger null hypersurface is singular at the junction of the two smooth surfaces (more precisely, the conformal metric is singular there). The direct operators $W_{pn}^\pm$ therefore solve the corresponding Goursat problems on these singular null hypersurfaces.

8.1 Theorem 2 in terms of principal null geodesics

We introduce the vector fields $v^{\pm}$, generating the outgoing and incoming principal null geodesics, normalized so that their flows preserve the foliation $\{\Sigma_t\}_t$:

$$v^{\pm} := \frac{\Delta}{r^2+a^2} V^{\pm} = \frac{\partial}{\partial t} \pm \frac{\Delta}{r^2+a^2}\frac{\partial}{\partial r} + \frac{a}{r^2+a^2}\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial r_*} + \frac{a}{r^2+a^2}\frac{\partial}{\partial \varphi}\,.$$

We introduce the spatial part $w^{\pm}$ of $v^{\pm}$:

$$w^{\pm} := \pm\frac{\partial}{\partial r_*} + \frac{a}{r^2+a^2}\frac{\partial}{\partial \varphi}\,.$$

The flow of the vector field $w^{\pm}$, acting on Σ, is the dynamics generated by the Hamiltonian $P_N^{\pm}$ (P_N for principal null), defined by

$$P_N^{\pm} = \mp D_{r_*} - \frac{a}{r^2+a^2} D_{\varphi}\,.$$

We denote by $F_{w^{\pm}}(t)$ the flow of $w^{\pm}$ on Σ and $F_{v^{\pm}}(t)$ the flow of $v^{\pm}$ on $\mathbb{R}\times\Sigma$. They are related as follows :

$$F_{v^{\pm}}(t)(t_0, r_0, \theta_0, \varphi_0) = (t+t_0, F_{w^{\pm}}(t)(r_0,\theta_0,\varphi_0))$$

and the group associated with $P_N^{\pm}$ is expressed in terms of $F_{w^{\pm}}$ as

$$\left(e^{itP_N^{\pm}} g\right)(r_0,\theta_0,\varphi_0) = g\left(F_{w^{\pm}}(-t)(r_0,\theta_0,\varphi_0)\right)\,.$$

We introduce the comparison operator

$$\mathbf{P_N} = \begin{pmatrix} P_N^- & 0 \\ 0 & P_N^+ \end{pmatrix} = \gamma D_{r_*} - \frac{a}{r^2+a^2} D_{\varphi}\,.$$

Its action on $\mathcal{H}$ is described in terms of the flows of $w^{\pm}$:

$$\left(e^{it\mathbf{P_N}} G\right)(r_0,\theta_0,\varphi_0) = \begin{pmatrix} g_0\left(F_{w^-}(-t)(r_0,\theta_0,\varphi_0)\right) \\ g_1\left(F_{w^+}(-t)(r_0,\theta_0,\varphi_0)\right) \end{pmatrix}\,,\quad G = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \in \mathcal{H}\,.$$

The operator $\mathbf{P_N}$ is self-adjoint on $\mathcal{H}$ and its spaces of incoming and outgoing data are $\mathcal{H}^-$ and $\mathcal{H}^+$. Moreover, the results of theorem 2 are still valid if, instead of $\mathbb{D}_H$ and $\mathbb{D}_\infty$, we use $\mathbf{P_N}$ as comparison dynamics in the neighbourhoods of both the horizon and infinity. This can also be expressed as follows :

Theorem 9. *The following strong limits*

$$\begin{aligned}
\mathfrak{W}^{\pm}_{H,pn} &:= s-\lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\mathbf{P_N}} P_{\mathcal{H}^{\mp}}\,, && (8.1)\\
\mathfrak{W}^{\pm}_{\infty,pn} &:= s-\lim_{t\to\pm\infty} e^{-it\not{D}_K} e^{it\mathbf{P_N}} P_{\mathcal{H}^{\pm}}\,, && (8.2)\\
\tilde{\mathfrak{W}}^{\pm}_{H,pn} &:= s-\lim_{t\to\pm\infty} e^{-it\mathbf{P_N}} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^{\pm})\,, && (8.3)\\
\tilde{\mathfrak{W}}^{\pm}_{\infty,pn} &:= s-\lim_{t\to\pm\infty} e^{-it\mathbf{P_N}} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^{\pm})\,. && (8.4)
\end{aligned}$$

exist and satisfy the same properties as the wave operators of theorem 2. The corresponding global wave operators are :

$$\begin{aligned} W^+_{pn} : \quad & \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H}\,, \\ & ((\psi_0, 0)\,, (0, \psi_1)) \longmapsto \mathfrak{W}^+_{H,pn}\,(\psi_0, 0) + \mathfrak{W}^+_{\infty,pn}\,(0, \psi_1)\ , \end{aligned} \tag{8.5}$$

$$\begin{aligned} W^-_{pn} : \quad & \mathcal{H}^+ \oplus \mathcal{H}^- \longrightarrow \mathcal{H} \\ & ((0, \psi_1)\,, (\psi_0, 0)) \longmapsto \mathfrak{W}^-_{H,pn}\,(0, \psi_1) + \mathfrak{W}^-_{\infty,pn}\,(\psi_0, 0)\ . \end{aligned} \tag{8.6}$$

$$\tilde{W}^+_{pn} : \mathcal{H} \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+\,,\ \tilde{W}^+_{pn}\Psi = \left(\tilde{\mathfrak{W}}^+_{H,pn}\Psi\,,\ \tilde{\mathfrak{W}}^+_{\infty,pn}\Psi\right)\ , \tag{8.7}$$

$$\tilde{W}^-_{pn} : \mathcal{H} \longrightarrow \mathcal{H}^+ \oplus \mathcal{H}^-\,,\ \tilde{W}^-_{pn}\Psi = \left(\tilde{\mathfrak{W}}^-_{H,pn}\Psi\,,\ \tilde{\mathfrak{W}}^-_{\infty,pn}\Psi\right)\ . \tag{8.8}$$

Proof. It is a straightforward consequence of theorem 2 and of the fact that $\mathbf{P_N}$ is a short-range perturbation of $\mathbb{D}_H$ as $r_* \to -\infty$, of $\mathbb{D}_\infty$ as $r_* \to +\infty$ and that all three Hamiltonians $\mathbf{P_N}$, $\mathbb{D}_H$ and $\mathbb{D}_\infty$ satisfy Huygens's principle. This, by Cook's method and the chain rule, allows us to prove existence of direct and inverse wave operators. □

Remark 8.1. *The wave operators in the above theorem can also be defined in terms of cut-off functions $j_\pm$ as in the proof of theorem 2 ; the two definitions are equivalent. We have for example*

$$\mathfrak{W}^+_{H,pn} := s - \lim_{t\to+\infty} e^{-it\not{D}_K} j_- e^{it\mathbf{P_N}}\,.$$

8.2 Inverse wave operators at the horizon as trace operators

8.2.1 Kerr-star and star-Kerr coordinates

A full account of these two coordinate systems can be found in [48]. The Kerr-star coordinate system $(t^*, r, \theta, \varphi^*)$ is based on incoming principal null geodesics, parametrized as the integral lines of the vector V^- (defined in (2.3)). The new coordinates t^* and φ^* are of the form

$$t^* = t + T(r)\,,\ \varphi^* = \varphi + \Lambda(r)\,,$$

where the functions T and Λ are required to satisfy

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{r^2+a^2}{\Delta}\,,\ \frac{\mathrm{d}\Lambda}{\mathrm{d}r} = \frac{a}{\Delta}\,. \tag{8.9}$$

The incoming principal null geodesics now appear as the r coordinate curves parametrized by $s = -r$:

$$\dot{r} = -1\,,\ \dot{\theta} = 0\,,\ \dot{t^*} = \dot{t} + \frac{\mathrm{d}T}{\mathrm{d}r}\dot{r} = 0\,,\ \dot{\varphi^*} = \dot{\varphi} + \frac{\mathrm{d}\Lambda}{\mathrm{d}r}\dot{r} = 0\,.$$

Remark 8.2. *In other words, in Kerr-star coordinates, the vector V^- takes the form*

$$V^- = -\frac{\partial}{\partial r}\,.$$

It will be useful in what follows[4] to express the Boyer-Lindquist coordinate vector fields in terms of the Kerr-star coordinate vector fields. In order to avoid confusion, we denote respectively

[4] When studying the behaviour of the Newman-Penrose tetrad l^a, n^a, m^a, $\bar{m}^a$ at null infinity.

