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A nonlinear Klein-Gordon equation on Kerr metrics

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Abstract

We consider the non linear Klein-Gordon equation $\Box u + m^2 u + \lambda |u|^2 u = 0$, with $\lambda \geq 0$, outside a Kerr black hole. We solve the global Cauchy problem for large data with minimum regularity. Then, using a Penrose compactification, we prove, in the massless case, the existence of smooth asymptotic profiles and Sommerfeld radiation conditions, at the horizon and at null infinity, for smooth solutions.

Résumé

Nous considérons l'équation de Klein-Gordon non linéaire $\Box u + m^2 u + \lambda |u|^2 u = 0$, avec $\lambda \geq 0$, à l'extérieur d'un trou noir de Kerr. Nous résolvons le problème de Cauchy global pour des données initiales grandes et de régularité minimale. Ensuite, à l'aide d'une compactification de Penrose, nous établissons dans le cas sans masse l'existence de profils asymptotiques réguliers et de conditions de radiation de type Sommerfeld, à l'horizon du trou noir et à l'infini isotrope, pour les solutions régulières.

1 Introduction

Schwarzschild and Kerr metrics are the simplest known examples of non flat solutions of the Einstein vaccum equations to be "physically relevant" in that they contain energy.

The Schwarzschild solution is the simplest of the two ; it describes the space-time geometry of a universe containing nothing but a static spherically symmetric uncharged black hole. Linear fields outside spherical black holes have been studied intensively over the past ten years (see A. Bachelot [1, 2], A. Bachelot and A. Motet-Bachelot [6], J. Dimock [11], J. Dimock and B. Kay [12], W.M. Jin [17], F. Melnyk [19] and the author [22]-[24]). The mechanisms of time dependent scattering are now well understood in many cases and we even have a clear interpretation of the Hawking effect for a spherically symmetric gravitational collapse (see A. Bachelot [3]-[5]). Much less is known about non linear fields on the Schwarzschild geometry : the non linear Dirac equation, Yang-Mills fields and non linear Klein-Gordon fields were studied respectively by A. Motet-Bachelot [20], W.T. Shu [28] and the author [21]. No complete scattering theory has to this day been developed for such equations ; the required Strichartz estimates are still lacking on this curved background.

The geometry of Kerr's space-time is much more complicated than that of Schwarzschild's space-time of which it is a generalization, but it is also richer and physically more realistic. Slow Kerr metrics are a description of a space-time containing only an uncharged rotating black hole. Leaving aside the questions related to the interior of the black hole (the instability of the Cauchy horizon, the singularity and the time machine in block III), the analysis of field equations outside the black hole faces an essential difficulty : this part of space-time, in contrast with the Schwarzschild exterior, is not stationary ; the effects of rotation become extreme in a toroidal region called the ergosphere, surrounding the horizon, where it is impossible for any material body to remain at rest in the perception of an observer static at infinity. The only Killing vector field to be timelike at infinity is timelike everywhere outside the black hole except inside the ergosphere where it becomes spacelike. For fields of half-integral spin, which possess a conserved current inducing a positive-definite inner product on each spacelike slice, independently of the existence of timelike Killing vector fields, one would not expect this to represent a serious difficulty. For Klein-Gordon or Maxwell's equations however, the absence of globally defined timelike Killing vector field implies the non existence of positive-definite conserved quantities. This is what allows the phenomenon of super-radiance (an analogue at the level of fields of the Penrose process for particles) by which a scalar or electromagnetic field can extract energy from the ergosphere. Because of this difficulty and of the complexity of the geometry, analytic studies of field equations on Kerr metrics have been scarse since the publication of S. Chandrasekhar's work [8] : the existence of smooth solutions to the Dirac and Maxwell systems was proved by A. De Vries [9, 10] ; recently, the author solved the Cauchy problem for Dirac fields in Sobolev and weighted Sobolev spaces using a $3+1$ decomposition of the geometry [24] and some results on the timelike asymptotic behaviour of Dirac fields have been obtained by J. Finster, N. Kamran, J. Smoller and S.-T. Yau [13] ; the only time-dependent scattering construction known to this day on such backgrounds was obtained by D. Häfner for the non super-radiant modes of Klein-Gordon fields [15].

