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Scattering of linear Dirac fields by a spherically symmetric Black-Hole

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Abstract - *We study the linear Dirac system outside a spherical Black-Hole. In the case of massless fields, we prove the existence and asymptotic completeness of classical wave operators at the horizon of the Black-Hole and at infinity.*

Résumé - On étudie le système linéaire de Dirac à l'extérieur d'un Trou Noir sphérique. Dans le cas des champs sans masse, on montre l'existence et la complétude asymptotique des opérateurs d'onde classiques à l'horizon du Trou Noir et à l'infini.

1 Introduction

We develop a time-dependent scattering theory for the linear Dirac system on Schwarzschild-type metrics. The first time-dependent scattering results on the Schwarzschild metric were obtained by J. Dimock [8]. Using the short range at infinity of the interaction between gravity and a massless scalar field, he proved the existence and asymptotic completeness of classical wave-operators for the wave equation. The case of the Maxwell system in which the interaction is pseudo long-range has been worked out by A. Bachelot [2], and for the Regge-Wheeler equation, a complete scattering theory has been developed by A. Bachelot and A. Motet-Bachelot [3]. Our purpose in this work is to study the classical wave operators and their asymptotic completeness for the linear massless Dirac system on a general "Schwarzschild-type" metric which covers all the usual cases of spherical black-holes. The main tools are Cook's method for the existence and the results obtained in [3] for the asymptotic completeness.

Let us consider the manifold $\mathbb{R}_t \times]0,+\infty[_r \times S^2_{\theta,\phi}$ endowed with the pseudo-riemannian metric

$$g_{\mu\nu}dx^\mu dx^\nu = F(r)e^{2\delta(r)}dt^2 - [F(r)^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2] \qquad (1)$$

where $F,\delta \in C^\infty(]0,+\infty[_r)$. We assume the existence of three values r_ν of r, $0 \leq r_- < r_0 < r_+ \leq +\infty$, which are the only possible zeros of F, such that

$$F(r_\nu) = 0 \;\;,\;\; F'(r_\nu) = 2\kappa_\nu, \;\; \kappa_\nu \neq 0 \;\;,\;\; if\ 0 < r_\nu < +\infty,$$
$$F(r) > 0\ for\ r \in]r_0, r_+[\;\;,\;\; F(r) < 0\ for\ r \in]r_-, r_0[.$$

When they are finite and non zero, r_-, r_0 and r_+ are the radii of the spheres called: horizon of the black-hole (r_0), Cauchy horizon (r_-) and cosmological horizon (r_+). κ_ν is the surface gravity at the horizon $\{r = r_\nu\}$. If r_+ is infinite, we assume moreover that

$$F(r) = 1 - \frac{r_1}{r} + O\left(r^{-2}\right) \;\;,\;\; r_1 > 0 \;\;,\;\; \delta(r) = \delta(+\infty) + o(r^{-1}) \;\;,\;\; r \to +\infty,$$
$$F'(r)\ ,\ \delta'(r) \;\; = O(r^{-2})\ ,\;\; r \to +\infty.$$

All these properties are satisfied by usual spherical black-holes (see [13]).

Notations: Let (M,g) be a Riemannian manifold, $\mathcal{C}_0^\infty(M)$ denotes the set of $\mathcal{C}^\infty$functions with compact support in M, $H^k(M,g)$, $k\in\mathbb{N}$ is the Sobolev space, completion of $\mathcal{C}_0^\infty(M)$ for the norm

$$\|f\|^2_{H^k(M)} = \sum_{j=0}^{k}\int_M \left\langle \nabla^j f, \nabla^j f\right\rangle d\mu,$$

where ∇^j, $d\mu$ and $<,>$ are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric g. We write $L^2(M,g)=H^0(M,g)$.

If E is a distribution space on M, E_{comp} represents the subspace of elements of E with compact support in M.

The 2-dimensional euclidian sphere S^2_ω is endowed with its usual metric

$$d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2\ ,\ 0\le\theta\le\pi\ ,\ 0\le\varphi<2\pi.$$

2 The covariant generalization of the linear Dirac system on Schwarzschild-type metrics

The covariant generalization of the Dirac system on the metric g has the form

$$(i\gamma^\mu\nabla_\mu - m)\,\Phi = 0\ ,\quad m\ge 0 \tag{2}$$

for a particle with mass m, where Φ is a Dirac 4-spinor, the γ^μ are the contravariant Dirac matrices on curved space-time and ∇_μ is the covariant derivation of spinor fields. We make the following choices of flat space-time Dirac matrices

$$\gamma_{\tilde 0} = \begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix} \qquad \gamma_{\tilde\alpha} = \begin{pmatrix}0 & \sigma_\alpha\\ -\sigma_\alpha & 0\end{pmatrix}\quad \alpha=1,2,3 \tag{3}$$

where

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\quad \sigma_1 = -\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad \sigma_2 = -\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\quad \sigma_3 = -\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4}$$

are the Pauli matrices, and of local Lorentz frame

$$e_{\tilde\alpha}{}^{\mu} = \begin{cases} |g^{\mu\mu}|^{\frac12} & if\quad \tilde\alpha=\mu,\\ 0 & if\quad \tilde\alpha\ne\mu.\end{cases} \tag{5}$$

We recall that flat space time Dirac matrices are a set of 4x4 matrices $\{\gamma_{\tilde\alpha}\}_{0\le\tilde\alpha\le3}$ such that

$$\left\{\gamma_{\tilde\alpha},\gamma_{\tilde\beta}\right\} = \gamma_{\tilde\alpha}\gamma_{\tilde\beta}+\gamma_{\tilde\beta}\gamma_{\tilde\alpha} = 2\eta_{\tilde\alpha\tilde\beta}\mathbb{I}\qquad (\tilde\alpha,\tilde\beta=0,1,2,3) \tag{6}$$

where

$$\eta_{\tilde\alpha\tilde\beta} = diag(1,-1,-1,-1) \tag{7}$$

is the Minkowski metric. The indices with a tilde refer to flat space-time and can be raised or lowered using $\eta_{\tilde\alpha\tilde\beta}$, whereas the indices without tilde refer to curved space-time and are raised or lowered using the metric g.

With these definitions, the γ^μ and ∇_μ are then defined by (see for example [5], [7])

$$\gamma^\mu = \gamma_{\tilde\alpha}e^{\tilde\alpha\mu} \tag{8}$$

and

$$\nabla_\mu = \partial_\mu + \frac12 G_{[\tilde\alpha\tilde\beta]}\omega^{\tilde\alpha\tilde\beta}{}_{\mu} \tag{9}$$

where

$$G_{[\tilde\alpha,\tilde\beta]} = \frac14\left[\gamma_{\tilde\alpha},\gamma_{\tilde\beta}\right] \equiv \frac14\left(\gamma_{\tilde\alpha}\gamma_{\tilde\beta}-\gamma_{\tilde\beta}\gamma_{\tilde\alpha}\right) \tag{10}$$

are the generators of the spinor representation of the proper Lorentz group and

$$\omega^{\tilde{\alpha}\tilde{\beta}}{}_{\mu}=\frac{1}{2}e^{\tilde{\alpha}\nu}\left(e^{\tilde{\beta}}{}_{\nu,\mu}-e^{\tilde{\beta}}{}_{\mu,\nu}\right)-\frac{1}{2}e^{\tilde{\beta}\nu}\left(e^{\tilde{\alpha}}{}_{\nu,\mu}-e^{\tilde{\alpha}}{}_{\mu,\nu}\right)+\frac{1}{2}e^{\tilde{\alpha}\nu}e^{\tilde{\beta}\sigma}\left(e^{\tilde{\gamma}}{}_{\nu,\sigma}-e^{\tilde{\gamma}}{}_{\sigma,\nu}\right)e_{\tilde{\gamma}\mu}=-\omega^{\tilde{\beta}\tilde{\alpha}}{}_{\mu} \tag{11}$$

are the coefficients of the spin connection, $_{,\mu}$ standing for the derivation with respect to the μ-th variable. We compute the a priori non zero components:

$$\begin{aligned}
\omega^{\tilde{t}\tilde{r}}{}_{t}&=\frac{1}{2}e^{\tilde{t}t}\left[\partial_t\left(e^{\tilde{r}}{}_t\right)-\partial_t\left(e^{\tilde{r}}{}_t\right)\right]-\frac{1}{2}e^{\tilde{r}r}\left[\partial_t\left(e^{\tilde{t}}{}_r\right)-\partial_r\left(e^{\tilde{t}}{}_t\right)\right]+\frac{1}{2}e^{\tilde{t}t}e^{\tilde{r}r}\left[\partial_r\left(e^{\tilde{t}}{}_t\right)-\partial_t\left(e^{\tilde{t}}{}_r\right)\right]e_{\tilde{t}t}\\
&=\frac{1}{2}e^{\tilde{r}r}\partial_r\left(e^{\tilde{t}}{}_t\right)\left(1+e^{\tilde{t}t}e_{\tilde{t}t}\right)=\frac{1}{2}(-F^{1/2})\partial_r(F^{1/2}e^{\delta})(1+F^{-1/2}e^{-\delta}F^{1/2}e^{\delta})=-\left(\frac{F'}{2}+F\delta'\right)e^{\delta},\\
\omega^{\tilde{t}\tilde{r}}{}_{r}&=\frac{1}{2}e^{\tilde{t}t}\left[\partial_r\left(e^{\tilde{r}}{}_t\right)-\partial_t\left(e^{\tilde{r}}{}_r\right)\right]-\frac{1}{2}e^{\tilde{r}r}\left[\partial_r\left(e^{\tilde{t}}{}_r\right)-\partial_r\left(e^{\tilde{t}}{}_r\right)\right]+\frac{1}{2}e^{\tilde{t}t}e^{\tilde{r}r}\left[\partial_r\left(e^{\tilde{r}}{}_t\right)-\partial_t\left(e^{\tilde{r}}{}_r\right)\right]e_{\tilde{r}r}=0,\\
\omega^{\tilde{t}\tilde{\theta}}{}_{t}&=\frac{1}{2}e^{\tilde{t}t}\left[\partial_t\left(e^{\tilde{\theta}}{}_t\right)-\partial_t\left(e^{\tilde{\theta}}{}_t\right)\right]-\frac{1}{2}e^{\tilde{\theta}\theta}\left[\partial_t\left(e^{\tilde{t}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{t}}{}_t\right)\right]+\frac{1}{2}e^{\tilde{t}t}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{t}}{}_t\right)-\partial_t\left(e^{\tilde{t}}{}_\theta\right)\right]e_{\tilde{t}t}=0,\\
\omega^{\tilde{t}\tilde{\theta}}{}_{\theta}&=\frac{1}{2}e^{\tilde{t}t}\left[\partial_\theta\left(e^{\tilde{\theta}}{}_t\right)-\partial_t\left(e^{\tilde{\theta}}{}_\theta\right)\right]-\frac{1}{2}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{t}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{t}}{}_\theta\right)\right]+\frac{1}{2}e^{\tilde{t}t}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{\theta}}{}_t\right)-\partial_t\left(e^{\tilde{\theta}}{}_\theta\right)\right]e_{\tilde{\theta}\theta}=0,\\
\omega^{\tilde{t}\tilde{\varphi}}{}_{t}&=\frac{1}{2}e^{\tilde{t}t}\left[\partial_t\left(e^{\tilde{\varphi}}{}_t\right)-\partial_t\left(e^{\tilde{\varphi}}{}_t\right)\right]-\frac{1}{2}e^{\tilde{\varphi}\varphi}\left[\partial_t\left(e^{\tilde{t}}{}_\varphi\right)-\partial_\varphi\left(e^{\tilde{t}}{}_t\right)\right]+\frac{1}{2}e^{\tilde{t}t}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{t}}{}_t\right)-\partial_t\left(e^{\tilde{t}}{}_\varphi\right)\right]e_{\tilde{t}t}=0,\\
\omega^{\tilde{t}\tilde{\varphi}}{}_{\varphi}&=\frac{1}{2}e^{\tilde{t}t}\left[\partial_\varphi\left(e^{\tilde{\varphi}}{}_t\right)-\partial_t\left(e^{\tilde{\varphi}}{}_\varphi\right)\right]-\frac{1}{2}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{t}}{}_\varphi\right)-\partial_\varphi\left(e^{\tilde{t}}{}_\varphi\right)\right]+\frac{1}{2}e^{\tilde{t}t}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{\varphi}}{}_t\right)-\partial_t\left(e^{\tilde{\varphi}}{}_\varphi\right)\right]e_{\tilde{\varphi}\varphi}=0,\\
\omega^{\tilde{r}\tilde{\theta}}{}_{r}&=\frac{1}{2}e^{\tilde{r}r}\left[\partial_r\left(e^{\tilde{\theta}}{}_r\right)-\partial_r\left(e^{\tilde{\theta}}{}_r\right)\right]-\frac{1}{2}e^{\tilde{\theta}\theta}\left[\partial_r\left(e^{\tilde{r}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{r}}{}_r\right)\right]+\frac{1}{2}e^{\tilde{r}r}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{r}}{}_r\right)-\partial_r\left(e^{\tilde{r}}{}_\theta\right)\right]e_{\tilde{r}r}=0,\\
\omega^{\tilde{r}\tilde{\theta}}{}_{\theta}&=\frac{1}{2}e^{\tilde{r}r}\left[\partial_\theta\left(e^{\tilde{\theta}}{}_r\right)-\partial_r\left(e^{\tilde{\theta}}{}_\theta\right)\right]-\frac{1}{2}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{r}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{r}}{}_\theta\right)\right]+\frac{1}{2}e^{\tilde{r}r}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{\theta}}{}_r\right)-\partial_r\left(e^{\tilde{\theta}}{}_\theta\right)\right]e_{\tilde{\theta}\theta}=F^{1/2},\\
\omega^{\tilde{r}\tilde{\varphi}}{}_{r}&=\frac{1}{2}e^{\tilde{r}r}\left[\partial_r\left(e^{\tilde{\varphi}}{}_r\right)-\partial_r\left(e^{\tilde{\varphi}}{}_r\right)\right]-\frac{1}{2}e^{\tilde{\varphi}\varphi}\left[\partial_r\left(e^{\tilde{r}}{}_\varphi\right)-\partial_\varphi\left(e^{\tilde{r}}{}_r\right)\right]+\frac{1}{2}e^{\tilde{r}r}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{r}}{}_r\right)-\partial_r\left(e^{\tilde{r}}{}_\varphi\right)\right]e_{\tilde{r}r}=0,\\
\omega^{\tilde{r}\tilde{\varphi}}{}_{\varphi}&=\frac{1}{2}e^{\tilde{r}r}\left[\partial_\varphi\left(e^{\tilde{\varphi}}{}_r\right)-\partial_r\left(e^{\tilde{\varphi}}{}_\varphi\right)\right]-\frac{1}{2}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{r}}{}_\varphi\right)-\partial_\varphi\left(e^{\tilde{r}}{}_\varphi\right)\right]+\frac{1}{2}e^{\tilde{r}r}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{\varphi}}{}_r\right)-\partial_r\left(e^{\tilde{\varphi}}{}_\varphi\right)\right]e_{\tilde{\varphi}\varphi}\\
&=F^{1/2}sin\theta,\\
\omega^{\tilde{\theta}\tilde{\varphi}}{}_{\theta}&=\frac{1}{2}e^{\tilde{\theta}\theta}\left[\partial_\theta\left(e^{\tilde{\varphi}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{\varphi}}{}_\theta\right)\right]-\frac{1}{2}e^{\tilde{\varphi}\varphi}\left[\partial_\theta\left(e^{\tilde{\theta}}{}_\varphi\right)-\partial_\varphi\left(e^{\tilde{\theta}}{}_\theta\right)\right]+\frac{1}{2}e^{\tilde{\theta}\theta}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{\theta}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{\theta}}{}_\varphi\right)\right]e_{\tilde{\theta}\theta}=0,\\
\omega^{\tilde{\theta}\tilde{\varphi}}{}_{\varphi}&=\frac{1}{2}e^{\tilde{\theta}\theta}\left[\partial_\varphi\left(e^{\tilde{\varphi}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{\varphi}}{}_\varphi\right)\right]-\frac{1}{2}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{\theta}}{}_\varphi\right)-\partial_\varphi\left(e^{\tilde{\theta}}{}_\varphi\right)\right]+\frac{1}{2}e^{\tilde{\theta}\theta}e^{\tilde{\varphi}\varphi}\left[\partial_\varphi\left(e^{\tilde{\varphi}}{}_\theta\right)-\partial_\theta\left(e^{\tilde{\varphi}}{}_\varphi\right)\right]e_{\tilde{\varphi}\varphi}\\
&=cos\theta.
\end{aligned}$$