$(\partial/\partial r)_{BL}$ *and* $(\partial/\partial r)_{K*}$ *the* r *coordinate vector fields in Boyer-Lindquist and Kerr-star coordinates respectively ; we do the same for the* θ *vector fields. We have :*

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}, \tag{8.10}$$

$$\left(\frac{\partial}{\partial \theta}\right)_{BL} = \left(\frac{\partial}{\partial \theta}\right)_{K*}, \tag{8.11}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varphi^*}, \tag{8.12}$$

$$\left(\frac{\partial}{\partial r}\right)_{BL} = \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t^*} + \left(\frac{\partial}{\partial r}\right)_{K*} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi^*}. \tag{8.13}$$

Kerr-star coordinates are defined globally on block I[5]. The Kerr metric in Kerr-star coordinates takes the form

$$g = g_{tt}\mathrm{d}t^{*2} + 2g_{t\varphi}\mathrm{d}t^*\mathrm{d}\varphi^* + g_{\varphi\varphi}\mathrm{d}\varphi^{*2} + g_{\theta\theta}\mathrm{d}\theta^2 - 2\mathrm{d}t^*\mathrm{d}r + 2a\sin^2\theta\,\mathrm{d}\varphi^*\mathrm{d}r\,, \tag{8.14}$$

where g_{tt}, $2g_{t\varphi}$, $g_{\theta\theta}$ and $g_{\varphi\varphi}$ are the coefficients of $\mathrm{d}t^2$, $\mathrm{d}t\mathrm{d}\varphi$, $\mathrm{d}\theta^2$ and $\mathrm{d}\varphi^2$ in the expression (2.1) of g in Boyer-Lindquist coordinates :

$$g_{tt} = 1 - \frac{2Mr}{\rho^2}\,,\ g_{t\varphi} = \frac{2aMr\sin^2\theta}{\rho^2}\,,\ g_{\theta\theta} = -\rho^2\,,\ g_{\varphi\varphi} = -\frac{\sigma^2}{\rho^2}\sin^2\theta\,.$$

The expression (8.14) shows that g can be extended smoothly across the horizon $\{r = r_+\}$. Besides, it does not degenerate there since its determinant is given by

$$\det(g) = -\rho^4\sin^2\theta$$

and does not vanish for $r = r_+$. Thus, we can add the horizon to block I as a smooth boundary. It is important at this point to understand the nature of the boundary we have just glued to block I. The hypersurface

$$\mathfrak{H}^+ := \mathbb{R}_{t^*} \times \{r = r_+\} \times S^2_{\theta,\varphi^*}$$

is reached along incoming null geodesics ; it is the horizon that is reached as $t \to +\infty$ by light rays or material bodies falling into the black hole and not the horizon seen as $\mathbb{R}_t \times \{r_+\}_r \times S^2_{\theta,\varphi}$ in Boyer-Lindquist coordinates. We refer to it as the future horizon. It is a smooth hypersurface in the space-time $(\mathcal{B}_I \cup \mathfrak{H}^+\,,\ g)$. We can easily show that it is a null hypersurface. The metric induced by g on hypersurfaces of constant r,

$$g_r = g_{tt}\mathrm{d}t^{*2} + 2g_{t\varphi}\mathrm{d}t^*\mathrm{d}\varphi^* + g_{\varphi\varphi}\mathrm{d}\varphi^{*2} - \rho^2\mathrm{d}\theta^2\,,$$

has determinant

$$\det(g_r) = -\rho^2\left(g_{tt}g_{\varphi\varphi} - (g_{t\varphi})^2\right) = \rho^2\Delta\sin^2\theta$$

and thus degenerates for $\Delta = 0$, i.e. at $\mathfrak{H}^+$. Since g does not degenerate, it follows that one of the generators of $\mathfrak{H}^+$ is null (i.e. $\mathfrak{H}^+$ is a null hypersurface).

Star-Kerr coordinates $({}^*t, r, \theta, {}^*\varphi)$ are constructed using the outgoing principal null geodesics parametrized as the integral lines of V^+. We have

$${}^*t = t - T(r)\,,\ {}^*\varphi = \varphi - \Lambda(r)\,,$$

with the same functions T and Λ as for Kerr-star coordinates. Consequently the outgoing principal null geodesics appear as the r coordinate curves parametrized by r.

[5]With the exception of the axis ($\theta = 0$ and $\theta = \pi$) ; this coordinate singularity, similar to that of spherical coordinates on $\mathbb{R}^3$, can be dealt with simply (see [48] lemma 2.2.2), we shall systematically ignore it.

Remark 8.3. *It is equivalent to say that in star-Kerr coordinates, we have*

$$V^+ = \frac{\partial}{\partial r}\,.$$

As we did for Kerr-star coordinates, we express the Boyer-Lindquist coordinate vector fields in terms of the star-Kerr coordinate vector fields :

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial {}^*t}\,, \tag{8.15}$$

$$\left(\frac{\partial}{\partial \theta}\right)_{BL} = \left(\frac{\partial}{\partial \theta}\right)_{*K}\,, \tag{8.16}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial {}^*\varphi}\,, \tag{8.17}$$

$$\left(\frac{\partial}{\partial r}\right)_{BL} = -\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial {}^*t} + \left(\frac{\partial}{\partial r}\right)_{*K} - \frac{a}{\Delta}\frac{\partial}{\partial {}^*\varphi}\,. \tag{8.18}$$

This coordinate system allows us to add the past horizon

$$\mathfrak{H}^- := \mathbb{R}_{{}^*t} \times \{r = r_+\}_r \times S^2_{\theta,{}^*\varphi}$$

as a smooth null boundary to block I. This is a white hole horizon from whence light rays (and in particular outgoing principal null geodesics) emerge. White holes are natural features of the maximal extension of space-times containing eternal black holes (for a description of maximal Kerr space-time, see [8] or [48], or [46] for a shorter account).

The future and past horizons, that we have understood as smooth null boundaries to block I, are both reached for infinite values of t, they do not contain any point of the horizon described in Boyer-Lindquist coordinates as $\mathbb{R}_t \times \{r_+\} \times S^2_{\theta,\varphi}$. In the next subsection, we describe a coordinate system that gives us a global description of the horizon, encompassing the future and past horizons as well as the Boyer-Lindquist horizon. It will be useful to know the behaviour of t^* and ${}^*t$ on both $\mathfrak{H}^+$ and $\mathfrak{H}^-$:

Properties. *We have constructed $\mathfrak{H}^+$ (resp. $\mathfrak{H}^-$) using a coordinate system $(t^*, r, \theta, \varphi^*)$ (resp. $({}^*t, r, \theta, {}^*\varphi)$) that is regular across the horizon along incoming (resp. outgoing) principal null geodesics. The Kerr-star variable t^*, defined by*

$$t^* = t + T(r)\,,\ \frac{\mathrm{d}T}{\mathrm{d}r} = \frac{r^2+a^2}{\Delta}\,,$$

is constant on incoming principal null geodesics. Along outgoing principal null geodesics, its behaviour is best described in terms of star-Kerr coordinates :

$$t^* = {}^*t + 2T(r)\,.$$

Hence, on an outgoing principal null geodesic, t^* tends to $+\infty$ as $r \to +\infty$ and to $-\infty$ as $r \to r_+$. Consequently, t^* is regular on $\mathfrak{H}^+$, takes all real values on $\mathfrak{H}^+$, and tends to $-\infty$ on $\mathfrak{H}^-$. Similarly, ${}^*t$ is regular on $\mathfrak{H}^-$, takes all real values on $\mathfrak{H}^-$, and tends to $+\infty$ on $\mathfrak{H}^+$.

8.2.2 Kruskal-Boyer-Lindquist coordinates

This coordinate system, discovered in 1967 by R.H. Boyer and R.W. Lindquist [8], is made of a combination of the two Kerr coordinate systems, modified in such a way that it is regular on both the future and the past horizons. We describe briefly its definition and properties ; for details, see [8] or [48]. The time and radial variables are replaced by

$$U = e^{-\kappa_+ {}^*t} \,,\; V = e^{\kappa_+ t^*} \,, \tag{8.19}$$

where κ_+ is the surface gravity at the outer horizon, given by (2.61). The coordinate θ is conserved since it is regular on all three blocks (except on the axes, but as remarked above, this singularity is no more serious than that of standard spherical coordinates and we ignore it). The longitude function is defined by

$$\varphi^\sharp = \frac{1}{2}\left(\varphi^* + {}^*\varphi - \frac{a}{r_+^2 + a^2}\left(t^* + {}^*t\right)\right) = \varphi - \frac{a}{r_+^2 + a^2}\, t \,. \tag{8.20}$$

It is chosen so that the principal null geodesics in the future and past horizons are coordinate curves. The functions $(U, V, \theta, \varphi^\sharp)$ form an analytic coordinate system on $\mathcal{B}_I \cup \mathfrak{H}^+ \cup \mathfrak{H}^- - (axes)$. In this coordinate system, we have

$$\mathcal{B}_I =]0, +\infty[_U \times]0, +\infty[_V \times S^2_{\theta,\varphi^\sharp} \,,$$
$$\mathfrak{H}^+ = \{0\}_U \times [0, +\infty[_V \times S^2_{\theta,\varphi^\sharp} \,,\; \mathfrak{H}^- = [0, +\infty[_U \times \{0\}_V \times S^2_{\theta,\varphi^\sharp} \,,$$

simply because t^* (resp. ${}^*t$) is regular at $\mathfrak{H}^+$ (resp. $\mathfrak{H}^-$), takes all real values on $\mathfrak{H}^+$ (resp. $\mathfrak{H}^-$), and tends to $-\infty$ (resp. $+\infty$) at $\mathfrak{H}^-$ (resp. $\mathfrak{H}^+$). In these new coordinates, the Kerr metric takes the form

$$\begin{aligned} g \;=\; & -\frac{G_+^2 a^2 \sin^2\theta}{4\kappa_+^2 \rho^2} \frac{(r - r_-)(r + r_+)}{(r^2 + a^2)\left(r_+^2 + a^2\right)} \left[\frac{\rho^2}{r^2 + a^2} + \frac{\rho_+^2}{r_+^2 + a^2}\right] \left(U^2 \mathrm{d}V^2 + V^2 \mathrm{d}U^2\right) \\ & -\frac{G_+(r - r_-)}{2\kappa_+^2 \rho^2} \left[\frac{\rho^4}{(r^2 + a^2)^2} + \frac{\rho_+^4}{\left(r_+^2 + a^2\right)^2}\right] \mathrm{d}U \mathrm{d}V \\ & -\frac{G_+ a \sin^2\theta}{\kappa_+ \rho^2 \left(r_+^2 + a^2\right)} \left[\rho_+^2 (r - r_-) + \left(r^2 + a^2\right)(r + r_+)\right] (U \mathrm{d}V - V \mathrm{d}U)\, \mathrm{d}\varphi^\sharp \\ & -\rho^2 \mathrm{d}\theta^2 - g_{\varphi\varphi} \left(\mathrm{d}\varphi^\sharp\right)^2 , \end{aligned} \tag{8.21}$$

where

$$\rho_+^2 = r_+^2 + a^2 \cos^2\theta \;,\;\; G_+ = \frac{r - r_+}{UV} = e^{-2\kappa_+ r} \left|r - r_-\right|^{r_-/r_+} \,.$$

The functions r and G_+ are analytic and non-vanishing on $[0, +\infty[_U \times [0, +\infty[_V$. The expression (8.21) shows that g is smooth on $\mathcal{B}_I \cup \mathfrak{H}^+ \cup \mathfrak{H}^-$ and can be extended smoothly on $[0, +\infty[_U \times [0, +\infty[_V \times S^2_{\theta,\varphi^\sharp}$. The 2-sphere $\{U = V = 0\}$, where the future and past horizons meet, is called the crossing sphere ; we denote it by S^2_{c}. It is a regular surface in the extended space-time

$$\left(\mathcal{B}_I^{\mathrm{KBL}} := [0, +\infty[_U \times [0, +\infty[_V \times S^2_{\theta,\varphi^\sharp} \,,\; g\right)$$

and it represents the whole horizon as described in Boyer-Lindquist coordinates (i.e. $\mathbb{R}_t \times \{r_+\}_r \times S^2_{\theta,\varphi}$). Hence, the Kruskal-Boyer-Lindquist coordinates give us a global description of the horizon

$$\mathfrak{H} = \mathfrak{H}^- \cup S^2_{\mathrm{c}} \cup \mathfrak{H}^+ = \left([0, +\infty[_U \times \{0\}_V \times S^2_{\theta,\varphi^\sharp}\right) \cup \left(\{0\}_U \times [0, +\infty[_V \times S^2_{\theta,\varphi^\sharp}\right)$$

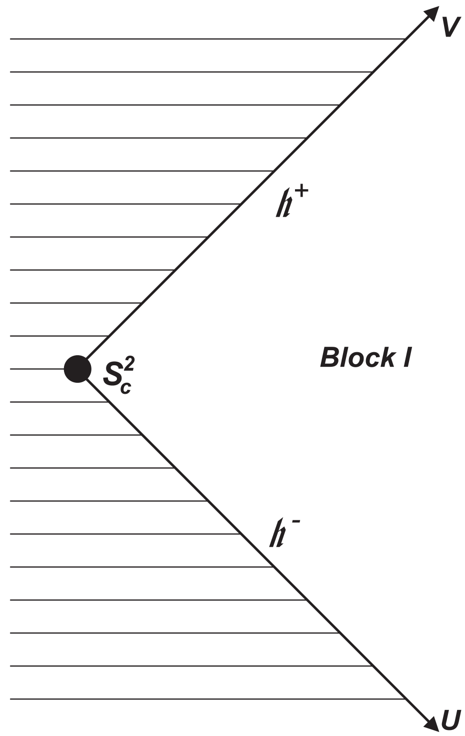


Figure 1: The extended space-time $\mathcal{B}_I^{\mathrm{KBL}}$ in Kruskal-Boyer-Lindquist coordinates.

as a union of two smooth null boundaries $\mathfrak{H}^+ \cup S^2_{\mathrm{c}}$ and $S^2_{\mathrm{c}} \cup \mathfrak{H}^-$ (see figure 1 for a picture of the extended space-time $\mathcal{B}_I^{\mathrm{KBL}}$). This allows us to construct a spin-frame that behaves smoothly at the horizon. As we shall see below, such a spin-frame will be fundamental for the interpretation of $\tilde{\mathfrak{W}}^{\pm}_{H,pn}$ as trace operators on the horizon. We define the new spin-frame by choosing a Newman-Penrose tetrad that is regular at the horizon. We start with the Kinnersley-type tetrad l^a, n^a, m^a, $\bar{m}^a$ defined by (2.24)-(2.27). We express its vectors in the Kruskal-Boyer-Lindquist basis and rescale the frame vectors when necessary, so that they neither vanish nor blow up at the horizon.

The relation between the coordinate vector fields of Kruskal-Boyer-Lindquist coordinates and of Boyer-Lindquist coordinates is given by

$$\frac{\partial}{\partial t} = \kappa_+ \left(-U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} \right) - \frac{a}{r_+^2 + a^2} \frac{\partial}{\partial \varphi^\sharp} \, , \quad \frac{\partial}{\partial r} = \kappa_+ \frac{r^2+a^2}{\Delta} \left(U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} \right) \, ,$$
$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} \, , \; \frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varphi^\sharp} \, .$$

This yields the expression of the null vectors (2.24)-(2.27) in terms of Kruskal-Boyer-Lindquist coordinates :

$$\begin{aligned}
l^a \frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\Delta\rho^2}} \left(2\kappa_+ \left(r^2+a^2\right) V \frac{\partial}{\partial V} - \frac{a\left(r^2 - r_+^2\right)}{r_+^2+a^2} \frac{\partial}{\partial \varphi^\sharp} \right) , \\
n^a \frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\Delta\rho^2}} \left(-2\kappa_+ \left(r^2+a^2\right) U \frac{\partial}{\partial U} - \frac{a\left(r^2 - r_+^2\right)}{r_+^2+a^2} \frac{\partial}{\partial \varphi^\sharp} \right) , \\
m^a \frac{\partial}{\partial x^a} &= \frac{1}{p\sqrt{2}} \left(ia\kappa_+ \sin\theta \left(-U\frac{\partial}{\partial U} + V \frac{\partial}{\partial V} \right) + \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi^\sharp} \right) .
\end{aligned}$$

Lemma 8.1. *The rescaled Newman-Penrose tetrad* $\mathfrak{l}^a$, $\mathfrak{n}^a$, m^a, $\bar{m}^a$, *defined by*

$$\mathfrak{l}^a = \frac{U}{\sqrt{\Delta}} e^{-\kappa_+ r} (r - r_-)^{M/r_+} l^a \,, \; \mathfrak{n}^a = \frac{V}{\sqrt{\Delta}} e^{-\kappa_+ r} (r - r_-)^{M/r_+} n^a \,,$$

is smooth on $\mathfrak{H} - (axes)$. *The spin-frame* ($\mathfrak{o}^A$, $\mathfrak{i}^A$) *associated to* $\mathfrak{l}^a$, $\mathfrak{n}^a$, m^a, $\bar{m}^a$, *is therefore regular (off axis) on* $\mathfrak{H}$ *and is given by*

$$\mathfrak{o}^A = \left(\frac{U}{\sqrt{\Delta}} e^{-\kappa_+ r} (r - r_-)^{M/r_+} \right)^{1/2} o^A \,, \; \mathfrak{i}^A = \left(\frac{V}{\sqrt{\Delta}} e^{-\kappa_+ r} (r - r_-)^{M/r_+} \right)^{1/2} \iota^A \,. \tag{8.22}$$

Proof. The vector m^a is clearly regular off axis on $\mathcal{B}_I^{\mathrm{KBL}}$. For $\mathfrak{l}^a$, we have

$$\begin{aligned} \mathfrak{l}^a \frac{\partial}{\partial x^a} &= e^{-\kappa_+ r} (r - r_-)^{M/r_+} \left(\frac{2\kappa_+ \left(r^2 + a^2\right)}{\sqrt{2\rho^2}} \frac{UV}{\Delta} \frac{\partial}{\partial V} + \frac{a \left(r_+^2 - r^2\right) U}{\left(r_+^2 + a^2\right) \sqrt{2\rho^2} \Delta} \frac{\partial}{\partial \varphi^\sharp} \right) \\ &= e^{-\kappa_+ r} (r - r_-)^{M/r_+} \left(\frac{2\kappa_+ \left(r^2 + a^2\right)}{\sqrt{2\rho^2}} \frac{1}{(r - r_-) G_+} \frac{\partial}{\partial V} - \frac{a \left(r + r_+\right) U}{\left(r_+^2 + a^2\right) (r - r_-) \sqrt{2\rho^2}} \frac{\partial}{\partial \varphi^\sharp} \right) \end{aligned}$$

and $\mathfrak{l}^a$ is therefore regular on $\mathcal{B}_I^{\mathrm{KBL}} - (axes)$. A similar calculation can be done for $\mathfrak{n}^a$. It is also easy to check, using the definition and properties of G_+, that the tetrad satisfies

$$\mathfrak{l}_a \mathfrak{n}^a = 1 = -m_a \bar{m}^a \,, \; \mathfrak{l}_a m^a = \mathfrak{n}_a m^a = 0 \,.$$

□

8.2.3 Interpretation of the traces on the horizon

Let us consider on Σ_0 some smooth compactly supported initial data $\chi_A \in \mathcal{C}_0^\infty(\Sigma_0 \,;\, \mathbb{S}_A)$ for the Weyl equation (2.5). Then (2.5) admits a unique solution $\phi_A \in \mathcal{C}^\infty(\mathbb{R}_t \,;\, \mathcal{C}_0^\infty(\Sigma \,;\, \mathbb{S}_A))$ such that $\phi_A|_{\Sigma_0} = \chi_A$. This solution can be extended uniquely as a smooth spinor-valued function on $\mathcal{B}_I^{\mathrm{KBL}}$, still denoted ϕ_A. This is proved by first applying standard regularity results for symmetric hyperbolic systems, then extending the space-time $(\mathcal{B}_I^{\mathrm{KBL}}, g)$ beyond the horizon (a natural way is to construct the maximal analytic extension of Kerr space-time) and finally applying once more the standard theorems for symmetric hyperbolic systems (for details, see [46]). It follows that for smooth solutions of (2.5), we can naturally define the trace of ϕ_A on $\mathfrak{H}^+$ and $\mathfrak{H}^-$. This trace, projected onto the spin-frame (8.22) can easily be compared with the limit of the vector Φ (of the components of ϕ_A in the spin-frame (o^A, ι^A)) along principal null geodesics as $r \to r_+$. This is expressed by the following proposition.