and we obtain the following expression for the linear massive Dirac equation outside a spherical black-hole:

$$\begin{aligned}
&\left\{\gamma^{\tilde{0}}\partial_t+Fe^{\delta}\gamma^{\tilde{1}}\left(\partial_r+\frac{1}{r}+\frac{F'}{4F}+\frac{\delta'}{2}\right)+\frac{F^{1/2}e^{\delta}}{r}\gamma^{\tilde{2}}\left(\partial_\theta+\frac{1}{2}cotg\theta\right)+\right.\\
&\left.\frac{F^{1/2}e^{\delta}}{rsin\theta}\gamma^{\tilde{3}}\partial_\varphi+iF^{1/2}e^{\delta}m\right\}\Phi=0.
\end{aligned} \tag{12}$$

We introduce the frame with respect to which we shall express the equation, $\mathcal{R}'=\left(\frac{1}{rsin\theta}\partial_\varphi,-\frac{1}{r}\partial_\theta,F^{1/2}\partial_r\right)$, image of $\mathcal{R}=\left(F^{1/2}\partial_r,\frac{1}{r}\partial_\theta,\frac{1}{rsin\theta}\partial_\varphi\right)$ by the spatial rotation f with Euler angles (see for example [15]) $(\varphi,\theta,\psi)=(0,\pi/2,\pi)$, and the Regge-Wheeler variable r_* defined by

$$\frac{dr}{dr_*}=Fe^{\delta}\qquad r\in]r_0,r_+[. \tag{13}$$

The spinor

$$\Psi=T_{(f^{-1})}rF^{1/4}e^{\delta/2}\Phi, \tag{14}$$

where $T_{(f^{-1})}$ is the spin transformation associated with the rotation f^{-1}, satisfies

$$\partial_t \Psi = iH\Psi \quad , \quad H = i\left[\gamma^{\tilde{0}}\gamma^{\tilde{3}}\partial_{r_*} - \frac{F^{1/2}e^{\delta}}{r}\gamma^{\tilde{0}}\gamma^{\tilde{2}}\left(\partial_\theta + \frac{1}{2}cotg\theta\right) + \frac{F^{1/2}e^{\delta}}{rsin\theta}\gamma^{\tilde{0}}\gamma^{\tilde{1}}\partial_\varphi + i\gamma^{\tilde{0}}F^{1/2}e^{\delta}m\right] \tag{15}$$

on the domain $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega$ representing the exterior of the black-hole in the variables (t, r_*, ω).

We recall (see [7]) that, given a spatial rotation f of angle θ around a unit vector $n = (n_1, n_2, n_3)$, its associated spin transformation T_f is

$$T_f = Exp\left\{\left[n_1 G_{[\tilde{2},\tilde{3}]} + n_2 G_{[\tilde{3},\tilde{1}]} + n_3 G_{[\tilde{1},\tilde{2}]}\right]\theta\right\} \tag{16}$$

where Exp is the exponential mapping.

3 Global Cauchy problem

We introduce the Hilbert space

$$\mathcal{H} = \left\{L^2\left(\mathbb{R}_{r_*} \times S^2_\omega; dr_*^2 + d\omega^2\right)\right\}^4. \tag{17}$$

Theorem 3.1. *Given $\Psi_0 \in \mathcal{H}$, equation (15) has a unique solution Ψ such that*

$$\Psi \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{H}\right) \quad , \quad \Psi\mid_{t=0} = \Psi_0. \tag{18}$$

Moreover, for any $t \in \mathbb{R}$

$$\|\Psi(t)\|_{\mathcal{H}} = \|\Psi_0\|_{\mathcal{H}}. \tag{19}$$

Proof: We show that the operator

$$\tilde{H} = H + \gamma^{\tilde{0}}F^{1/2}e^{\delta}m \tag{20}$$

is self-adjoint with dense domain on $\mathcal{H}$. We decompose $\mathcal{H}$ using generalized spherical functions of weights $1/2$ and $-1/2$. Let

$$\mathcal{I} = \{(l, m, n)\ ;\ 2l, 2m, 2n \in \mathbb{Z}\ ;\ l - |m|, l - |n| \in \mathbb{N}\} \tag{21}$$

and for any half-integer m

$$\mathcal{I}_m = \{(l, n)\ ;\ (l, m, n) \in \mathcal{I}\}. \tag{22}$$

For $(l, m, n) \in \mathcal{I}$, we define the function T^l_{mn} of $(\varphi_1, \theta, \varphi_2)$, $\varphi_1, \varphi_2 \in [0, 2\pi[$, $\theta \in [0, \pi]$, by

$$T^l_{mn}(\varphi_1, \theta, \varphi_2) = e^{-im\varphi_2}u^l_{mn}(\theta)e^{-in\varphi_1} \tag{23}$$

where u^l_{mn} satisfies the following ordinary differential equations

$$\frac{d^2u^l_{mn}}{d\theta^2} + cotg\theta\frac{du^l_{mn}}{d\theta} + \left[l(l+1) - \frac{n^2 - 2mncos\theta + m^2}{sin^2\theta}\right]u^l_{mn} = 0, \tag{24}$$

$$\frac{du^l_{mn}}{d\theta} - \frac{n - mcos\theta}{sin\theta}u^l_{mn} = -i\left[(l+m)(l-m+1)\right]^{1/2}u^l_{m-1,n}, \tag{25}$$

$$\frac{du^l_{mn}}{d\theta} + \frac{n - mcos\theta}{sin\theta}u^l_{mn} = -i\left[(l+m+1)(l-m)\right]^{1/2}u^l_{m+1,n} \tag{26}$$

and the normalization condition

$$\int_0^\pi \left|u^l_{mn}(\theta)\right|^2 sin\theta d\theta = \frac{1}{4\pi^2}. \tag{27}$$

We know from [12], that $\left\{T^l_{mn}\right\}_{(l,m,n)\in\mathcal{I}_{\frac{1}{2}}}$ is a Hilbert basis of

$$L^2\left([0, 2\pi[_{\varphi_1} \times [0, \pi]_\theta \times [0, 2\pi[_{\varphi_2}; sin^2\theta d\varphi_1^2 + d\theta^2 + d\varphi_2^2\right). \tag{28}$$

Thus, for any half-integer m,

$$\left\{T^l_{mn}(\varphi, \theta, 0) = e^{-in\varphi}u^l_{mn}(\theta)\right\}_{(l,n)\in\mathcal{I}_m}$$

is a Hilbert basis of $L^2(S^2_\omega; d\omega^2)$. In particular,

$$\mathcal{H} = \bigoplus_{(l,n)\in\mathcal{I}_{\frac{1}{2}}} \mathcal{H}_{ln} \tag{29}$$

where

$$\mathcal{H}_{ln} = \left\{ {}^t\left(f_1 T^l_{-\frac{1}{2},n}, f_2 T^l_{\frac{1}{2},n}, f_3 T^l_{-\frac{1}{2},n}, f_4 T^l_{\frac{1}{2},n}\right) \; ; \; f_i \in L^2(\mathbb{R}_{r_*}; dr_*^2) \; , \; i=1,2,3,4 \right\}, \tag{30}$$

or equivalently,

$$\mathcal{H}_{ln} = \left[L^2(\mathbb{R}_{r_*}; dr_*^2)\right]^4 \otimes F_{ln} \quad ; \quad F_{ln} = {}^t\left(T^l_{-\frac{1}{2},n}, T^l_{\frac{1}{2},n}, T^l_{-\frac{1}{2},n}, T^l_{\frac{1}{2},n}\right) \tag{31}$$

where the $T^l_{\pm\frac{1}{2},n}$ are seen as functions of only φ, θ. Let

$$\Psi = {}^t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}.$$

Denoting $\alpha = F^{1/2} e^{\delta}$, the four components of $\tilde{H}\Psi$ are

$$i\partial_{r_*} f_3 T^l_{-\frac{1}{2},n} - \frac{\alpha}{r} f_4 \left(\partial_\theta + \frac{1}{2} cotg\theta\right) T^l_{\frac{1}{2},n} + i\frac{\alpha}{rsin\theta} f_4 \partial_\varphi T^l_{\frac{1}{2},n},$$

$$-i\partial_{r_*} f_4 T^l_{\frac{1}{2},n} + \frac{\alpha}{r} f_3 \left(\partial_\theta + \frac{1}{2} cotg\theta\right) T^l_{-\frac{1}{2},n} + i\frac{\alpha}{rsin\theta} f_3 \partial_\varphi T^l_{-\frac{1}{2},n},$$

$$i\partial_{r_*} f_1 T^l_{-\frac{1}{2},n} - \frac{\alpha}{r} f_2 \left(\partial_\theta + \frac{1}{2} cotg\theta\right) T^l_{\frac{1}{2},n} + i\frac{\alpha}{rsin\theta} f_2 \partial_\varphi T^l_{\frac{1}{2},n},$$

$$-i\partial_{r_*} f_2 T^l_{\frac{1}{2},n} + \frac{\alpha}{r} f_1 \left(\partial_\theta + \frac{1}{2} cotg\theta\right) T^l_{-\frac{1}{2},n} + i\frac{\alpha}{rsin\theta} f_1 \partial_\varphi T^l_{-\frac{1}{2},n}.$$