Proposition 8.1. *Given* $\phi_A \in \mathcal{C}^\infty(\mathcal{B}_I^{\mathrm{KBL}})$ *a smooth solution of (2.5), we denote by* $\mathfrak{f}$ *the vector*

$$\mathfrak{f} = \begin{pmatrix} \mathfrak{f}_0 := \phi_A \mathfrak{o}^A \\ \mathfrak{f}_1 := \phi_A \mathfrak{i}^A \end{pmatrix} .$$

The trace of $\mathfrak{f}$ *on* $\mathfrak{H}^+$ *is related to the limit of* Φ *in the future along incoming principal null geodesics as follows :*

$$\begin{aligned} \mathfrak{f}_0(0, V, \theta, \varphi^\sharp) &= \lim_{r \to r_+} \left(\frac{U}{\sqrt{\Delta}} e^{-\kappa_+ r} (r - r_-)^{M/r_+} \right)^{1/2} \phi_0 \left(\gamma^-_{V,\theta,\varphi^\sharp}(r) \right) , \\ \mathfrak{f}_1(0, V, \theta, \varphi^\sharp) &= \lim_{r \to r_+} \left(\frac{V}{\sqrt{\Delta}} e^{-\kappa_+ r} (r - r_-)^{M/r_+} \right)^{1/2} \phi_1 \left(\gamma^-_{V,\theta,\varphi^\sharp}(r) \right) , \end{aligned}$$

where $\gamma^-_{V,\theta,\varphi^\sharp}(r)$ *is the incoming radial null geodesic, parametrized by* r*, that encounters* $\mathfrak{H}^+$ *at the point of Kruskal-Boyer-Lindquist coordinates* $(0,V,\theta,\varphi^\sharp)$ *; the description of this geodesic in Kerr-star coordinates is*

$$\gamma^-_{V,\theta,\varphi^\sharp}(r) = \left(t^* = \frac{1}{\kappa_+}\log(V), r, \theta, \varphi^* = \varphi^\sharp + \Lambda(r_+) + \frac{a}{r_+^2+a^2}\left(t^* - T(r_+)\right)\right),$$

T *and* Λ *being the functions defining the Kerr-star and star-Kerr coordinate systems, satisfying (8.9). Similarly, the trace of* $\mathfrak{f}$ *on* $\mathfrak{H}^-$ *is related to the limit of* Φ *in the past along outgoing principal null geodesics as follows :*

$$\begin{aligned}
\mathfrak{f}_0(U,0,\theta,\varphi^\sharp) &= \lim_{r\to r_+}\left(\frac{U}{\sqrt{\Delta}}e^{-\kappa_+ r}(r-r_-)^{M/r_+}\right)^{1/2}\phi_0\left(\gamma^+_{U,\theta,\varphi^\sharp}(r)\right),\\
\mathfrak{f}_1(U,0,\theta,\varphi^\sharp) &= \lim_{r\to r_+}\left(\frac{V}{\sqrt{\Delta}}e^{-\kappa_+ r}(r-r_-)^{M/r_+}\right)^{1/2}\phi_1\left(\gamma^+_{U,\theta,\varphi^\sharp}(r)\right),
\end{aligned}$$

where $\gamma^+_{U,\theta,\varphi^\sharp}(r)$ *is the outgoing radial null geodesic, parametrized by* r*, that encounters* $\mathfrak{H}^-$ *at the point of Kruskal-Boyer-Lindquist coordinates* $(U,0,\theta,\varphi^\sharp)$ *; the description of this geodesic in star-Kerr coordinates is*

$$\gamma^+_{U,\theta,\varphi^\sharp}(r) = \left({}^*t = -\frac{1}{\kappa_+}\log(U), r, \theta, {}^*\varphi = \varphi^\sharp - \Lambda(r_+) + \frac{a}{r_+^2+a^2}\left({}^*t + T(r_+)\right)\right).$$

Proof. It is an immediate consequence of (8.22), the definition of Kruskal-Boyer-Lindquist coordinates and the fact that in Kerr-star coordinates (resp. star-Kerr coordinates), the incoming (resp. outgoing) principal null geodesics are the r coordinate lines. □

The trace of $\mathfrak{f}$ on $\mathfrak{H}^\pm$ can then be related to the limit of the vector Ψ (defined in (2.51)), solution of equation (2.56), along incoming and outgoing principal null geodesics.

Corollary 8.1. *The vector field* Ψ *extends as a smooth vector field on* $\mathcal{B}_I^{\mathrm{KBL}}$ *and its trace on* $\mathfrak{H}^\pm$ *is naturally defined as the limit of* Ψ *as* $r\to r_+$ *along incoming or outgoing principal null geodesics. The trace of* Ψ *on* $\mathfrak{H}^+$ *is given in terms of the trace of* $\mathfrak{f}$ *on* $\mathfrak{H}^+$ *by*

$$\begin{aligned}
\Psi_{0}|_{\mathfrak{H}^+}(0,V,\theta,\varphi^\sharp) &:= \lim_{r\to r_+}\Psi_0\left(\gamma^-_{V,\theta,\varphi^\sharp}(r)\right)\\
&= \left((r_+-r_-)\,G_+(r_+)\right)^{1/4}\sqrt{pV}\,\mathfrak{f}_0(0,V,\theta,\varphi^\sharp), \qquad (8.23)\\
\Psi_{1}|_{\mathfrak{H}^+}(0,V,\theta,\varphi^\sharp) &:= \lim_{r\to r_+}\Psi_1\left(\gamma^-_{V,\theta,\varphi^\sharp}(r)\right) = 0, \qquad (8.24)
\end{aligned}$$

and the trace of Ψ *on* $\mathfrak{H}^-$ *is given in terms of the trace of* $\mathfrak{f}$ *on* $\mathfrak{H}^-$ *by*

$$\begin{aligned}
\Psi_{0}|_{\mathfrak{H}^-}(U,0,\theta,\varphi^\sharp) &:= \lim_{r\to r_+}\Psi_0\left(\gamma^+_{U,\theta,\varphi^\sharp}(r)\right) = 0, \qquad (8.25)\\
\Psi_{1}|_{\mathfrak{H}^-}(U,0,\theta,\varphi^\sharp) &:= \lim_{r\to r_+}\Psi_1\left(\gamma^+_{U,\theta,\varphi^\sharp}(r)\right)\\
&= \left((r_+-r_-)\,G_+(r_+)\right)^{1/4}\sqrt{\bar{p}U}\,\mathfrak{f}_1(U,0,\theta,\varphi^\sharp), \qquad (8.26)
\end{aligned}$$

with

$$(r_+-r_-)\,G_+(r_+) = 2\sqrt{M^2-a^2}\,e^{-2\kappa_+ r_+}.$$

Proof. The relation between Ψ and $\mathfrak{f}$ is given by, using the two expressions of G_+ :

$$\begin{aligned}\Psi &= \mathbf{U}\left(\frac{\Delta\,\sigma^2\rho^2}{(r^2+a^2)^2}\right)^{\frac14}\begin{pmatrix}\sqrt{\frac{\sqrt{\Delta}}{U}e^{\kappa_+ r}\,(r-r_-)^{-M/r_+}} & 0\\ 0 & \sqrt{\frac{\sqrt{\Delta}}{V}e^{\kappa_+ r}\,(r-r_-)^{-M/r_+}}\end{pmatrix}\mathfrak{f}\\ &= \mathbf{U}\begin{pmatrix}\sqrt{\frac{\sigma\rho}{r^2+a^2}V}\,((r-r_-)\,G_+)^{1/4} & 0\\ 0 & \sqrt{\frac{\sigma\rho}{r^2+a^2}U}\,((r-r_-)\,G_+)^{1/4}\end{pmatrix}\mathfrak{f}\,.\end{aligned}$$

The matrix

$$\mathbf{U}\begin{pmatrix}\sqrt{\frac{\sigma\rho}{r^2+a^2}V}\,((r-r_-)\,G_+)^{1/4} & 0\\ 0 & \sqrt{\frac{\sigma\rho}{r^2+a^2}U}\,((r-r_-)\,G_+)^{1/4}\end{pmatrix}$$

is smooth on $\mathcal{B}_I^{\mathrm{KBL}}$, hence Ψ extends as a smooth vector field on $\mathcal{B}_I^{\mathrm{KBL}}$. Besides, on the horizon, $\mathbf{U}$ reduces to

$$\begin{pmatrix}\sqrt{\frac{p}{\rho}} & 0\\ 0 & \sqrt{\frac{\bar p}{\rho}}\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{r_+ + ia\cos\theta}{\sqrt{r_+^2+a^2\cos^2\theta}}} & 0\\ 0 & \sqrt{\frac{r_+ - ia\cos\theta}{\sqrt{r_+^2+a^2\cos^2\theta}}}\end{pmatrix}$$

and for $r=r_+$, we have $\sigma = r^2+a^2$. These identities and the facts that $U=0$ on $\mathfrak{H}^+$ and $V=0$ on $\mathfrak{H}^-$ imply (8.23)-(8.26). $\square$