Relations (25) and (26) yield

$$\left(\partial_\theta + \frac{1}{2} cotg\theta\right) T^l_{\frac{1}{2},n} = \frac{n}{sin\theta} T^l_{\frac{1}{2},n} - i\left(l + \frac{1}{2}\right) T^l_{-\frac{1}{2},n}, \tag{32}$$

$$\left(\partial_\theta + \frac{1}{2} cotg\theta\right) T^l_{-\frac{1}{2},n} = \frac{-n}{sin\theta} T^l_{-\frac{1}{2},n} - i\left(l + \frac{1}{2}\right) T^l_{\frac{1}{2},n} \tag{33}$$

and we also have

$$\partial_\varphi T^l_{\pm\frac{1}{2},n}(\varphi,\theta,0) = -in T^l_{\pm\frac{1}{2},n}(\varphi,\theta,0). \tag{34}$$

Thus, the four components of $\tilde{H}\Psi$ are

$$\begin{array}{l} \left(i\partial_{r_*} f_3 + i\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_4\right) T^l_{-\frac{1}{2},n}, \\ \left(-i\partial_{r_*} f_4 - i\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_3\right) T^l_{\frac{1}{2},n}, \\ \left(i\partial_{r_*} f_1 + i\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_2\right) T^l_{-\frac{1}{2},n}, \\ \left(-i\partial_{r_*} f_2 - i\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_1\right) T^l_{\frac{1}{2},n}. \end{array}$$

We see that on $\mathcal{H}_{ln}$, $\tilde{H}$ has the form

$$\tilde{H} \mid_{\mathcal{H}_{ln}} = \left(i\partial_{r_*} L + \frac{\alpha}{r}\left(l + \frac{1}{2}\right) M\right)_{r_*} \otimes \mathbb{1}_{\theta,\varphi} \tag{35}$$

where the matrices L et M, defined by

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

are hermitian and L is invertible. Since the function αr^{-1} belongs to $L^\infty(\mathbb{R}_{r_*})$, $\tilde{H}\mid_{\mathcal{H}_{ln}}$ is self-adjoint with domain

$$D_{ln} = [D(i\partial_{r_*})]^4 \otimes F_{ln} \simeq \left[H^1(\mathbb{R}_{r_*}; dr_*^2)\right]^4 \otimes F_{ln} \tag{37}$$

dense in $\mathcal{H}_{ln}$. On D_{ln}, we choose the following norm

$$\Psi = {}^t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in D_{ln} \quad , \quad \|\Psi\|^2_{D_{ln}} = \|\Psi\|^2_{(L^2(\mathbf{R}))^4} + \left\| \left(i\partial_{r_*} L + \frac{\alpha}{r}\left(l+\frac{1}{2}\right) M\right)\Psi \right\|^2_{(L^2(\mathbf{R}))^4} \tag{38}$$

and we introduce the dense subspace of $\mathcal{H}$

$$D(H) = \left\{ \Psi = \sum_{(l,n)\in\mathcal{I}_{\frac{1}{2}}} \Psi_{ln} \; ; \; \Psi_{ln} \in D_{ln} \; , \; \sum_{(l,n)\in\mathcal{I}_{\frac{1}{2}}} \|\Psi_{ln}\|^2_{D_{ln}} < +\infty \right\}. \tag{39}$$

$\tilde{H}$ is self-adjoint on $\mathcal{H}$ with domain $D(H)$, $\gamma^{\tilde{0}}\alpha m$ is self-adjoint and bounded on $\mathcal{H}$, therefore, H is self-adjoint on $\mathcal{H}$ with dense domain $D(H)$. Theorem 3.1 follows from Stone's theorem.

Q.E.D.

4 Wave operators at the horizon

When $r \to r_0$, the operator H has the formal limit

$$H_0 = i\gamma^{\tilde{0}}\gamma^{\tilde{3}}\partial_{r_*} \tag{40}$$

which is a self-adjoint operator on $\mathcal{H}$ with dense domain

$$D(H_0) = \left\{ H^1\left[\left(\mathbb{R}_{r_*}; dr_*^2\right); L^2\left(S^2_\omega; d\omega^2\right)\right]\right\}^4. \tag{41}$$

The spectrum of H_0 is purely absolutely continuous. We define the subspaces of incoming and outgoing waves associated with H_0:

$$\mathcal{H}_0^\pm = \left\{\Psi = {}^t\left(u^1, u^2, u^3, u^4\right) \; , \; u^3 = \mp u^1 \; , \; u^4 = \pm u^2\right\}. \tag{42}$$

$\mathcal{H}_0^\pm$ as well as the $\mathcal{H}_{ln}$ remain stable under H_0 and we have

$$\mathcal{H} = \mathcal{H}_0^+ \oplus \mathcal{H}_0^- \; , \quad \forall \Psi_0 \in \mathcal{H}_0^\pm, \quad \left(e^{iH_0t}\Psi_0\right)(r_*, \omega) = \Psi_0\left(r_* \pm t, \omega\right). \tag{43}$$

Since we want to compare H with H_0 in the neighbourhood of the horizon, we introduce the cut-off function

$$\begin{gathered} \chi_o \in \mathcal{C}^\infty(\mathbb{R}_{r_*}) \quad , \quad 0 \le \chi_o \le 1, \\ \exists a, b \in \mathbb{R} \quad , \quad a < b \;\; \textit{such that} \\ \textit{for} \;\; r_* < a \;\; \chi_o(r_*) = 1 \quad ; \quad \textit{for} \;\; r_* > b \;\; \chi_o(r_*) = 0 \end{gathered} \tag{44}$$

together with the identifying operator

$$\mathcal{J}_0 : \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H} \\ \Psi & \longmapsto & \chi_0\Psi. \end{array} \tag{45}$$

We consider the classical wave operators

$$W_0^\pm \Psi_0 = \underset{t\to\pm\infty}{s-lim} \; e^{-iHt}\mathcal{J}_0 e^{iH_0t}\Psi_0 \quad \textit{in } \mathcal{H}. \tag{46}$$

Theorem 4.1. *The operator W_0^+ (resp. W_0^-) is well-defined from $\mathcal{H}_0^+$ (resp. $\mathcal{H}_0^-$) to $\mathcal{H}$, is independent of the choice of χ_o satisfying (44), moreover*

$$\forall \Psi_0 \in \mathcal{H}_0^\pm \; , \quad \left\|W_0^\pm\Psi_0\right\|_{\mathcal{H}} = \|\Psi_0\|_{\mathcal{H}}. \tag{47}$$

Proof: We apply Cook's method. $\mathcal{J}_0$ being a bounded operator, it suffices to prove that for

$$\Psi_0 \in \mathcal{D}^{\pm}_{ln} \ ; \quad \mathcal{D}^{\pm}_{ln} = \mathcal{H}^{\pm}_0 \cap \mathcal{H}_{ln} \cap \left[\mathcal{C}^{\infty}_0 \left(\mathbb{R}_{r_*} \times S^2_{\omega} \right) \right]^4 \quad , \quad (l,n) \in \mathcal{I}_{\frac{1}{2}} \tag{48}$$

we have

$$\left\| \left(H\mathcal{I}_0 - \mathcal{I}_0 H_0 \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} \in L^1 \left(\pm t > 0 \right). \tag{49}$$

Let for $(l,n) \in \mathcal{I}_{\frac{1}{2}}$

$$\Psi_0 \in \mathcal{D}^{+}_{ln} \ , \quad Supp \Psi_0 \subset [-R, R]_{r_*} \times S^2_{\omega} \quad , \quad R > 0, \tag{50}$$

then

$$He^{iH_0 t} \Psi_0 = \left(i\partial_{r_*} + \frac{\alpha}{r} \left(l + \frac{1}{2} \right) M - \alpha m \gamma^{\tilde{0}} \right) \Psi_0 (r_* + t),$$

and

$$H_0 e^{iH_0 t} \Psi_0 = i\partial_{r_*} L \Psi_0 (r_* + t).$$

Ψ_0 being compactly supported, for t large enough,

$$\begin{aligned} \left\| \left(H\mathcal{I}_0 - \mathcal{I}_0 H_0 \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} &= \left\| \left(\frac{\alpha}{r} \left(l + \frac{1}{2} \right) M - \alpha m \gamma^{\tilde{0}} \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} \\ &\leq \left\| \left(l + \frac{1}{2} \right) \frac{\alpha}{r} + \alpha m \right\|_{L^{\infty}(-R-t, R-t)} \left\| \Psi_0 \right\|_{\mathcal{H}}. \end{aligned}$$

α is rapidly decreasing in r_* when $r \to r_0$, therefore

$$\left\| \left(H\mathcal{I}_0 - \mathcal{I}_0 H_0 \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} \in L^1 \left(t > 0 \right)$$

and W_0^+ is well-defined. The same proof can of course be applied to W_0^-. Furthermore, if $\Psi_0 \in \mathcal{H}_0^{\pm}$, we get from (43) that the energy of $e^{iH_0 t}\Psi_0$ in a domain of $\mathbb{R}_{r_*} \times S^2_{\omega}$ bounded to the left in r_* vanishes when t tends to infinity, which gives (47). If now we consider two different cut-off functions χ_o and χ_o', and the associated identifying operators $\mathcal{J}_0$ and $\mathcal{J}_0'$, the difference $\chi_o - \chi_o'$ is compactly supported, thus

$$\left\| e^{-iHt} \mathcal{J}_0 e^{iH_0 t} \Psi_0 - e^{-iHt} \mathcal{J}_0' e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} \to 0 \quad , \quad t \to \pm\infty.$$

Q.E.D.

Remark 4.1. *In the case where r_+ is finite, we construct in the same way classical wave operators at the cosmological horizon*

$$W_1^{\pm} \Psi_0 = \underset{t \to \pm\infty}{s - lim} \quad e^{-iHt} \mathcal{J}_1 e^{iH_0 t} \Psi_0 \quad in \ \mathcal{H} \tag{51}$$

where the identifying operator $\mathcal{J}_1$ is defined by

$$\mathcal{J}_1 : \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H} \\ \Psi & \longmapsto & \chi_1 \Psi, \end{array} \tag{52}$$

χ_1 being a cut-off function

$$\begin{array}{c} \chi_1 \in \mathcal{C}^{\infty} \left(\mathbb{R}_{r_*} \right), \quad 0 \leq \chi_1 \leq 1, \\ \exists a, b \in \mathbb{R} \quad , \quad a < b \ \ such \ that \\ for \quad r_* < a \quad \chi_1(r_*) = 0 \quad ; \quad for \quad r_* > b \quad \chi_1(r_*) = 1. \end{array} \tag{53}$$

W_1^+ (resp. W_1^-) is an isometry from $\mathcal{H}_0^-$ (resp. $\mathcal{H}_0^+$) to $\mathcal{H}$ and is independent of the choice of χ_1 satisfying (53).

5 Wave operators at infinity (massless case)

In all this paragraph, we shall assume that $r_+ = +\infty$; the metric (1) is then asymptotically flat in the neighbourhood of infinity and we choose to compare H to an operator H_∞ which is equivalent to the hamiltonian operator for the Dirac equation on the Minkowski space-time. We also make the hypothesis that $m = 0$ in order to avoid long range perturbations at infinity. Let us consider on the Minkowski metric

$$ds^2_{\mathcal{M}} = dt^2 - dx^2 - dy^2 - dz^2 \quad ; \quad x, y, z \in \mathbb{R} \tag{54}$$

the massless Dirac system

$$\left\{\gamma^{\tilde{0}}\partial_t + \gamma^{\tilde{1}}\partial_x + \gamma^{\tilde{2}}\partial_y + \gamma^{\tilde{3}}\partial_z\right\}\Phi = 0. \tag{55}$$

The associated hamiltonian operator, defined by

$$H_{\mathcal{M}} = i\gamma^{\tilde{0}}\left\{\gamma^{\tilde{1}}\partial_x + \gamma^{\tilde{2}}\partial_y + \gamma^{\tilde{3}}\partial_z\right\}, \tag{56}$$

is self-adjoint with dense domain on $\left[L^2\left(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z\right)\right]^4$ and if $\Phi \in \mathcal{C}\left(\mathbb{R}_t; \left[L^2\left(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z\right)\right]^4\right)$ is a solution of (55), its energy in a compact domain goes to zero when t goes to $\pm\infty$. In addition, for any $\Phi_0 \in \left[L^2\left(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z\right)\right]^4$ with a compact support contained in

$$B(0,R) = \left\{(x,y,z);\ 0 \le \rho < R\ ,\ \ \rho = \left(x^2 + y^2 + z^2\right)^{1/2}\right\}, \tag{57}$$

the solution Φ of (55) associated with the initial data Φ_0 satisfies

$$\Phi(t,x,y,z) = 0 \quad for \ \ 0 \le \rho \le |t| - R. \tag{58}$$

At the point of spherical coordinates (ρ, θ, φ), we apply the spatial rotation f with Euler angles $(\pi/2, \theta, \pi - \varphi)$. The local frame $(\partial_x, \partial_y, \partial_z)$ is thus transformed by f^{-1} into

$$(\partial_{x^1}, \partial_{x^2}, \partial_{x^3}) = \left(\frac{1}{\rho sin\theta}\partial_\varphi, \frac{-1}{\rho}\partial_\theta, \partial_\rho\right). \tag{59}$$

The spinor

$$\Psi = \rho T_f \Phi, \tag{60}$$

where T_f is the spin transformation associated with f defined in (16), satisfies

$$\partial_t\Psi = iH_\infty\Psi \quad , \quad H_\infty = i\left[\gamma^{\tilde{0}}\gamma^{\tilde{3}}\partial_\rho - \frac{1}{\rho}\gamma^{\tilde{0}}\gamma^{\tilde{2}}\left(\partial_\theta + \frac{1}{2}cotg\theta\right) + \frac{1}{\rho sin\theta}\gamma^{\tilde{0}}\gamma^{\tilde{1}}\partial_\varphi\right]. \tag{61}$$

The operator H_∞ on

$$\mathcal{H}_\infty = \left\{L^2\left([0, +\infty[_\rho \times S^2_\omega\ ;\ \ d\rho^2 + d\omega^2\right)\right\}^4 \tag{62}$$

is unitarily equivalent to $H_{\mathcal{M}}$ on

$$\left\{L^2\left(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z \ \ ; \ \ dx^2 + dy^2 + dz^2\right)\right\}^4.$$

Therefore, H_∞ is self-adjoint with dense domain on $\mathcal{H}_\infty$ and if $\Psi \in \mathcal{C}\left(\mathbb{R}_t, \mathcal{H}_\infty\right)$ satisfies (61), then its energy in a compact domain goes to zero when t goes to $\pm\infty$. Moreover, for

$$\Psi_0 \in \mathcal{H}_\infty \quad ; \quad Supp(\Psi_0) \subset B(0,R)$$

$\Psi(t) = e^{iH_\infty t}\Psi_0$ satisfies

$$\Psi(t,\rho,\theta,\varphi) = 0 \quad for \ \ 0 \le \rho \le |t| - R. \tag{63}$$

In order to avoid artificial long-range interactions, we choose

$$\rho = r_* \ge 0 \tag{64}$$

and we introduce the cut-off function

$$\begin{array}{c}\chi_\infty \in \mathcal{C}^\infty\left([0,+\infty[_{r_*}\right) \quad , \quad 0 \leq \chi_\infty \leq 1, \\ \exists \;\; 0<a<b<+\infty \quad such \ that \\ for \ 0 \leq r_* \leq a \;\; \chi_\infty(r_*)=0 \quad , \quad for \ r_* \geq b \;\; \chi_\infty(r_*)=1\end{array} \tag{65}$$

together with the identifying operator

$$\mathcal{J}_\infty : \mathcal{H}_\infty \longrightarrow \mathcal{H} \;\; ; \;\; for \ \Psi \in \mathcal{H}_\infty \quad \left\{ \begin{array}{l} (\mathcal{J}\Psi)\mid_{\{r_*\geq 0\}} = \chi_\infty \Psi, \\ \\ (\mathcal{J}\Psi)\mid_{\{r_* \leq 0\}} = 0. \end{array}\right. \tag{66}$$

We define the classical wave operators

$$W_\infty^\pm \Psi_0 = \begin{array}{c} s-lim \\ t\rightarrow \pm\infty \end{array} \quad e^{-iHt}\mathcal{J}_\infty e^{iH_\infty t}\Psi_0 \quad in \ \mathcal{H}. \tag{67}$$

Theorem 5.1. *The operators $W_\infty^\pm$ are well-defined from $\mathcal{H}_\infty$ to $\mathcal{H}$, are independent of the choice of χ_∞ and*

$$\forall \Psi_0 \in \mathcal{H}_\infty \ , \quad \left\| W_\infty^\pm \Psi_0 \right\|_{\mathcal{H}} = \left\| \Psi_0 \right\|_{\mathcal{H}_\infty} . \tag{68}$$

Proof: For $(l,n) \in \mathcal{I}_{\frac{1}{2}}$, we introduce the subspaces of $\mathcal{H}_\infty$

$$\mathcal{D}_{ln}^\infty = \left\{ \Psi = {}^t\left(f_1,f_2,f_3,f_4\right) \otimes F_{ln} \in \mathcal{H}_\infty ; \;\; 1 \leq i \leq 4 \;\; f_i \in \mathcal{C}_0^\infty\left(\mathbb{R}_{r_*}^+\right)\right\} \tag{69}$$

the direct sum of which is dense in $\mathcal{H}_\infty$. For $\Psi_0 \in \mathcal{D}_{ln}^\infty$,

$$H_\infty \mid_{\mathcal{D}_{ln}^\infty} = \left(i\partial_{r_*} L + \frac{1}{r_*}\left(l+\frac{1}{2}\right) M\right)_{r_*} \otimes \mathbb{1}_\omega \tag{70}$$

where the matrices L and M are defined by (36), and

$$\mathcal{J}_\infty \Psi_0 \in \mathcal{H}_{ln}. \tag{71}$$

$\mathcal{J}_\infty$ being a bounded operator, it suffices to prove that for

$$\Psi_0 \in \mathcal{D}_{ln}^\infty \;\; ; \;\; Supp(\Psi_0) \subset \;\; B(0,R), \tag{72}$$

we have

$$\left\| \left(H\mathcal{J}_\infty - \mathcal{J}_\infty H_\infty\right) e^{iH_\infty t}\Psi_0 \right\|_{\mathcal{H}} \in L^1(\mathbb{R}_t). \tag{73}$$

(63) yields

$$e^{iH_\infty t}\Psi_0 = 0 \;\; in \;\; \left\{(t,r_*,\theta,\varphi); \;\; 0 \leq r_* \leq |t| - R\right\}. \tag{74}$$

Thus, for $|t|$ large enough

$$\begin{aligned}\left\| \left(H\mathcal{J}_\infty - \mathcal{J}_\infty H_\infty\right) e^{iH_\infty t}\Psi_0 \right\|_{\mathcal{H}} &= \left\| \left(\frac{\alpha}{r} - \frac{1}{r_*}\right)\left(l+\frac{1}{2}\right) M e^{iH_\infty t}\Psi_0 \right\|_{\mathcal{H}} \\ &\leq \left(l+\frac{1}{2}\right) \left\|\Psi_0\right\|_{\mathcal{H}_\infty} \left\| \frac{\alpha}{r} - \frac{1}{r_*}\right\|_{L^\infty([|t|+R,+\infty[_{r_*})} .\end{aligned}$$

We study the asymptotic behavior of

$$\frac{\alpha}{r} - \frac{1}{r_*} = \frac{1}{r_*}\left(F^{1/2} e^\delta \frac{r_*}{r} - 1\right)$$

when r_* goes to $+\infty$. The Regge-Wheeler variable r_* is defined with respect to r by

$$r_* = \frac{1}{2\kappa_0}\left\{ Log|r-r_0| - \int_{r_0}^r \left[\frac{1}{r-r_0} - \frac{2\kappa_0}{Fe^\delta}\right] dr\right\} \tag{75}$$

where $2\kappa_0 = F'(r_0)$. For r larger than r_0+1, we have

$$r_* = C + \int_{r_0+1}^{r} F^{-1}e^{-\delta}dr \tag{76}$$

where

$$2\kappa_0 C = -\int_{r_0}^{r_0+1}\left[\frac{1}{r-r_0} - \frac{2\kappa_0}{Fe^{\delta}}\right]dr. \tag{77}$$

F and δ satisfy

$$\delta(r) = o(r^{-1}) \;\; ; \;\; F(r) = 1 - \frac{r_1}{r} + O(r^{-2}) \quad r_1 > 0 \;\; ; \;\; r \to +\infty$$

and therefore

$$F^{-1}(r)e^{-\delta(r)} = 1 + \frac{r_1}{r} + o(r^{-1}),$$

$$r_* = r + r_1 Log(r) + o(Log(r)),$$

$$F^{1/2}(r)e^{\delta(r)} = 1 - \frac{r_1}{2r} + o(r^{-1})$$

which implies

$$F^{1/2}(r)e^{\delta(r)}\frac{r_*}{r} - 1 = r_1\frac{Log(r)}{r} + o\left(\frac{Log(r)}{r}\right) = O(r^{-1/2}) = O(r_*^{-1/2}).$$

The operators $W_\infty^\pm$ are thus well-defined. The fact that they are isometries and do not depend on the choice of the cut-off function can be verified using exactly the same remarks as in the case of the horizon.

Q.E.D.

6 Asymptotic completeness of operators $W_0^\pm$ and $W_\infty^\pm$ (massless case)

We assume again that $m=0$ and $r_+ = +\infty$. We introduce the inverse wave operators at the horizon and at infinity, defined for $\Psi_0 \in \mathcal{H}$ by

$$\tilde{W}_0^\pm \Psi_0 = \underset{t\to\pm\infty}{s-lim} \quad e^{-iH_0 t}\mathcal{J}_0^* e^{iHt}\Psi_0 \quad in\ \mathcal{H}, \tag{78}$$

$$\tilde{W}_\infty^\pm \Psi_0 = \underset{t\to\pm\infty}{s-lim} \quad e^{-iH_\infty t}\mathcal{J}_\infty^* e^{iHt}\Psi_0 \quad in\ \mathcal{H}_\infty, \tag{79}$$

where $\mathcal{J}_0^*$ and $\mathcal{J}_\infty^*$ are respectively the adjoints of $\mathcal{J}_0$ and $\mathcal{J}_\infty$. We also define the wave operators W^+ and W^- by

$$\Psi_0 \in \mathcal{H}_0^\pm \quad , \quad \Psi_\infty \in \mathcal{H}_\infty \qquad W^\pm(\Psi_0, \Psi_\infty) = W_0^\pm\Psi_0 + W_\infty^\pm\Psi_\infty \tag{80}$$

as well as the inverse wave operators $\tilde{W}^+$, $\tilde{W}^-$

$$\Psi_0 \in \mathcal{H} \qquad \tilde{W}^\pm\Psi_0 = \left(\tilde{W}_0^\pm\Psi_0, \tilde{W}_\infty^\pm\Psi_0\right). \tag{81}$$

Eventually, we define the scattering operator

$$S = \tilde{W}^+ W^-. \tag{82}$$

Theorem 6.1. *Operators $\tilde{W}_0^\pm$ (resp. $\tilde{W}_\infty^\pm$) are well defined from $\mathcal{H}$ into $\mathcal{H}_0^\pm$ (resp. from $\mathcal{H}$ into $\mathcal{H}_\infty$), are independent of the choice of χ_o (resp. χ_∞) and their norm is lower or equal to 1. Moreover*

$W^\pm$ is an isometry of $\mathcal{H}_0^\pm \times \mathcal{H}_\infty$ onto $\mathcal{H}$.
$\tilde{W}^\pm$ is an isometry of $\mathcal{H}$ onto $\mathcal{H}_0^\pm \times \mathcal{H}_\infty$.
S is an isometry of $\mathcal{H}_0^- \times \mathcal{H}_\infty$ onto $\mathcal{H}_0^+ \times \mathcal{H}_\infty$.

Proof: For any solution Ψ of (15) in $\mathcal{C}(\mathbb{R}_t;\mathcal{H}_{ln})$, $(l,n),\in\mathcal{I}_{\frac{1}{2}}$, we construct asymptotic profiles at the horizon and at infinity. The idea is that each component of Ψ satisfies an equation of the form

$$\left(\partial_t^2-\partial_{r_*}^2+V(r_*)\right)f=0 \tag{83}$$

where the potential V has the following properties

$$\begin{aligned} V=V_+-V_- \quad &; \quad V_+,V_-\geq 0,\\ V_+(r_*)\leq C(1+|r_*|)^{-1-\varepsilon} \quad &, \quad \varepsilon>0,\\ V_-(r_*)\leq C(1+|r_*|)^{-2-\varepsilon} \quad &, \quad \varepsilon>0. \end{aligned} \tag{84}$$

We then apply the scattering results of [3]. This suffices to define $\tilde{W}_0^{\pm}$, but to prove the existence of $\tilde{W}_\infty^{\pm}$, we need to recover a solution of $(\partial_t-iH_\infty)\Psi=0$ from the asymptotic profile at infinity.