Definition 8.1. *We define the trace operators :*

$$T^+_{\mathfrak{H}} \;:\; \begin{array}{ccc}\mathcal{C}_0^\infty(\Sigma_0\,;\,\mathbb{C}^2) & \longrightarrow & \mathcal{C}^\infty(\mathfrak{H}^+\,;\,\mathbb{C})\\ \Psi_{\Sigma_0} & \longmapsto & {\psi_0}_{|_{\mathfrak{H}^+}}\end{array}\,, \tag{8.27}$$

$$T^-_{\mathfrak{H}} \;:\; \begin{array}{ccc}\mathcal{C}_0^\infty(\Sigma_0\,;\,\mathbb{C}^2) & \longrightarrow & \mathcal{C}^\infty(\mathfrak{H}^-\,;\,\mathbb{C})\\ \Psi_{\Sigma_0} & \longmapsto & {\psi_1}_{|_{\mathfrak{H}^-}}\end{array}\,, \tag{8.28}$$

where

$$\Psi = \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}$$

is the solution of (2.56) in $\mathcal{C}^\infty\left(\mathcal{B}_I^{\mathrm{KBL}}\right)$ associated with Ψ_{Σ_0}.

We are now in position, for any initial data for equation (2.56) in $\mathcal{H}$, to prove the existence of a trace on $\mathfrak{H}^\pm$ of the corresponding solution of (2.56) and to relate this trace to the image of the initial data by the inverse wave operator $\tilde{\mathfrak{W}}^\pm_{H,pn}$.

Theorem 10. *We consider the $\mathcal{C}^\infty$ diffeomorphisms $\mathfrak{F}^\pm_{\mathfrak{H}}$ from $\mathfrak{H}^\pm$ onto Σ_0 defined by identifying points along incoming (resp. outgoing) principal null geodesics. Their expressions in terms of Kruskal-Boyer-Lindquist coordinates on $\mathfrak{H}^\pm$ and Boyer-Lindquist coordinates on Σ_0 are as follows : first for $\mathfrak{F}^+_{\mathfrak{H}}$, we have*

$$r\left(\mathfrak{F}^+_{\mathfrak{H}}(0,V,\theta,\varphi^\sharp)\right) = T^{-1}\left(\frac{1}{\kappa_+}\mathrm{Log}(V)\right), \tag{8.29}$$

$$\theta\left(\mathfrak{F}^+_{\mathfrak{H}}(0,V,\theta,\varphi^\sharp)\right) = \theta\,, \tag{8.30}$$

$$\begin{aligned}\varphi\left(\mathfrak{F}^+_{\mathfrak{H}}(0,V,\theta,\varphi^\sharp)\right) &= \varphi^\sharp - \frac{a}{r_+^2+a^2}\left(r\left(\mathfrak{F}^+_{\mathfrak{H}}(0,V,\theta,\varphi^\sharp)\right) - r_+\right)\\ &\quad -\frac{2Ma}{r_+^2+a^2}\mathrm{Log}\left(\frac{r\left(\mathfrak{F}^+_{\mathfrak{H}}(0,V,\theta,\varphi^\sharp)\right)-r_+}{r_+-r_-}\right),\end{aligned} \tag{8.31}$$

T^{-1} being the inverse of the function T defined in (8.9) ; and for $\mathfrak{F}_{\mathfrak{H}}^-$, we have similar formulae

$$\begin{aligned} r\left(\mathfrak{F}_{\mathfrak{H}}^-(U,0,\theta,\varphi^\sharp)\right) &= T^{-1}\left(\frac{1}{\kappa_+}\mathrm{Log}(U)\right), && (8.32)\\ \theta\left(\mathfrak{F}_{\mathfrak{H}}^-(U,0,\theta,\varphi^\sharp)\right) &= \theta\,, && (8.33)\\ \varphi\left(\mathfrak{F}_{\mathfrak{H}}^-(U,0,\theta,\varphi^\sharp)\right) &= \varphi^\sharp - \frac{a}{r_+^2+a^2}\left(r\left(\mathfrak{F}_{\mathfrak{H}}^-(U,0,\theta,\varphi^\sharp)\right)-r_+\right) \\ &\quad -\frac{2Ma}{r_+^2+a^2}\mathrm{Log}\left(\frac{r\left(\mathfrak{F}_{\mathfrak{H}}^-(U,0,\theta,\varphi^\sharp)\right)-r_+}{r_+-r_-}\right). && (8.34)\end{aligned}$$

The trace operators $\mathcal{T}_{\mathfrak{H}}^{\pm}$ (defined in (8.27)-(8.28)) extend in a unique manner as bounded operators[6] from $\mathcal{H}$ to $L^2(\mathfrak{H}^\pm;\mathrm{dVol}_{\mathfrak{H}^\pm})$, where the measure $\mathrm{dVol}_{\mathfrak{H}^\pm}$ on $\mathfrak{H}^\pm$ is the pull-back of the volume measure $\mathrm{d}r_*\mathrm{d}\omega$ on Σ_0 by the diffeomorphism $\mathfrak{F}_{\mathfrak{H}}^\pm$. They are related to the inverse wave operators $\tilde{\mathfrak{W}}_{H,pn}^{\pm}$ via the diffeomorphisms $\mathfrak{F}_{\mathfrak{H}}^\pm$ as follows :

$$\left(\tilde{\mathfrak{W}}_{H,pn}^{\pm}\Psi_{|\Sigma_0}\right)(r,\theta,\varphi) = \left(\mathcal{T}_{\mathfrak{H}}^{\pm}\Psi_{|\Sigma_0}\right)\left(\left(\mathfrak{F}_{\mathfrak{H}}^{\pm}\right)^{-1}(r,\theta,\varphi)\right), \qquad (8.35)$$

that is to say, $\mathcal{T}_{\mathfrak{H}}^{\pm}$ is the pull-back $\left(\mathfrak{F}_{\mathfrak{H}}^{\pm}\right)^*\tilde{\mathfrak{W}}_{H,pn}^{\pm}$ of the inverse wave operator $\tilde{\mathfrak{W}}_{H,pn}^{\pm}$ by the diffeomorphism $\mathfrak{F}_{\mathfrak{H}}^{\pm}$.

Proof. We first establish the expressions of the diffeomorphisms $\mathfrak{F}_{\mathfrak{H}}^{\pm}$. We start with $\mathfrak{F}_{\mathfrak{H}}^{+}$. Given $V_+>0$ and $(\theta_+,\varphi_+^\sharp)\in S^2$, we denote by $(t_0=0,r_0,\theta_0,\varphi_0)$ the Boyer-Lindquist coordinates of $\mathfrak{F}_{\mathfrak{H}}^{+}(0,V_+,\theta_+,\varphi_+^\sharp)$ and we calculate them in terms of $V_+,\theta_+,\varphi_+^\sharp$. Along an incoming principal null geodesic, t^*, θ and φ^* are constant. This already gives us (8.30). Using the definitions of t^* and of V, we have, on the incoming principal null geodesic going from $(t_0,r_0,\theta_0,\varphi_0)$ to $(0,V_+,\theta_+,\varphi_+^\sharp)$:

$$t^* = \frac{1}{\kappa_+}\mathrm{Log}(V_+) = t_0 + T(r_0) = T(r_0)$$

which entails (8.29). In order to calculate φ_0, we express the vector V^- in terms of Kruskal-Boyer-Lindquist coordinates (using the expression of n^a in these coordinates) :

$$V^- = -2\kappa_+\frac{r^2+a^2}{\Delta}U\frac{\partial}{\partial U} - \frac{a\left(r^2-r_+^2\right)}{\Delta\left(r_+^2+a^2\right)}\frac{\partial}{\partial\varphi^\sharp}\,.$$

This shows that along an incoming principal null geodesic,

$$\dot{\varphi^\sharp} = -\frac{a\left(r^2-r_+^2\right)}{\Delta\left(r_+^2+a^2\right)} = -\frac{a}{\left(r_+^2+a^2\right)}\frac{r+r_+}{r-r_-}\,.$$

Consequently, we must have

$$\begin{aligned}\varphi_+^\sharp - \varphi^\sharp(t_0,r_0,\theta_0,\varphi_0) &= -\frac{a}{\left(r_+^2+a^2\right)}\int_{r_0}^{r_+}\frac{r+r_+}{r-r_-}\,\mathrm{d}r \\ &= \frac{a}{\left(r_+^2+a^2\right)}(r_0-r_+) + \frac{2Ma}{\left(r_+^2+a^2\right)}\mathrm{Log}\left(\frac{r_0-r_-}{r_+-r_-}\right).\end{aligned}$$

[6] These operators can also be understood as trace operators acting on solutions of (2.56) in $\mathcal{C}(\mathbb{R}_t;\mathcal{H})$, instead of on their initial data.

Moreover, the definition (8.20) of $\varphi^\sharp$ shows that this coordinate function coincides with the Boyer-Lindquist function φ on Σ_0, which gives (8.31). The verification of (8.32)-(8.34) is done in the same manner, working with outgoing null geodesics and star-Kerr coordinates instead.