Firstly, we study some spectral properties of the operator H:

Proposition 6.1. *The point spectrum of H is empty.*

A straightforward consequence of proposition 6.1 is

Corollary 6.1. *For $k\in\mathbb{N}$, the direct sum of the sets*

$$\mathcal{E}_{ln}^k=\left\{H^k\Psi;\ \Psi={}^t(f_1,f_2,f_3,f_4)\otimes F_{ln}\in\mathcal{H}_{ln}\ ,\ \ 1\leq i\leq 4\ \ f_i\in\mathcal{C}_0^\infty(\mathbb{R}_{r_*})\right\}\ ;\ (l,n)\in\mathcal{I}_{\frac{1}{2}} \tag{85}$$

is dense in $\mathcal{H}$.

Proof of proposition 6.1: Let

$$\Psi_{ln}=\phi\otimes F_{ln}\in\mathcal{H}_{ln}\ ;\ \phi={}^t(f_1,f_2,f_3,f_4)\in\left[L^2(\mathbb{R},dr_*^2)\right]^4 \tag{86}$$

such that

$$H\Psi_{ln}=\lambda\Psi_{ln}\ ;\quad \lambda\in\mathbb{R}. \tag{87}$$

Equation (87) is equivalent to

$$\begin{aligned} f_1'&=-\beta_l f_2-i\lambda f_3,\\ f_2'&=-\beta_l f_1+i\lambda f_4,\\ f_3'&=-\beta_l f_4-i\lambda f_1,\\ f_4'&=-\beta_l f_3+i\lambda f_2, \end{aligned} \qquad \beta_l(r_*)=\left(l+\frac{1}{2}\right)\frac{F^{1/2}e^\delta}{r}. \tag{88}$$

We first consider the case $\lambda=0$. Putting

$$\begin{aligned} g_1=f_1+f_2 \quad &,\quad g_2=f_2-f_1,\\ g_3=f_3+f_4 \quad &,\quad g_4=f_4-f_3, \end{aligned} \tag{89}$$

we see that g_1 and g_3 are solutions of

$$g'=-\beta_l.g, \tag{90}$$

while g_2 and g_4 satisfy

$$f'=\beta_l.f. \tag{91}$$

Thus $\lambda=0$ is an eigenvalue for H if and only if there exists $l=\frac{1}{2}+k$, $k\in\mathbb{N}$, such that both equations (90) and (91) have solutions in $L^2\left(\mathbb{R}_{r_*};dr_*^2\right)$. β_l being smooth on $\mathbb{R}$, any solution of (90) or (91) in $L^1_{loc}(\mathbb{R})$ is necessarily smooth. Moreover, β_l decreases exponentially when r_* goes to $-\infty$, thus

$$\forall r_*^1\in\mathbb{R}\quad \beta_l\in L^1\left(]-\infty,r_*^1[\right) \tag{92}$$

and both integral equations

$$f(r_*)=1+\int_{-\infty}^{r_*}\beta_l.f\,dr_*, \tag{93}$$

$$g(r_*) = 1 - \int_{-\infty}^{r_*} \beta_l . g dr_* \tag{94}$$

have a unique solution in $L^\infty\left(\left]-\infty, r_{r_*}^1\right[\right)$, which can be extended on $\mathbb{R}$ as a smooth but not square integrable function. Therefore, (90) and (91) have no non trivial solution in $L^2(\mathbb{R})$ and $\lambda = 0$ is not an eigenvalue for H.

If now we suppose $\lambda \neq 0$, the components of ϕ satisfy

$$\begin{aligned} f_1'' &= \left(\beta_l^2 - \lambda^2\right) f_1 - \beta_l' f_2, \\ f_2'' &= \left(\beta_l^2 - \lambda^2\right) f_2 - \beta_l' f_1, \\ f_3'' &= \left(\beta_l^2 - \lambda^2\right) f_3 - \beta_l' f_4, \\ f_4'' &= \left(\beta_l^2 - \lambda^2\right) f_4 - \beta_l' f_3. \end{aligned} \tag{95}$$

Functions $g_1 = f_1 + f_2$ and $g_3 = f_3 + f_4$ are eigenvectors in $L^2(\mathbb{R})$ for the operator

$$L_1 = -\partial_{r_*}^2 + \beta_l^2(r_*) - \beta_l'(r_*) \tag{96}$$

associated with the eigenvalue $\lambda^2 > 0$, whereas $g_2 = f_2 - f_1$ and $g_4 = f_4 - f_3$ are eigenvectors in $L^2(\mathbb{R})$ for the operator

$$L_2 = -\partial_{r_*}^2 + \beta_l^2(r_*) + \beta_l'(r_*) \tag{97}$$

associated with the eigenvalue $\lambda^2 > 0$. It is easily seen that potentials

$$V_1(r_*) = \beta_l^2(r_*) - \beta_l'(r_*) \tag{98}$$

and

$$V_2(r_*) = \beta_l^2(r_*) + \beta_l'(r_*) \tag{99}$$

satisfy (84). Therefore, the operators L_1 and L_2 are of the same type as the second order operators studied in [3] and have no strictly positive eigenvalue.

Q.E.D.

Proof of corollary 6.1: For $(l,n) \in \mathcal{I}_{\frac{1}{2}}$ and $k \in \mathbb{N}$, if

$$\Psi = \phi \otimes F_{ln} \in \mathcal{H}_{ln} \quad ; \quad \phi \in \left[\mathcal{C}_0^\infty\left(\mathbb{R}_{r_*}\right)\right]^4,$$

then Ψ belongs to $D\left(H^k \mid_{\mathcal{H}_{ln}}\right)$. $\mathcal{E}_{ln}^k$ is well-defined and is a subset of $\mathcal{H}_{ln}$. To prove corollary 6.1 it suffices to establish that for $(l,n) \in \mathcal{I}_{\frac{1}{2}}$ and $k \in \mathbb{N}$, $\mathcal{E}_{ln}^k$ is dense in $\mathcal{H}_{ln}$. Let

$$\Psi_0 = \phi_0 \otimes F_{ln} \in \mathcal{H}_{ln}$$

be orthogonal to $\mathcal{E}_{ln}^k$. Then, for $\phi \in \left[\mathcal{C}_0^\infty\left(\mathbb{R}_{r_*}\right)\right]^4$

$$\left(\phi_0, H^k \mid_{\mathcal{H}_{ln}} \phi\right)_{L^2(\mathbb{R}_{r_*})} = 0,$$

$H^k \mid_{\mathcal{H}_{ln}}$ being here considered as an operator on $\left[L^2(\mathbb{R}_{r_*})\right]^4$. We have

$$H^k \mid_{\mathcal{H}_{ln}} \phi_0 = 0 \quad in \quad \left[\mathcal{D}'\left(\mathbb{R}_{r_*}\right)\right]^4 \tag{100}$$

where $\mathcal{D}'(\mathbb{R}_{r_*})$ is the space of distributions on $\mathbb{R}_{r_*}$. From (100), we deduce that Ψ_0 belongs to $D(H^k \mid_{\mathcal{H}_{ln}})$ and

$$H^k \Psi_0 = 0 \quad in \ \mathcal{H}_{ln}. \tag{101}$$

We know by proposition 6.1 that (101) has no non-trivial solution in $\mathcal{H}_{ln}$. Thus $\mathcal{E}_{ln}^k$ is dense in $\mathcal{H}_{ln}$.

Q.E.D.

We also study the spectral properties of operators L_1, L_2. We recall their definition for $l - 1/2 \in \mathbb{N}$

$$i = 1,2 \quad L_i = -\partial_{r_*}^2 + V_i(r_*) \;\; ; \;\; V_i(r_*) = \beta_l^2(r_*) + (-1)^i \beta_l'(r_*). \tag{102}$$

Proposition 6.2. *For $l - 1/2 \in \mathbb{N}$, the spectrum of operators L_1 and L_2 is purely absolutely continuous.*

Proof: We already know that potentials V_1 and V_2 satisfy (84), which, from [3] implies that the singular spectrum of L_1 and L_2 is empty, that their absolutely continuous spectrum is $[0, +\infty[$ and that their point spectrum contains at the most a finite number of negative or zero eigenvalues, all of them being simple. Furthermore, V_1 and V_2 decrease exponentially when $r_* \to -\infty$ and 0 is not an eigenvalue. We show that L_1 and L_2 do not have any strictly negative eigenvalue either by a method similar to the one used in [3]. We recall that for $l - 1/2 \in \mathbb{N}$, equations

$$1 \le i \le 2 \quad L_i f = 0 \tag{103}$$

both have on $\mathbb{R}_{r_*}$ a unique continuous strictly positive solution, given respectively by (93) and (94). We consider the general case of a potential

$$V \in L^\infty(\mathbb{R}_{r_*}) \cap L^2(\mathbb{R}_{r_*}) \tag{104}$$

such that there exists a function g, continuous and strictly positive on $\mathbb{R}_{r_*}$, satisfying

$$L_V g = 0 \;\; ; \;\; L_V = -\partial_{r_*}^2 + V. \tag{105}$$

Let $f \in L^2(\mathbb{R}_{r_*})$ be such that

$$L_V f = -\lambda f \quad , \quad \lambda > 0, \tag{106}$$

which implies

$$f \in H^2(\mathbb{R}_{r_*}). \tag{107}$$

We define the cut-off function

$$\chi \in \mathcal{C}_0^\infty(\mathbb{R}_{r_*}) \,, \quad for \; |r_*| \le \frac{1}{2} \quad \chi(r_*) = 1 \,, \quad for \; |r_*| \ge 1 \quad \chi(r_*) = 0. \tag{108}$$

Putting for $n \ge 1$

$$f_n(r_*) = \chi\left(\frac{r_*}{n}\right) f(r_*), \tag{109}$$

we easily see that

$$\int_{[-n,n]} \left(|f_n'|^2 + V |f_n|^2 \right) dr_* = -\lambda \int_{[-\frac{n}{2}, \frac{n}{2}]} |f|^2 \, dr_* + o(1) \,. \tag{110}$$

Thus, for n large enough

$$\int_{[-n,n]} \left[|f_n'|^2 + V |f_n|^2 \right] dr_* < 0.$$

The operator $-\partial_{r_*}^2 + V$ on $L^2([-n,n])$ with domain $\{y \in H^2([-n,n]); \; y(\pm n) = 0\}$ has a strictly negative eigenvalue $-\lambda_n$ associated with an eigenvector u

$$\left\{ \begin{array}{l} -u'' + Vu = -\lambda_n u \; ; \quad -n < r_* < n, \\ \\ u(-n) = u(n) = 0. \end{array} \right. \tag{111}$$

Even if it means changing u into $-u$, there exist α and β such that

$$\begin{gathered} -n \le \alpha < \beta \le n, \\ u(\alpha) = u(\beta) = 0 \,, \quad u'(\alpha) > 0 \,, \quad u'(\beta) < 0, \\ u > 0 \;\; for \;\; \alpha < r_* < \beta. \end{gathered} \tag{112}$$

We denote

$$I = \int_\alpha^\beta (u'g - ug')' \, dr_*.$$

On the one hand, we can write

$$I = u'(\beta)g(\beta) - u'(\alpha)g(\alpha),$$

g being strictly positive on $\mathbb{R}$, (112) yields

$$I < 0.$$

On the other hand

$$(u'g - ug')' = u''g - g''u = -\lambda_n ug,$$

thus

$$I = \lambda_n \int_\alpha^\beta ug dr_* > 0.$$

We end up with a contradiction, which means that L_V has no strictly negative eigenvalue.

Q.E.D.

We now prove the existence of the inverse wave operators $\tilde{W}_0^\pm$ and $\tilde{W}_\infty^\pm$. For $(l,n) \in \mathcal{I}_{\frac{1}{2}}$, we consider the orthogonal decomposition of $\mathcal{H}_{ln}$

$$\mathcal{H}_{ln} = \mathcal{H}_{ln}^+ \oplus \mathcal{H}_{ln}^- \quad , \quad \mathcal{H}_{ln}^\pm = \left\{ \Psi = {}^t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln} \; ; \quad f_2 = \mp f_1 \; , \; f_4 = \pm f_3 \right\}. \tag{113}$$

Each $\mathcal{H}_{ln}^\pm$ is stable under H and by corollary 6.1, for $(l,n) \in \mathcal{I}_{\frac{1}{2}}$, $k \in \mathbb{N}$, the sets

$$\mathcal{E}_{ln}^{k\pm} = \mathcal{E}_{ln}^k \cap \mathcal{H}_{ln}^\pm = \left\{ H^k \Psi ; \; \Psi = {}^t(f_1, \mp f_1, f_3, \pm f_3) \otimes F_{ln} \in \mathcal{H}_{ln}^\pm ; \quad f_1, f_3 \in \mathcal{C}_0^\infty(\mathbb{R}_{r_*}) \right\} \tag{114}$$

are respectively dense in $\mathcal{H}_{ln}^+$ and $\mathcal{H}_{ln}^-$. For $\Psi_0 \in \mathcal{E}_{ln}^{2\pm}$ we establish the existence of the strong limits (78) and (79) defining $\tilde{W}_0^\pm \Psi_0$ and $\tilde{W}_\infty^\pm \Psi_0$. The following lemma guarantees the existence of asymptotic profiles for Ψ_0. The details of its proof will be given after the proof of theorem 6.1.