We now justify the existence of trace operators for minimum regularity solutions and their relation to the inverse wave operators. We consider some initial data $\Psi_{\Sigma_0} \in \mathcal{C}_0^\infty(\Sigma_0)$ and Ψ the associated solution of (2.56). As observed in remark 8.1, we have

$$s - \lim_{t\to+\infty} e^{-it\mathbf{P_N}} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^+) = s - \lim_{t\to+\infty} e^{-it\mathbf{P_N}} j_- e^{it\not{D}_K}$$

where $j_- \in \mathcal{C}^\infty(\mathbb{R})$, $\mathrm{supp} j_- \subset \mathbb{R}^-$ and $j_- \equiv 1$ on $]-\infty, -\varepsilon[$, $\varepsilon > 0$ and small. We denote by $\Psi(t) \in \mathcal{C}_0^\infty(\Sigma)$ the restriction to Σ_t $(\simeq \Sigma)$ of the solution Ψ. As remarked above, we have for $(r_0, \theta_0, \varphi_0)$ given in Σ,

$$\begin{aligned}
\left(e^{-it\mathbf{P_N}} j_- e^{it\not{D}_K} \Psi_{\Sigma_0}\right)(r_0,\theta_0,\varphi_0) &= \begin{pmatrix} (j_-\psi_0(t))\,(F_{w^-}(t)(r_0,\theta_0,\varphi_0)) \\ (j_-\psi_1(t))\,(F_{w^+}(t)(r_0,\theta_0,\varphi_0)) \end{pmatrix} \\
&= \begin{pmatrix} j_-(r_0+t)\psi_0(t)\,(F_{w^-}(t)(r_0,\theta_0,\varphi_0)) \\ j_-(r_0-t)\psi_1(t)\,(F_{w^+}(t)(r_0,\theta_0,\varphi_0)) \end{pmatrix} \\
&= \begin{pmatrix} \psi_0(t)\,(F_{w^-}(t)(r_0,\theta_0,\varphi_0)) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \psi_0\,(F_{v^-}(t)(0,r_0,\theta_0,\varphi_0)) \\ 0 \end{pmatrix}
\end{aligned}$$

for t large enough. The line

$$\{F_{v^-}(t)(0,r_0,\theta_0,\varphi_0)\,,\ t\in\mathbb{R}\}$$

is exactly the principal null geodesic $\gamma^-_{(\mathfrak{F}^+_{\mathfrak{H}})^{-1}(r_0,\theta_0,\varphi_0)}$ going through the point $(0,r_0,\theta_0,\varphi_0)$ of Σ_0, parametrized by t instead of r. This change of parameter is analytic and given by :

$$t\left(\gamma^-_{(\mathfrak{F}^+_{\mathfrak{H}})^{-1}(r_0,\theta_0,\varphi_0)}(r)\right) = -r_*(r) + r_*(r_0)\,,$$

in other words

$$F_{v^-}(t)(0,r_0,\theta_0,\varphi_0) = \gamma^-_{(\mathfrak{F}^+_{\mathfrak{H}})^{-1}(r_0,\theta_0,\varphi_0)}\left(r(-t+r_*(r_0))\right)\,.$$

As t tends to $+\infty$ along this line, r tends to r_+, hence, we find

$$\lim_{t\to+\infty}\left(e^{-it\mathbf{P_N}} j_- e^{it\not{D}_K}\Psi_{\Sigma_0}\right)(r_0,\theta_0,\varphi_0) = \begin{pmatrix} \lim_{r\to r_+}\psi_0\left(\gamma^-_{(\mathfrak{F}^+_{\mathfrak{H}})^{-1}(r_0,\theta_0,\varphi_0)}(r)\right) \\ \\ 0 \end{pmatrix}.$$

This shows that $\tilde{\mathfrak{W}}^+_{H,pn}\Psi_{\Sigma_0}$ and $\mathcal{T}^+_{\mathfrak{H}}(\Psi_{\Sigma_0})\circ\left(\mathfrak{F}^+_{\mathfrak{H}}\right)^{-1}$ coincide for smooth compactly supported data Ψ_{Σ_0}. Since $\tilde{\mathfrak{W}}^+_{H,pn}$ extends by density to a bounded operator from $\mathcal{H}$ to $\mathcal{H}^+$, this entails that $\mathcal{T}^+_{\mathfrak{H}}$ also extends to a bounded operator from $\mathcal{H}$ to $L^2(\mathfrak{H}^+\,;\ \mathrm{dVol}_{\mathfrak{H}^+})$. A similar construction can be done for $\tilde{\mathfrak{W}}^-_{H,pn}$ and $\mathcal{T}^-_{\mathfrak{H}}$. □

Remark 8.4. *The equality $\mathcal{T}^+_{\mathfrak{H}} = \left(\mathfrak{F}^+_{\mathfrak{H}}\right)^*\tilde{\mathfrak{W}}^+_{H,pn}$ can be explained in a more visual manner for smooth compactly supported data. As $t\to+\infty$, the hypersurfaces Σ_t accumulate on the horizon and the limit of the quantity $j_-e^{it\not{D}_K}\Psi_{\Sigma_0}$ is simply the trace of the solution Ψ on $\mathfrak{H}^+$ (see figure*

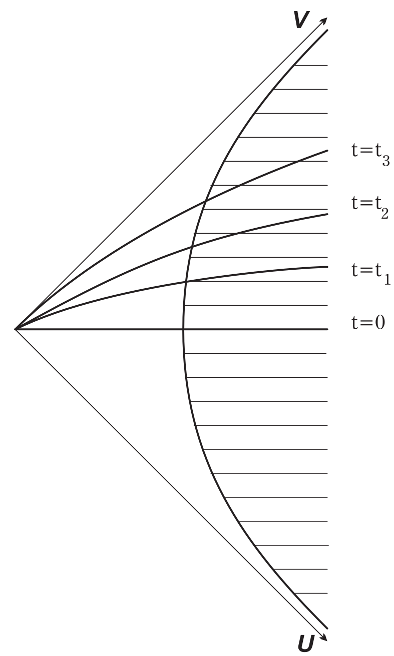


Figure 2: The hypersurfaces Σ_t in $(U, V, \theta, \varphi^\sharp)$ coordinates and the effect of the function j_-. We have represented the hypersurface $t = 0$ and three hypersurfaces of constant positive t for $t_3 > t_2 > t_1$. The effect of the cut-off function j_- is to obliterate all that happens in the striped region, corresponding to $r_* > R$.

2 for a description of the way the hypersurfaces accumulate on the horizon and of the effect of the cut-off function j_-). The operator $e^{-it\mathbf{P_N}}$ pulls back for a time interval t the first component of Ψ along the incoming principal null congruence and its second component along the outgoing null congruence. At the limit $t \to +\infty$, acting on the trace of Ψ on $\mathfrak{H}^+$, whose second component is zero, it simply pulls back the trace of Ψ onto Σ_0 along the incoming null congruence.

8.3 Inverse wave operators at infinity as trace operators

8.3.1 Penrose compactification of Block I

The Penrose compactification of the exterior of a Kerr black hole is done using two independent and symmetric constructions, one based on Kerr-star, the other on star-Kerr coordinates. We describe explicitely only the first of these two constructions, following [46].

Past null infinity is defined as the set of limit points of incoming principal null geodesics as $r \to +\infty$. This rather abstract definition of a 3-surface, describing the congruence of incoming principal null geodesics, can be given a precise meaning using Kerr-star coordinates. We consider the expression (8.14) of the Kerr metric in Kerr-star coordinates and replace the variable r by $w = 1/r$. In these new variables, the exterior of the black hole is described as

$$\mathcal{B}_I = \mathbb{R}_{t^*} \times \left]0, \frac{1}{r_+}\right[_w \times S^2_{\theta,\varphi^*} \,.$$

The conformally rescaled metric

$$\hat{g} = \Omega^2 g\,,\ \Omega = w = \frac{1}{r} \tag{8.36}$$

takes the form

$$\begin{aligned}\hat{g} \;=\; & \left(w^2 - \frac{2Mw^3}{1+a^2w^2\cos^2\theta}\right)\mathrm{d}t^{*2} + \frac{4Maw^3\sin^2\theta}{1+a^2w^2\cos^2\theta}\,\mathrm{d}t^*\mathrm{d}\varphi^* \\ & - \left(1+a^2w^2+\frac{2Ma^2w^3\sin^2\theta}{1+a^2w^2\cos^2\theta}\right)\sin^2\theta\,\mathrm{d}\varphi^{*2} \\ & - \left(1+a^2w^2\cos^2\theta\right)\mathrm{d}\theta^2 + 2\mathrm{d}t^*\mathrm{d}w - 2a\sin^2\theta\,\mathrm{d}\varphi^*\mathrm{d}w\,.\end{aligned}$$

This expression shows that $\hat{g}$ can be extended smoothly on the domain

$$\mathbb{R}_{t^*} \times \left[0, \frac{1}{r_+}\right]_w \times S^2_{\theta,\varphi^*}\,.$$

The hypersurface

$$\mathscr{I}^- := \mathbb{R}_{t^*} \times \{w=0\} \times S^2_{\theta,\varphi^*}$$

can thus be added to the rescaled space-time as a smooth hypersurface, describing past null infinity as defined above. This hypersurface is indeed null since

$$\hat{g}_{|w=0} = -\mathrm{d}\theta^2 - \sin^2\theta\,\mathrm{d}\varphi^{*2}$$

is degenerate (recall that $\mathscr{I}^-$ is a 3-surface) and

$$\det(\hat{g}) = -w^4\rho^4\sin^2\theta = -\left(1+a^2w^2\cos^2\theta\right)^2\sin^2\theta$$

does not vanish for $w=0$.