Lemma 6.1. *Given $\Psi_0 \in \mathcal{E}_{ln}^{2\pm}$, $(l,n) \in \mathcal{I}_{\frac{1}{2}}$, there exists*

$$\Psi_1 \in \left[\mathcal{C}\left(\mathbb{R}_t; H^1(\mathbb{R}_{r_*})\right) \cap \mathcal{C}^1\left(\mathbb{R}_t; L^2(\mathbb{R}_{r_*})\right) \right]^4 \otimes F_{ln} \tag{115}$$

such that

$$\partial_t \Psi_1 = iH_0 \Psi_1, \tag{116}$$

and

$$\underset{t \to +\infty}{s - lim} \; \left\| e^{iHt}\Psi_0 - \Psi_1(t) \right\|_{\mathcal{H}} = 0. \tag{117}$$

Any solution of (116) in $\mathcal{C}(\mathbb{R}_t; \mathcal{H})$ and in particuliar Ψ_1 can be expressed in the form

$$\Psi_1(t) = e^{iH_0 t}\Psi_0^+ + e^{iH_0 t}\Psi_0^- \tag{118}$$

where

$$\Psi_0^+ \in \mathcal{H}_0^+ \quad , \quad \Psi_0^- \in \mathcal{H}_0^-. \tag{119}$$

Thus, for a cut-off function χ_o satisfying (44), we have

$$\underset{t \to +\infty}{lim} \; \left\| \mathcal{J}_0 \Psi_1(t) - e^{iH_0 t}\Psi_0^+ \right\|_{\mathcal{H}} = 0. \tag{120}$$

That is to say that for $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l,n) \in \mathcal{I}_{\frac{1}{2}}$, $\varepsilon = +, -$, there exists

$$\Psi_0^+ \in \mathcal{H}_0^+ \cap \mathcal{H}_{ln}^\varepsilon \tag{121}$$

such that

$$\underset{t \to +\infty}{lim} \; \left\| \mathcal{J}_0 e^{iHt}\Psi_0 - e^{iH_0 t}\Psi_0^+ \right\|_{\mathcal{H}} = 0. \tag{122}$$

and of course, we can similarly prove the existence of

$$\Psi_0^- \in \mathcal{H}_0^- \cap \mathcal{H}_{ln}^{\varepsilon} \tag{123}$$

such that

$$\lim_{t \to -\infty} \left\| \mathcal{J}_0 e^{iHt} \Psi_0 - e^{iH_0 t} \Psi_0^- \right\|_{\mathcal{H}} = 0. \tag{124}$$

From (121) to (124), we conclude that $\tilde{W}_0^{\pm} \Psi_0$ is well-defined for $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, $\varepsilon = +, -$, and

$$\tilde{W}_0^{\pm} \Psi_0 \in \mathcal{H}_0^{\pm} \quad , \quad \left\| \tilde{W}_0^{\pm} \Psi_0 \right\|_{\mathcal{H}_0} \leq \|\Psi_0\|_{\mathcal{H}}. \tag{125}$$

Then, corollary 6.1 yields that the operator $\tilde{W}_0^+$ (resp. $\tilde{W}_0^-$) is well-defined from $\mathcal{H}$ to $\mathcal{H}_0^+$ (resp. $\mathcal{H}_0^-$) and its norm is lower or equal to 1.

In order to prove the existence of $\tilde{W}_\infty^+$, we need to compare in the neighbourhood of the future infinity the outgoing part of $\Psi_1(t)$ with a solution of

$$(\partial_t - iH_\infty) \Psi = 0. \tag{126}$$

Lemma 6.2. *The operator* W_0^∞

$$W_0^\infty \Psi_0 = \underset{t \to +\infty}{s - lim} \; e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iH_0 t} \Psi_0 \tag{127}$$

is well-defined from $\mathcal{H}_0^-$ *to* $\mathcal{H}_\infty$ *and is independent of the choice of* χ_∞ *satisfying (65). Of course* W_0^∞ *is defined as well from* $\mathcal{H}_0^+$ *to* $\mathcal{H}_\infty$ *and for* $\Psi_0 \in \mathcal{H}_0^+$

$$W_0^\infty \Psi_0 = 0.$$

Lemma 6.2, and (118), (119) yield the existence of

$$\Psi_\infty^+ \in \mathcal{H}_\infty \tag{128}$$

such that

$$\lim_{t \to +\infty} \left\| \mathcal{J}_\infty^* \Psi_1(t) - e^{iH_\infty t} \Psi_\infty^+ \right\|_{\mathcal{H}_\infty} = 0 \tag{129}$$

and therefore

$$\lim_{t \to +\infty} \left\| \mathcal{J}_\infty^* e^{iHt} \Psi_0 - e^{iH_\infty t} \Psi_\infty^+ \right\|_{\mathcal{H}_\infty} = 0. \tag{130}$$

which enables us to define $\tilde{W}_\infty^+$ on $\mathcal{E}_{ln}^{2\pm}$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and by density on $\mathcal{H}$. The same thing can be done for $\tilde{W}_\infty^-$. Let χ_∞ and χ'_∞ be two cut-off functions satisfying (65) and $\mathcal{J}_\infty$ and $\mathcal{J}'_\infty$ the associated identifying operators. For $t \in \mathbb{R}$, $\Psi_0 \in \mathcal{H}$

$$\left\| e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iHt} \Psi_0 - e^{-iH_\infty t} \mathcal{J}'^*_\infty e^{iHt} \Psi_0 \right\|_{\mathcal{H}_\infty} \leq \left\| (\chi_\infty - \chi'_\infty) e^{iHt} \Psi_0 \right\|_{\mathcal{H}},$$

and

$$\lim_{t \to \pm\infty} \left\| e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iHt} \Psi_0 - e^{-iH_\infty t} \mathcal{J}'^*_\infty e^{iHt} \Psi_0 \right\|_{\mathcal{H}_\infty} = 0.$$

Thus, the operators $\tilde{W}_\infty^{\pm}$ are independent of the choice of χ_∞ and by a similar argument, $\tilde{W}_0^{\pm}$ are independent of the choice of χ_o.

We still have to prove that $W^{\pm}$ and $\tilde{W}^{\pm}$ are bijective isometries, which yields that S is a bijective isometry by construction. Let $\Psi \in \mathcal{H}$ and

$$\Psi_0^{\pm} = \tilde{W}_0^{\pm} \Psi \quad , \quad \Psi_\infty^{\pm} = \tilde{W}_\infty^{\pm} \Psi. \tag{131}$$

For χ_o satisfying (44) and χ_∞ satisfying (65), we have

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_0 \left(e^{iHt}\Psi - e^{iH_0t}\Psi_0^\pm \right) \right\|_{\mathcal{H}} = 0, \tag{132}$$

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_\infty \mathcal{J}_\infty^* e^{iHt}\Psi - \mathcal{J}_\infty e^{iH_\infty t}\Psi_\infty^\pm \right\|_{\mathcal{H}} = 0, \tag{133}$$

$\mathcal{J}_\infty \mathcal{J}_\infty^*$ being simply the multiplication by χ_∞. The local energy of $e^{iHt}\Psi$ goes to 0 when t goes to $\pm\infty$, therefore

$$\lim_{t\to\pm\infty} \left\| \left(\chi_o + \chi_\infty - 1\right) e^{iHt}\Psi \right\|_{\mathcal{H}} = 0. \tag{134}$$

(132), (133) and (134) imply

$$\lim_{t\to\pm\infty} \left\| e^{iHt}\Psi - \mathcal{J}_0 e^{iH_0t}\Psi_0^\pm - \mathcal{J}_\infty e^{iH_\infty t}\Psi_\infty^\pm \right\|_{\mathcal{H}} = 0, \tag{135}$$

which means

$$W^\pm \tilde{W}^\pm = \mathbb{I}_{\mathcal{H}}. \tag{136}$$

If on the other hand we consider

$$\Psi_0^\pm \in \mathcal{H}_0^\pm \quad , \quad \Psi_\infty^\pm \in \mathcal{H}_\infty \tag{137}$$

and put

$$\Psi = W^\pm \left(\Psi_0^\pm, \Psi_\infty^\pm \right), \tag{138}$$

we have (135) from which we get

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_0^* \left(e^{iHt}\Psi - \mathcal{J}_0 e^{iH_0t}\Psi_0^\pm - \mathcal{J}_\infty e^{iH_\infty t}\Psi_\infty^\pm \right) \right\|_{\mathcal{H}} = 0 \tag{139}$$

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_\infty^* \left(e^{iHt}\Psi - \mathcal{J}_0 e^{iH_0t}\Psi_0^\pm - \mathcal{J}_\infty e^{iH_\infty t}\Psi_\infty^\pm \right) \right\|_{\mathcal{H}_\infty} = 0. \tag{140}$$

The local energy of $e^{iH_0t}\Psi_0^\pm$ and $e^{iH_\infty t}\Psi_\infty^\pm$ goes to 0 when $|t|$ goes to $+\infty$, therefore (139) and (140) yield

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_0^* e^{iHt}\Psi - e^{iH_0t}\Psi_0^\pm \right\|_{\mathcal{H}} = 0 \tag{141}$$

and

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_\infty^* e^{iHt}\Psi - e^{iH_\infty t}\Psi_\infty^\pm \right\|_{\mathcal{H}_\infty} = 0, \tag{142}$$

thus

$$\tilde{W}^\pm W^\pm = \mathbb{I}_{\mathcal{H}_0^\pm \times \mathcal{H}_\infty}. \tag{143}$$

(136) and (143) show that $W^\pm$ and $\tilde{W}^\pm$ are all bijections and if we choose χ_o and χ_∞ such that their supports have no intersection, we deduce from (135)

$$\|\Psi\|_{\mathcal{H}} = \left\|\Psi_0^\pm\right\|_{\mathcal{H}} + \left\|\Psi_\infty^\pm\right\|_{\mathcal{H}_\infty}. \tag{144}$$

Q.E.D.

Proof of lemma 6.1: Let $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l,n) \in \mathcal{I}_{\frac{1}{2}}$, $\varepsilon = +,-$. There exists

$$\Psi_0' = {}^t\left(f_1, -\varepsilon f_1, f_3, \varepsilon f_3\right) \otimes F_{ln} \in \mathcal{E}_{ln}^{1\varepsilon} \tag{145}$$

such that

$$\Psi_0 = iH\Psi_0' \tag{146}$$

and

$$\Psi_0'' = {}^t\left(g_1, -\varepsilon g_1, g_3, \varepsilon g_3\right) \otimes F_{ln} \in \mathcal{E}_{ln}^{0\varepsilon} \tag{147}$$

such that

$$\Psi_0' = -iH\Psi_0''. \tag{148}$$

We denote

$$\tilde{\Psi} = e^{iHt}\Psi_0' \ ; \quad \tilde{\Psi} = \tilde{\phi} \otimes F_{ln} = {}^t\left(\phi_1, -\varepsilon\phi_1, \phi_3, \varepsilon\phi_3\right) \otimes F_{ln} \tag{149}$$

and

$$\Psi = \partial_t \tilde{\Psi} = iH\tilde{\Psi}. \tag{150}$$

On the one hand, applying $\partial_t + iH$ to equation

$$\left(\partial_t - iH\right)\tilde{\Psi} = 0,$$

we obtain

$$\left(\partial_t^2 - H^2\right)\tilde{\Psi} = 0$$

which, taking into account the fact that $\tilde{\Psi}$ takes its values in $\mathcal{H}_{ln}$ can also be written

$$\left(\partial_t^2 - \partial_{r_*}^2 + \beta_l^2 + \varepsilon\beta_l'\right)\phi_1 = 0, \tag{151}$$

$$\left(\partial_t^2 - \partial_{r_*}^2 + \beta_l^2 - \varepsilon\beta_l'\right)\phi_3 = 0. \tag{152}$$

On the other hand

$$\phi_1 \mid_{t=0} = f_1 \quad ; \quad \phi_3 \mid_{t=0} = f_3 \quad ; \quad f_1, f_3 \in \mathcal{C}_0^\infty(\mathbb{R}_{r_*}) \tag{153}$$

and since $\Psi_0 = H^2\Psi_0''$

$$\partial_t\phi_1 \mid_{t=0} = \left(-\partial_{r_*}^2 + \beta_l^2 + \varepsilon\beta_l'\right)g_1 \quad , \quad g_1 \in \mathcal{C}_0^\infty(\mathbb{R}_{r_*}) \tag{154}$$

$$\partial_t\phi_3 \mid_{t=0} = \left(-\partial_{r_*}^2 + \beta_l^2 - \varepsilon\beta_l'\right)g_3 \quad , \quad g_3 \in \mathcal{C}_0^\infty(\mathbb{R}_{r_*}). \tag{155}$$

The scattering results obtained in [3] together with proposition 6.2 imply that for any solution

$$f \in \mathcal{C}\left(\mathbb{R}_t; H^1\left(\mathbb{R}_{r_*}\right)\right) \cap \mathcal{C}^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{r_*}\right)\right)$$

of equation

$$\left(\partial_t^2 - \partial_{r_*}^2 + \beta_l^2 + \eta\beta_l'\right)f = 0 \quad , \quad \eta = +, -$$

with initial data

$$f \mid_{t=0} = \mu_1 \quad , \quad \partial_t f \mid_{t=0} = \left(-\partial_{r_*}^2 + \beta_l^2 + \eta\beta_l'\right)\mu_2$$

such that

$$i = 1, 2 \quad \mu_i \in L^2(\mathbb{R}_{r_*}) \ ; \quad \left(-\partial_{r_*}^2 + \beta_l^2 + \eta\beta_l'\right)\mu_i \in L^2(\mathbb{R}_{r_*}),$$

there exists a solution

$$f_1 \in \mathcal{C}\left(\mathbb{R}_t; H^1\left(\mathbb{R}_{r_*}\right)\right) \cap \mathcal{C}^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{r_*}\right)\right) \tag{156}$$

of

$$\left(\partial_t^2 - \partial_{r_*}^2\right)f_1 = 0 \tag{157}$$

such that

$$\begin{array}{c} lim \\ t \to +\infty \end{array} \quad \|f(t) - f_1(t)\|_{H^1(\mathbb{R}_{r_*})} + \|\partial_t f(t) - \partial_t f_1(t)\|_{L^2(\mathbb{R}_{r_*})}.$$

$\tilde{\Psi}$ is the solution of (15) with initial data

$$\Psi_0' \in \left[\mathcal{C}_0^\infty\left(\mathbb{R}_{r_*}\right)\right]^4 \otimes F_{ln}$$

therefore in particular,

$$\phi_1, \phi_2 \in \mathcal{C}\left(\mathbb{R}_t; H^1\left(\mathbb{R}_{r_*}\right)\right) \cap \mathcal{C}^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{r_*}\right)\right)$$

and (151) to (155) yield the existence of

$$\tilde{\Psi}_1 \in \left[\mathcal{C}\left(\mathbb{R}_t; H^1(\mathbb{R}_{r_*})\right) \cap \mathcal{C}^1\left(\mathbb{R}_t; L^2(\mathbb{R}_{r_*})\right)\right]^4 \otimes F_{ln}$$

such that

$$\left(\partial_t^2 - \partial_{r_*}^2\right)\tilde{\Psi}_1 = 0$$

and

$$\lim_{t\to+\infty} \left\| e^{iHt}\tilde{\Psi}_0 - \tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0 \quad , \quad \lim_{t\to+\infty} \left\| \partial_{r_*}\left(e^{iHt}\tilde{\Psi}_0 - \tilde{\Psi}_1 \right) \right\|_{\mathcal{H}} = 0$$

$$\lim_{t\to+\infty} \left\| \partial_t\left(e^{iHt}\tilde{\Psi}_0 - \tilde{\Psi}_1 \right) \right\|_{\mathcal{H}} = 0$$

from which we deduce

$$\lim_{t\to+\infty} \left\| e^{iHt}\Psi_0 - \partial_t\tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0. \tag{158}$$

Ψ_0 being an element of $\mathcal{E}_{ln}^{2\varepsilon} \subset \mathcal{E}_{ln}^{1\varepsilon}$, we can apply the previous construction to Ψ_0. We find that there exists

$$\Psi_1 \in \left[\mathcal{C}\left(\mathbb{R}_t; H^1(\mathbb{R}_{r_*})\right) \cap \mathcal{C}^1\left(\mathbb{R}_t; L^2(\mathbb{R}_{r_*})\right)\right]^4 \otimes F_{ln}$$

solution of

$$\left(\partial_t^2 - \partial_{r_*}^2\right)\Psi_1 = 0$$

such that

$$\lim_{t\to+\infty} \left\| e^{iHt}\Psi_0 - \Psi_1 \right\|_{\mathcal{H}} = 0 \quad , \quad \lim_{t\to+\infty} \left\| \partial_{r_*}\left(e^{iHt}\Psi_0 - \Psi_1 \right) \right\|_{\mathcal{H}} = 0, \tag{159}$$

$$\lim_{t\to+\infty} \left\| \partial_t\left(e^{iHt}\Psi_0 - \Psi_1 \right) \right\|_{\mathcal{H}} = 0. \tag{160}$$

From (159) and (160) we deduce

$$\lim_{t\to+\infty} \left\| (\partial_t - iH_0)\left(e^{iHt}\Psi_0 - \Psi_1 \right) \right\|_{\mathcal{H}} = 0. \tag{161}$$

$e^{iHt}\Psi_0$ being a solution of (15) in $\mathcal{C}\left(\mathbb{R}_t; \mathcal{H}_{ln}\right)$, we have

$$(\partial_t - iH)\, e^{iHt}\Psi_0 = (\partial_t - iH_0 - i\beta_l M)\, e^{iHt}\Psi_0 = 0 \tag{162}$$

and by (158)

$$\lim_{t\to+\infty} \left\| i\beta_l M\left(e^{iHt}\Psi_0 - \partial_t\tilde{\Psi}_1 \right) \right\|_{\mathcal{H}} = 0.$$

$\partial_t\tilde{\Psi}_1$ is identically zero in

$$\left\{(t, r_*, \omega);\ |r_*| \leq |t| - R\ ,\ \ \omega \in S^2\right\},$$

which is not true in general for $\tilde{\Psi}_1$, therefore

$$\lim_{t\to+\infty} \left\| i\beta_l M \partial_t\tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0$$

and

$$\lim_{t\to+\infty} \left\| i\beta_l M e^{iHt}\Psi_0 \right\|_{\mathcal{H}} = 0. \tag{163}$$

(161), (162) and (163) give

$$\lim_{t \to +\infty} \|(\partial_t - iH_0)\,\Psi_1\|_{\mathcal{H}} = 0$$

and $(\partial_t - iH_0)\,\Psi_1$ being an element of $\mathcal{C}\left(\mathbb{R}_t;\mathcal{H}\right)$ and satisfying

$$(\partial_t + iH_0)\left[(\partial_t - iH_0)\,\Psi_1\right] = 0$$

we must have

$$(\partial_t - iH_0)\,\Psi_1 = 0.$$

Q.E.D.

Proof of lemma 6.2: Let

$$\Psi_0 \in \mathcal{H}_0^- \cap \mathcal{E}_{ln}^{0\varepsilon} \quad , \quad (l,n) \in \mathcal{I}_{\frac{1}{2}} \quad , \quad \varepsilon = +,- \tag{164}$$

with

$$Supp(\Psi_0) \subset [-R,R]_{r_*} \times S^2_{\theta,\varphi} \quad , \quad R > 0. \tag{165}$$

Ψ_0 can be written

$$\Psi_0 = \ {}^t\left(f_0, -\varepsilon f_0, f_0, \varepsilon f_0\right) \otimes F_{ln} \quad , \quad f_0 \in \mathcal{C}_0^\infty(\mathbb{R}_{r_*}) \quad Supp f_0 \subset [-R,R] \tag{166}$$

and

$$e^{iH_0t}\Psi_0 = \ {}^t\left(f, -\varepsilon f, f, \varepsilon f\right) \otimes F_{ln} \ , \quad f(t,r_*) = f_0(r_* - t). \tag{167}$$

f is the solution of

$$\left(\partial_t^2 - \partial_{r_*}^2\right) f = 0 \tag{168}$$

associated with the initial data

$$f\,|_{t=0} = f_0 \quad , \quad \partial_t f\,|_{t=0} = -\partial_{r_*} f_0. \tag{169}$$

Instead of applying Cook's method to operators H_∞ and H_0, which would give an apparently long-range perturbation at infinity, we work on the second order scalar equations and establish the existence of g_η solution of

$$\left\{ \begin{array}{l} \left(\partial_t^2 - \partial_{r_*}^2 + V_\eta(r_*)\right) g_\eta = 0 \\ V_\eta(r_*) = \chi_\infty(r_*)\frac{1}{r_*^2}\left(\left(l+\frac{1}{2}\right)^2 + \eta\left(l+\frac{1}{2}\right)\right) \quad , \quad \eta = +,-, \end{array} \right. \tag{170}$$

where χ_∞ is a cut-off function satisfying (65); the solution g_η being such that

$$\lim_{t \to +\infty} \|\partial_t\left(g_\eta - f\right)\|_{L^2(\mathbf{R})} = 0 \quad , \quad \lim_{t \to +\infty} \|\partial_{r_*}\left(g_\eta - f\right)\|_{L^2(\mathbf{R})} = 0, \tag{171}$$

$$\lim_{t \to +\infty} \left\| \frac{l+\frac{1}{2}}{r}\left(g_\eta - f\right)\right\|_{L^2(\mathbf{R})} = 0. \tag{172}$$

In the case where $l = 1/2$ and $\eta = -$, equations (168) and (170) are the same and it suffices to take $g_- = f$. Let us now assume

$$\left(l + \frac{1}{2}\right)^2 + \eta\left(l + \frac{1}{2}\right) > 0. \tag{173}$$

We write equations (168) and (170) in their hamiltonian form

$$\partial_t \begin{pmatrix} f \\ \partial_t f \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ -\partial_{r_*}^2 & 0 \end{pmatrix} \begin{pmatrix} f \\ \partial_t f \end{pmatrix} = -A_0 \begin{pmatrix} f \\ \partial_t f \end{pmatrix}, \tag{174}$$

$$\partial_t \begin{pmatrix} g \\ \partial_t g \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ -\partial_{r_*}^2 + V_\eta & 0 \end{pmatrix} \begin{pmatrix} g \\ \partial_t g \end{pmatrix} = -A_\eta \begin{pmatrix} g \\ \partial_t g \end{pmatrix}. \tag{175}$$

The operator iA_0 is skew-adjoint with dense domain on

$$\mathbb{H}_0 = BL^1(\mathbb{R}_{r_*}) \times L^2(\mathbb{R}_{r_*}) \tag{176}$$

completion of $[\mathcal{C}_0^\infty(\mathbb{R}_{r_*})]^2$ for the norm

$$\| \,^t(f_1, f_2)\|^2_{\mathbb{H}_0} = \int_{\mathbb{R}} \left\{ |\partial_{r_*} f_1|^2 + |f_2|^2 \right\} dr_* \tag{177}$$

and iA_η is skew-adjoint with dense domain (cf. [3]) on

$$\mathbb{H} = \mathbb{H}_1 \times L^2(\mathbb{R}_{r_*}) \tag{178}$$

completion of $[\mathcal{C}_0^\infty(\mathbb{R}_{r_*})]^2$ for the norm

$$\| \,^t(g_1, g_2)\|^2_{\mathbb{H}} = \int_{\mathbb{R}} \left\{ |\partial_{r_*} g_1|^2 + |g_2|^2 + V_\eta |g_1|^2 \right\} dr_*. \tag{179}$$

Under assumption (173), the norm (179) is equivalent to

$$|||\,^t(g_1, g_2)|||^2 = \| \,^t(g_1, g_2)\|^2_{\mathbb{H}_0} + \left\| \frac{\left(l + \frac{1}{2}\right)\chi_\infty}{r_*} g_1 \right\|^2_{L^2(\mathbb{R}_{r_*})}. \tag{180}$$

Moreover, any solution $\,^t(g, \partial_t g) \in \mathcal{C}(\mathbb{R}_t; \mathbb{H})$ of (170) satisfies the following energy estimate: for $r_*^1 < r_*^2$ and $t \in \mathbb{R}$

$$\int_{r_*^1 < r_* < r_*^2} \left\{ |\partial_{r_*} g(t)|^2 + |\partial_t g(t)|^2 + V_\eta(r_*)|g(t)|^2 \right\} dr_* \tag{181}$$

$$\leq \int_{r_*^1 - |t| < r_* < r_*^2 + |t|} \left\{ |\partial_{r_*} g(0)|^2 + |\partial_t g(0)|^2 + V_\eta(r_*)|g(0)|^2 \right\} dr_*$$

which is very easily obtained by multiplying (170) by $\partial_t g$ and integrating by parts on the domain

$$\Omega_{t, r_*^1, r_*^2} = \left\{ (\tau, r_*);\ \tau \in (0, t),\ r_*^1 - |t - \tau| < r_* < r_*^2 + |t - \tau| \right\}. \tag{182}$$

f_0 being in $\mathcal{C}_0^\infty(\mathbb{R}_{r_*})$, we can consider that

$$e^{-A_0 t} \left[\,^t(f_0, -\partial_{r_*} f_0) \right] \in \mathcal{C}(\mathbb{R}_t; \mathbb{H})$$

and we apply Cook's method to prove the existence in $\mathbb{H}$ of the limit

$$\begin{pmatrix} g_{0\eta} \\ g_{1\eta} \end{pmatrix} = \underset{t \to +\infty}{s - lim}\ e^{A_\eta t} e^{-A_0 t} \begin{pmatrix} f_0 \\ -\partial_{r_*} f_0 \end{pmatrix}. \tag{183}$$