Similarly, using star-Kerr instead of Kerr-star coordinates, we describe future null infinity, the set of limit points as $r \to +\infty$ of outgoing principal null geodesics, as

$$\mathscr{I}^+ := \mathbb{R}_{{}^*t} \times \{w=0\} \times S^2_{\theta,{}^*\varphi}\,.$$

The Penrose compactification of block I is then the space-time

$$\left(\overline{\mathcal{B}_I}\,,\ \hat{g}\right)\,,\ \overline{\mathcal{B}_I} = \mathcal{B}_I \cup \mathfrak{H}^+ \cup S^2_{\mathrm{c}} \cup \mathfrak{H}^- \cup \mathscr{I}^+ \cup \mathscr{I}^-\,,$$

$\hat{g}$ being defined by (8.36). In spite of the terminology used, the compactified space-time is not compact. There are three "points" missing to the boundary : i_+, or future timelike infinity, defined as the limit point of uniformly timelike curves as $t \to +\infty$, i_-, past timelike infinity, symmetric of i_+ in the distant past, and i_0, spacelike infinity, the limit point of uniformly spacelike curves as $r \to +\infty$. These "points", that can be described as 2-spheres, or even blown up further, are singularities of the rescaled metric. See figure 3 for a representation of the compactified block I. We conclude this paragraph with a useful result concerning the behaviour at null infinity of the Newman-Penrose tetrad l^a, n^a, m^a, $\bar{m}^a$:

Proposition 8.2. *Each vector field of the Newman-Penrose tetrad l^a, n^a, m^a, $\bar{m}^a$ extends as a smooth vector field over $\mathcal{B}_I \cup \mathscr{I}^+ \cup \mathscr{I}^-$. All the frame vectors vanish on $\mathscr{I}^-$ (resp. $\mathscr{I}^+$) except l^a (resp. n^a) which coincides there with the future oriented null generator up $\mathscr{I}^-$ (resp. $\mathscr{I}^+$). Consequently, the spinor fields o^A and ι^A extend as smooth spinor fields on $\mathcal{B}_I \cup \mathscr{I}^+ \cup \mathscr{I}^-$, o^A does not vanish on $\mathscr{I}^-$ but vanishes on $\mathscr{I}^+$ while ι^A does not vanish on $\mathscr{I}^+$ but vanishes on $\mathscr{I}^-$. The spin-frame $(\hat{o}^A, \hat{\iota}^A) = (\Omega^{-1}o^A, \iota^A)$ is a smooth non-degenerate normalized (relative to the metric $\hat{g}$) spin-frame over $\mathcal{B}_I \cup \mathscr{I}^+$. Symmetrically, the spin-frame $(\check{o}^A, \check{\iota}^A) = (o^A, \Omega^{-1}\iota^A)$ is a smooth non-degenerate normalized spin-frame over $\mathcal{B}_I \cup \mathscr{I}^-$.*

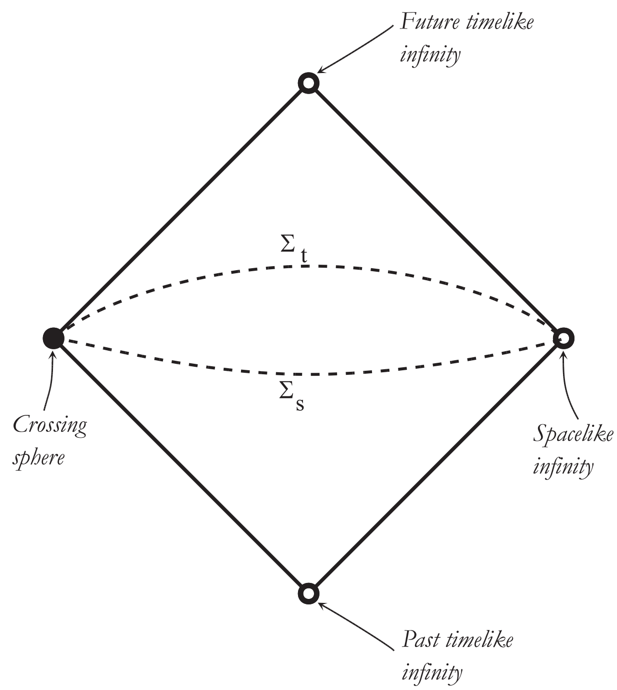


Figure 3: The Penrose compactification of block I, with two hypersurfaces Σ_s and Σ_t, $t > s$.

Remark 8.5. *These properties are well-known for more general space-times admitting a regular null infinity (see [50], Vol. 2). We prove them here explicitely.*

Proof.
We first express each of the frame vectors in Kerr-star coordinates, using (8.10)-(8.13) :

$$\begin{aligned}
l^a\frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\Delta\rho^2}}V^+ \\
&= \frac{1}{\sqrt{2\Delta\rho^2}}V^- + \sqrt{\frac{2\Delta}{\rho^2}}\left(\frac{\partial}{\partial r}\right)_{BL} \\
&= -\frac{1}{\sqrt{2\Delta\rho^2}}\left(\frac{\partial}{\partial r}\right)_{K*} + \sqrt{\frac{2\Delta}{\rho^2}}\left(\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t^*} + \left(\frac{\partial}{\partial r}\right)_{K*} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi^*}\right), \\
n^a\frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\Delta\rho^2}}V^- = -\frac{1}{\sqrt{2\Delta\rho^2}}\left(\frac{\partial}{\partial r}\right)_{K*}, \\
m^a\frac{\partial}{\partial x^a} &= \frac{1}{p\sqrt{2}}\left(ia\sin\theta\frac{\partial}{\partial t^*} + \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi^*}\right).
\end{aligned}$$

Going over to coordinates $t^*, w, \theta, \varphi^*$, we obtain :

$$\begin{aligned}
l^a\frac{\partial}{\partial x^a} &= \frac{1-2\Delta}{\sqrt{2\Delta\rho^2}}w^2\left(\frac{\partial}{\partial w}\right)_{K*} + \sqrt{\frac{2\Delta}{\rho^2}}\left(\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t^*} + \frac{a}{\Delta}\frac{\partial}{\partial\varphi^*}\right), \\
n^a\frac{\partial}{\partial x^a} &= \frac{1}{\sqrt{2\Delta\rho^2}}w^2\left(\frac{\partial}{\partial w}\right)_{K*}, \\
m^a\frac{\partial}{\partial x^a} &= \frac{1}{p\sqrt{2}}\left(ia\sin\theta\frac{\partial}{\partial t^*} + \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi^*}\right).
\end{aligned}$$

All three vector fields are smooth on $\mathcal{B}_I \cup \mathscr{I}^+ \cup \mathscr{I}^-$; l^a is the only one not to vanish on $\mathscr{I}^-$ and

$$l^a \frac{\partial}{\partial x^a}\Big|_{w=0} = \sqrt{2}\frac{\partial}{\partial t^*} \,,$$

i.e., on $\mathscr{I}^-$, l^a is the future oriented null generator up $\mathscr{I}^-$. A similar calculation can be done for $\mathscr{I}^+$ using star-Kerr coordinates and identities (8.15)-(8.18). We obtain in particular that

$$n^a \frac{\partial}{\partial x^a}\Big|_{w=0} = \sqrt{2}\frac{\partial}{\partial {}^*t} \,.$$

The properties of the spin-frame $(\check{o}^A, \check{\iota}^A) = (o^A, \Omega^{-1}\iota^A)$ are easily proved by noticing that the vectors l^a, $\Omega^{-2}n^a$, $\Omega^{-1}m^a$ and $\Omega^{-1}\bar{m}^a$ are all smooth and non vanishing over $\mathscr{I}^-$, and define on $\mathcal{B}_I \cup \mathscr{I}^-$ a normalized Newman-Penrose tetrad for the metric $\hat{g}$. The same can be done for $(\hat{o}^A, \hat{\iota}^A)$ on $\mathscr{I}^+$. □

8.3.2 Interpretation of the traces on $\mathscr{I}^\pm$

The conformal invariance of the Dirac equation entails that a spinor field $\phi_A \in L^2_{\mathrm{loc}}(\mathcal{B}_I\, ;\, \mathbb{S}_A)$ satisfies equation (2.5) if and only if the rescaled spinor field $\hat{\phi}_A = \Omega^{-1}\phi_A \in L^2_{\mathrm{loc}}(\mathcal{B}_I\, ;\, \mathbb{S}_A)$ satisfies on $\mathcal{B}_I$

$$\hat{\nabla}^{AA'}\hat{\phi}_A = 0\,, \tag{8.37}$$

where $\hat{\nabla}_a$ is the covariant derivative associated with the rescaled metric $\hat{g}$. We consider for (2.5) initial data $\phi_A(0) \in \mathcal{C}_0^\infty(\Sigma_0)$ and $\phi_A \in \mathcal{C}^\infty(\mathbb{R}_t\, ,\, \mathcal{C}_0^\infty(\Sigma\, ;\, \mathbb{S}_A))$ the associated solution. The support of the spinor field $\hat{\phi}_A$ remains far from i_0 and therefore, by standard arguments[7] of regularity of the solution of Dirac's (or Weyl's) equation on a smooth space-time, $\hat{\phi}_A$ extends as a solution of (8.37) in $\mathcal{C}^\infty\left(\overline{\mathcal{B}_I}\right)$. Now, the vector field Ψ, defined in relation to ϕ_A by (2.51), is related to $\hat{\phi}_A$ as follows :

$$\Psi = \mathbf{U}\left(\frac{\Delta\sigma^2\rho^2}{(r^2+a^2)^2}\right)^{1/4}\begin{pmatrix} o^A\phi_A \\ \iota^A\phi_A \end{pmatrix} = \mathbf{U}\Omega\left(\frac{\Delta\sigma^2\rho^2}{(r^2+a^2)^2}\right)^{1/4}\begin{pmatrix} o^A\hat{\phi}_A \\ \iota^A\hat{\phi}_A \end{pmatrix}.$$

The matrix $\mathbf{U}$, defined in (2.50), as well as its inverse, can be extended smoothly over $\overline{\mathcal{B}_I}$, and the same is true of the quantity

$$\Omega\left(\frac{\Delta\sigma^2\rho^2}{(r^2+a^2)^2}\right)^{1/4}.$$

Moreover, by proposition 8.2, o^A and ι^A extend as smooth spinor-fields on $\mathcal{B}_I \cup \mathscr{I}^+ \cup \mathscr{I}^-$. All this entails that the vector field Ψ extends as a smooth vector field over $\overline{\mathcal{B}_I}$ (the regularity over $\mathfrak{H}$ was shown in the previous subsection).