We shall denote

$$\phi_0 = \,^t(f_0, -\partial_{r_*} f_0) \quad , \quad \phi_\infty = \,^t(g_{0\eta}, g_{1\eta}). \tag{184}$$

We have

$$\left\| \partial_t \left(e^{A_\eta t} e^{-A_0 t} \phi_0 \right) \right\|_{\mathbb{H}} = \left\| (A_\eta - A_0)\, e^{-A_0 t} \phi_0 \right\|_{\mathbb{H}} = \|V_\eta(r_*) f_0(r_* - t)\|_{L^2(\mathbb{R}_{r_*})} \leq \|f_0\|_{L^2(\mathbb{R}_{r_*})} \|V_\eta\|_{L^\infty(r_* > t - R)}$$

and for r_* large enough

$$V_\eta(r_*) = C r_*^{-2} \quad , \quad C > 0, \tag{185}$$

thus

$$\left\| \partial_t \left(e^{A_\eta t} e^{-A_0 t} \phi_0 \right) \right\|_{\mathbb{H}} = O(t^{-2}) \quad ; \quad t \to +\infty,$$

and

$$\left\| \partial_t \left(e^{A_\eta t} e^{-A_0 t} \phi_0 \right) \right\|_{\mathbb{H}} \in L^1(t > 0).$$

The limit (183)is therefore well-defined and if g_η is the solution of (170) such that

$$\begin{pmatrix} g_\eta(t) \\ \partial_t g_\eta(t) \end{pmatrix} = e^{-A_\eta t} \phi_\infty, \tag{186}$$

then

$$\lim_{t \to +\infty} \left\| {}^t(g_\eta, \partial_t g_\eta) - {}^t(f, \partial_t f) \right\|_{\mathbf{H}} = 0. \tag{187}$$

This last limit together with the equivalence of norms (179) and (180) gives (171) and (172). Moreover, for $r_* < t - R$

$$g_\eta(t, r_*) = 0 \quad and \quad \partial_t g_\eta(t, r_*) = 0. \tag{188}$$

Indeed, for $t \in \mathbb{R}$, $\varepsilon > 0$ we choose $\tau \in \mathbb{R}$ such that

$$\left\| \phi_\infty - e^{iA_\eta \tau} e^{-iA_0 \tau} \phi_0 \right\|_{\mathbf{H}} \leq \varepsilon \quad , \quad \tau \geq t. \tag{189}$$

For ${}^t(f_1, f_2) \in \mathbb{H}$, we denote

$$\mathcal{L}\left({}^t(f_1, f_2) \right) = |\partial_{r_*} f_1|^2 + V_\eta |f_1|^2 + |f_2|^2. \tag{190}$$

Let us consider

$$\int_{r_* < t-R} \mathcal{L}\left(e^{-iA_\eta t} \phi_\infty \right) dr_* \leq \int_{r_* < t-R} \mathcal{L}\left[e^{-iA_\eta t} \left(\phi_\infty - e^{iA_\eta \tau} e^{-iA_0 \tau} \phi_0 \right) \right] dr_*$$

$$+ \int_{r_* < t-R} \mathcal{L}\left(e^{-iA_\eta (t-\tau)} e^{-iA_0 \tau} \phi_0 \right) dr_*.$$

(181) and (189) yield

$$\int_{r_* < t-R} \mathcal{L}\left(e^{-iA_\eta t} \phi_\infty \right) dr_* \leq \varepsilon^2 + \int_{r_* < \tau - R} \mathcal{L}\left(e^{-iA_0 \tau} \phi_0 \right) dr_*$$

and this last integral is zero since

$$Supp\left(e^{-iA_0 \tau} \phi_0 \right) \subset [\tau - R, \tau + R].$$

(188) is therefore satisfied and for t large enough g_η is a solution of

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left(\left(l + \frac{1}{2} \right)^2 + \eta \left(l + \frac{1}{2} \right) \right) \right] g_\eta = 0. \tag{191}$$

Let us now introduce

$$\tilde{\Psi}_\infty(t) = {}^t\left(g_{-\varepsilon}(t), -\varepsilon g_{-\varepsilon}(t), g_\varepsilon(t), \varepsilon g_\varepsilon(t) \right) \otimes F_{ln}. \tag{192}$$

There exists $t_0 > 0$ such that, for $t \geq t_0$, g_ε and $g_{-\varepsilon}$ satisfy

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left(\left(l + \frac{1}{2} \right)^2 + \varepsilon \left(l + \frac{1}{2} \right) \right) \right] g_\varepsilon = 0, \tag{193}$$

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left(\left(l + \frac{1}{2} \right)^2 - \varepsilon \left(l + \frac{1}{2} \right) \right) \right] g_{-\varepsilon} = 0 \tag{194}$$

with

$$g_\varepsilon, g_{-\varepsilon} \in \mathcal{C}\left([t_0, +\infty[; \mathbb{H}_1 \right) \quad , \quad \partial_t g_\varepsilon, \partial_t g_{-\varepsilon} \in \mathcal{C}\left([t_0, +\infty[; L^2(\mathbb{R}_{r_*}) \right). \tag{195}$$

Moreover, for $t \geq t_0$

$$Supp\left(g_\varepsilon(t), g_{-\varepsilon}(t), \partial_t g_\varepsilon(t), \partial_t g_{-\varepsilon}(t) \right) \quad \subset \quad [t - R, +\infty[\ \subset \ [0, +\infty[. \tag{196}$$

Thus, the quantities

$$\partial_t \tilde{\Psi}_\infty, \ \partial_{r_*} \tilde{\Psi}_\infty, \ \left(l + \frac{1}{2} \right) r_*^{-1} \tilde{\Psi}_\infty$$

belong to $\mathcal{C}\left([t_0,+\infty[;\mathcal{H}\right)$ and (171), (172) yield

$$\begin{array}{c} lim \\ t\rightarrow+\infty \end{array} \left\|\partial_t\left(\tilde{\Psi}_\infty(t)-e^{iH_0t}\Psi_0\right)\right\|_{\mathcal{H}}=0 \qquad \begin{array}{c} lim \\ t\rightarrow+\infty \end{array} \left\|\partial_{r_*}\left(\tilde{\Psi}_\infty(t)-e^{iH_0t}\Psi_0\right)\right\|_{\mathcal{H}}=0, \tag{197}$$

$$\begin{array}{c} lim \\ t\rightarrow+\infty \end{array} \left\|\left(l+\frac{1}{2}\right)r_*^{-1}\left(\tilde{\Psi}_\infty(t)-e^{iH_0t}\Psi_0\right)\right\|_{\mathcal{H}}=0. \tag{198}$$

In particular, we have

$$\begin{array}{c} lim \\ t\rightarrow+\infty \end{array} \left\|\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\left(\tilde{\Psi}_\infty(t)-e^{iH_0t}\Psi_0\right)\right\|_{\mathcal{H}}=0. \tag{199}$$

Since $e^{iH_0t}\Psi_0$ is a solution of

$$\left(\partial_t+L\partial_{r_*}\right)e^{iH_0t}\Psi_0=0,$$

we have

$$\left\|\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)e^{iH_0t}\Psi_0\right\|_{\mathcal{H}}=\left(l+\frac{1}{2}\right)\left\|r_*^{-1}e^{iH_0t}\Psi_0\right\|_{\mathcal{H}}=O(t^{-1}) \quad t\rightarrow+\infty$$

and therefore

$$\begin{array}{c} lim \\ t\rightarrow+\infty \end{array} \left\|\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\tilde{\Psi}_\infty(t)\right\|_{\mathcal{H}}=0. \tag{200}$$

We introduce

$$\Psi_\infty=\tilde{\Psi}_\infty\mid_{\{r_*\geq0\}}. \tag{201}$$

The quantities

$$\partial_t\Psi_\infty \quad , \quad \partial_{r_*}\Psi_\infty \quad , \quad \left(l+\frac{1}{2}\right)r_*^{-1}\Psi_\infty$$

belong to $\mathcal{C}\left([t_0,+\infty[;\mathcal{H}_\infty^{\varepsilon ln}\right)$ where, for $(l,n)\in\mathcal{I}_{\frac{1}{2}}$ and $\varepsilon=+,-$

$$\mathcal{H}_\infty^{\varepsilon ln}=\left\{\ {}^t(f,-\varepsilon f,g,\varepsilon g)\otimes F_{ln}\in\mathcal{H}_\infty\right\}. \tag{202}$$

From (200), we get

$$\begin{array}{c} lim \\ t\rightarrow+\infty \end{array} \left\|\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\Psi_\infty(t)\right\|_{\mathcal{H}_\infty}=0 \tag{203}$$

and, the function

$$\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\Psi_\infty\in\mathcal{C}\left([t_0,+\infty[;\mathcal{H}_\infty^{\varepsilon ln}\right)$$

satisfies

$$\left(\partial_t-L\partial_{r_*}+i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\left[\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\Psi_\infty\right]=0. \tag{204}$$

Therefore, we must have for $t\geq t_0$

$$\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\Psi_\infty(t)=0 \quad in\ \mathcal{H}_\infty.$$

$\mathbb{H}_1$ being a distribution space, we can write in the sense of distributions for $t\geq t_0$

$$\partial_t\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\Psi_\infty(t)=\left(\partial_t+L\partial_{r_*}-i\left(l+\frac{1}{2}\right)r_*^{-1}M\right)\partial_t\Psi_\infty(t)=0 \quad in\ \mathcal{H}_\infty,$$

which implies that $\partial_t \Psi_\infty$ is a solution in $\mathcal{C}\left([t_0, +\infty[; \mathcal{H}_\infty^{\varepsilon l n}\right)$ of

$$(\partial_t - iH_\infty)\,\Psi = 0.$$

This solution can be extended to $\mathcal{C}\left(\mathbb{R}_t; \mathcal{H}_\infty^{\varepsilon l n}\right)$ and we denote

$$\Psi_\infty^0 = e^{-iH_\infty t_0} \partial_t \Psi_\infty(t_0) \tag{205}$$

its initial data at $t = 0$. From (196), (197), we get

$$\lim_{t \to +\infty} \left\| e^{iH_\infty t} \Psi_\infty^0 - \mathcal{J}_\infty^* \partial_t \left(e^{iH_0 t} \Psi_0\right) \right\|_{\mathcal{H}_\infty} = 0. \tag{206}$$

The value of $\partial_t \left(e^{iH_0 t}\Psi_0\right)$ at $t = 0$ is $iH_0\Psi_0$. H_0 is a self-adjoint operator with dense domain on $\mathcal{H}$, its point spectrum is empty and the spaces $\mathcal{H}_0^\pm$, $\mathcal{H}_{ln}^\pm$ are invariant under H_0. Therefore the direct sum of the sets

$$\left\{H_0\Psi_0;\ \ \Psi_0 \in \mathcal{H}_0^- \cap \mathcal{E}_{ln}^{0\varepsilon}\right\}\ ;\quad (l,n) \in \mathcal{I}_{\frac{1}{2}},\ \varepsilon = +,- \tag{207}$$

is dense in $\mathcal{H}_0^-$. (206) shows that for an initial data $H_0\Psi_0$ in a set of type (207), the limit

$$\Psi_\infty^0 = \underset{t \to +\infty}{s-\lim}\ e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iH_0 t} H_0 \Psi_0 \tag{208}$$

exists in $\mathcal{H}_\infty$. The operator W_0^∞ is consequently well-defined from $\mathcal{H}_0$ into $\mathcal{H}_\infty$. Since the local energy of the solution $e^{iH_0 t}H_0\Psi_0$ goes to zero when $|t|$ goes to $+\infty$, the limit Ψ_∞^0 is independent of the choice of χ_∞ satisfying (65).

Q.E.D.

7 Conclusion

The scattering theory developed in this paper is only valid for the linear massless Dirac system. In the case of a massive field and when space-time is asymptotically flat, the mass of the field induces long-range perturbations at infinity and classical wave operators will probably not exist. However, using the methods developed by J. Dollard and G. Velo [10] and by V. Enss and B. Thaller [11] about the relativistic Coulomb scattering of Dirac fields as well as the works of A. Bachelot [1] and J. Dimock and B. Kay [9] on the Klein-Gordon equation on the Schwarzschild metric, it must be possible to show the existence and asymptotic completeness of Dollard-modified wave operators at infinity.

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