Remark 8.6. *Noting that* $\mathbf{U}$ *reduces to the identity matrix on* $\mathscr{I}^\pm$*, proposition 8.2 entails that the trace of* Ψ *on* $\mathscr{I}^+$ *is simply the second component of the trace of* $\hat{\phi}_A$ *on* $\mathscr{I}^+$ *in the spin-frame* $(\hat{o}^A, \hat{\iota}^A)$ *that is regular and non-degenerate on* $\mathscr{I}^+$*. Similarly, the trace of* Ψ *on* $\mathscr{I}^-$ *is the first component of the trace of* $\hat{\phi}_A$ *on* $\mathscr{I}^-$ *in the spin-frame* $(\check{o}^A, \check{\iota}^A)$ *that is regular and non-degenerate on* $\mathscr{I}^-$*.*

[7] These arguments are explained in details for a non-linear wave equation in [47]. Their application to the Dirac case is identical.

Arguments similar to the ones used for the horizon now allow us to interpret the inverse wave operators at infinity as trace operators on $\mathscr{I}^{\pm}$:

Theorem 11. *We denote by* $\mathfrak{F}_{\mathscr{I}}^{+}$ *(resp.* $\mathfrak{F}_{\mathscr{I}}^{-}$*) the* $\mathcal{C}^\infty$ *diffeomorphism from* $\mathscr{I}^+$ *(resp.* $\mathscr{I}^-$*) onto* Σ_0 *defined by identifying points along outgoing (resp. incoming) principal null geodesics in* $\overline{\mathcal{B}_I}$. *They have the following explicit expressions (*$\mathfrak{F}_{\mathscr{I}}^{+}$ *being defined in terms of star-Kerr coordinates and* $\mathfrak{F}_{\mathscr{I}}^{-}$ *in terms of Kerr-star coordinates) :*

$$\begin{aligned} \mathfrak{F}_{\mathscr{I}}^{+}\left({}^*t\,,\ w=0\,,\ \theta\,,\ {}^*\varphi\right) &= \left(t=0\,,\ r=T^{-1}(-{}^*t)\,,\ \theta\,,\ \varphi={}^*\varphi+\Lambda(T^{-1}(-{}^*t))\right)\,, \\ \mathfrak{F}_{\mathscr{I}}^{-}\left(t^*\,,\ w=0\,,\ \theta\,,\ \varphi^*\right) &= \left(t=0\,,\ r=T^{-1}(t^*)\,,\ \theta\,,\ \varphi=\varphi^*-\Lambda(T^{-1}(t^*))\right)\,. \end{aligned}$$

We define, on $\mathscr{I}^{\pm}$, *volume measures* $\mathrm{dVol}_{\mathscr{I}^{\pm}}$ *as the pull backs of the measure* $\mathrm{d}r_*\mathrm{d}\omega$ *on* Σ_0 *by the diffeomorphisms* $\mathfrak{F}_{\mathscr{I}}^{\pm}$. *The trace operators* $\mathcal{T}_{\mathscr{I}}^{\pm}$, *that to initial data* $\Psi_{\Sigma_0}\in\mathcal{C}_0^\infty(\Sigma_0)$ *for (2.56), associate the smooth trace on* $\mathscr{I}^{\pm}$ *of the associated solution* Ψ, *extend as bounded operators from* $\mathcal{H}$ *to* $L^2\left(\mathscr{I}^{\pm};\mathrm{dVol}_{\mathscr{I}^{\pm}}\right)$. *They are related to the inverse wave operators* $\tilde{\mathfrak{W}}_{\infty,pn}^{\pm}$ *via the diffeomorphisms* $\mathfrak{F}_{\mathscr{I}}^{\pm}$ *:*

$$\left(\tilde{\mathfrak{W}}_{\infty,pn}^{\pm}\Psi_{\Sigma_0}\right)(r,\theta,\varphi)=\left(\mathcal{T}_{\mathscr{I}}^{\pm}\Psi_{\Sigma_0}\right)\left(\left(\mathfrak{F}_{\mathscr{I}}^{\pm}\right)^{-1}(r,\theta,\varphi)\right)\,. \tag{8.38}$$

In other words, $\mathcal{T}_{\mathscr{I}}^{\pm}=\left(\mathfrak{F}_{\mathscr{I}}^{\pm}\right)^*\tilde{\mathfrak{W}}_{\infty,pn}^{\pm}$.

8.4 The Goursat problem

Putting together theorems 10 and 11, we obtain the interpretation of the scattering theory for equation (2.56) as the solution of a singular Goursat problem on $\overline{\mathcal{B}_I}$:

Theorem 12. *The global inverse wave operator* $\tilde{W}_{pn}^{+}$ *(resp.* $\tilde{W}_{pn}^{-}$*) is a representation of the trace operator that, to initial data for (2.56), associates the trace of the solution* Ψ *on the null hypersurface in* $\overline{\mathcal{B}_I}$, *singular at its vertex,* $\mathfrak{H}^+\cup\mathscr{I}^+$ *(resp.* $\mathfrak{H}^-\cup\mathscr{I}^-$*). The scattering theory of theorem 9 states that these operators are isomorphisms (even isometries) from* $\mathcal{H}$ *onto* $\mathcal{H}^-\oplus\mathcal{H}^+$ *(resp.* $\mathcal{H}^+\oplus\mathcal{H}^-$*), i.e. that the solutions are completely and uniquely determined by their trace on* $\mathfrak{H}^+\cup\mathscr{I}^+$ *(resp.* $\mathfrak{H}^-\cup\mathscr{I}^-$*). This is exactly saying that the Goursat problem for (2.56) is well posed on* $\mathfrak{H}^+\cup\mathscr{I}^+$ *(resp.* $\mathfrak{H}^-\cup\mathscr{I}^-$*). The direct wave operator* W_{pn}^{+} *(resp.* W_{pn}^{-}*) then solves this Goursat problem by associating to the null data, the initial data on* Σ_0 *of the unique corresponding solution.*

More precisely, we define the future trace operator

$$\mathbb{T}_F\ :\ \begin{array}{ccl} \mathcal{H} & \longrightarrow & L^2\left(\mathfrak{H}^+;\mathrm{dVol}_{\mathfrak{H}^+}\right)\oplus L^2\left(\mathscr{I}^+;\mathrm{dVol}_{\mathscr{I}^+}\right)=:\mathcal{H}_F \\ \Psi_{\Sigma_0} & \longmapsto & \left(\mathcal{T}_{\mathfrak{H}}^{+}\Psi_{\Sigma_0}\,,\ \mathcal{T}_{\mathscr{I}}^{+}\Psi_{\Sigma_0}\right)=\left(\left(\mathfrak{F}_{\mathfrak{H}}^{+}\right)^*\tilde{\mathfrak{W}}_{H,pn}^{+}\Psi_{\Sigma_0}\,,\ \left(\mathfrak{F}_{\mathscr{I}}^{+}\right)^*\tilde{\mathfrak{W}}_{\infty,pn}^{+}\Psi_{\Sigma_0}\right)\,. \end{array}$$

$\mathbb{T}_F$ *is an isomorphism, i.e. for each* $\Phi\in\mathcal{H}_F$ *there exists a unique* $\Psi\in\mathcal{C}(\mathbb{R}_t;\mathcal{H})$ *solution of (2.56) such that* $\Phi=\mathbb{T}_F\Psi(0)$. *A similar formulation is valid for the past.*

Remark 8.7. *It is interesting to remember that the hypersurface* $\mathfrak{H}^+\cup\mathscr{I}^+$ *(resp.* $\mathfrak{H}^-\cup\mathscr{I}^-$*) on which the Goursat problem is solved, is singular at its vertex because the conformal metric is singular there. This means that there is no choice of conformal factor* Ω *that would make the corresponding rescaled metric regular and non degenerate at* $i_\pm$. *In the time-dependent scattering theory that we have constructed here, this singularity is not really seen ; we have two separate asymptotic regions and* $i_\pm$ *are considered as points at infinity on* $\mathscr{I}^{\pm}$ *and* $\mathfrak{H}^{\pm}$ *(similarly* i_0 *is understood as a point at infinity on* $\mathscr{I}^{\pm}$*).*

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