A complete understanding of super-radiance such as could be obtained through a time-dependent scattering theory for Klein-Gordon or Maxwell's equations seems yet remote. In the present contribution, we propose a first study of the non linear Klein-Gordon equation

$$\Box_g u + m^2 u + \lambda |u|^2 u = 0\,,\ \lambda \geq 0\,, \tag{1}$$

outside a slow Kerr black hole and observe that super-radiance is no obstacle to controlling the non linearity locally uniformly in time or asymptotically along null rays. More precisely, we obtain two types of results :

- The well-posedness of the global Cauchy problem for large weakly regular initial data. The proof is similar to that of F. Cagnac and Y. Choquet-Bruhat [7] who solved the global Cauchy problem for the above equation on globally hyperbolic space-times with uniformly timelike time coordinate curves. This last hypothesis is not satisfied near the horizon in black hole space-times. An earlier work by the author [21] extended the results of [7] to the space-time outside a spherical black hole and we now obtain analogous theorems for a Kerr black hole. The fundamental geometrical tool is the $3+1$ (or ADM) decomposition of the exterior ; this is sometimes referred to as the point of view of locally non rotating observers and was explained in details in [24]. We obtain a description of the exterior of the black hole as a space-time $(\mathcal{M}, g)$ with

$$\mathcal{M} = \mathbb{R}_t \times \Sigma\,,\ \Sigma \simeq \mathbb{R}^3 \setminus \bar{B}(0,1)\,,\ g = N^2 \mathrm{d}t^2 - h(t)\,,$$

the lapse function N and the spacelike metric h being smooth and satisfying

- $N \to 0$ at the horizon (the boundary of Σ) and $N \to 1$ at infinity ;
- $h(t)$ is equivalent to the euclician metric on the exterior of the unit ball in $\mathbb{R}^3$, the equivalence being uniform in space and locally uniform in time.

This allows us to formulate (1) as an evolution equation with a natural Hilbert space framework. The Hilbert norm is not a conserved quantity for the linear evolution but is easily controlled by an energy estimate ; this solves the linear Cauchy problem for finite energy solutions. The flat Sobolev embedding $H^1 \hookrightarrow L^6$ gives the required control on the non linear term and global existence then follows from an energy estimate.

- In the massless case, we study the asymptotic behaviour of smooth solutions using R. Penrose's technique of conformal compactification (see for example [26]). We obtain the existence of smooth asymptotic profiles at the horizon and at null infinity (denoted $\mathfrak{I}$). It is also shown that solutions satisfy radiation conditions of Sommerfeld type asymptotically : the conditions obtained at the horizon differ from those at infinity by a rotation imposed on the field, the rotation speed being exactly that of the horizon as observed from infinity.

The results obtained here extend those of [21] to Kerr geometry but also improve their precision : the asymptotic profiles at null infinity in [21] were constructed as functions of very low regularity ; the description of null infinity adopted here is based on a better choice of coordinates (inspired by R. Penrose's treatment of the Schwarzschild null infinity in [26] and [27] vol. II) and allows us to show that the profiles are $\mathcal{C}^\infty$ functions on $\mathfrak{I}$.

The paper is organized as follows. In section 2, we recall briefly the principles of the $3+1$ decomposition of the exterior of the black hole, then we use this decomposition to express equation (1) as an evolution equation with an elliptic spacelike part, thus ensuring the existence of a natural Hilbert space in which finite energy solutions take their values. Section 3 contains the global existence and uniqueness theorems for the linear and non linear evolutions ; both smooth and minimum regularity solutions are considered. Finally, in section 4, we explain the Penrose compactification of the exterior of a Kerr black hole and use it to describe, in terms of asymptotic profiles and radiation conditions, the behaviour at the horizon and at null infinity of smooth solutions to the massless equation.

Notations : some of our equations will be expressed using the abstract index formalism of R. Penrose and W. Rindler [27]. Abstract tensor indices are denoted by light face lower case latin letters ; they are a notational device for keeping track of the nature of objects in the course of calculations, they do not imply any reference to a coordinate basis, all expressions and calculations involving them are perfectly intrinsic. For example, on a space-time $(\mathcal{M}, g)$, g_{ab} will refer to the space-time metric as an intrinsic symmetric tensor field of valence $\left[\begin{array}{c}0\\2\end{array}\right]$, i.e. a section of $\mathrm{T}^*\mathcal{M} \odot \mathrm{T}^*\mathcal{M}$ and g^{ab} will refer to the inverse metric as an intrinsic symmetric tensor field of valence $\left[\begin{array}{c}2\\0\end{array}\right]$, i.e. a section of $\mathrm{T}\mathcal{M} \odot \mathrm{T}\mathcal{M}$ (where $\odot$ denotes the symmetric tensor product, $\mathrm{T}\mathcal{M}$ the tangent bundle to our space-time manifold $\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ its cotangent bundle). Concrete indices defining components in reference to a basis are represented by bold face lower case latin letters and take their values in $\{0, 1, 2, 3\}$. Lower case greek letters will be used here only for denoting spacelike concrete indices in the framework of the $3+1$ decomposition. They take their values in $\{1,2,3\}$. We adopt the Einstein convention for indices appearing twice (once up, once down) in the same term : for abstract indices, the index is considered as contracted, signifying the action of a 1-form on a vector ; for concrete indices, the sum is taken over the possible values of the index (from 0 to 3 for latin letters, from 1 to 3 for greek letters).

Given a smooth manifold $\mathcal{M}$, we denote by $\mathcal{C}_0^\infty(\mathcal{M})$ the space of smooth functions compactly supported on $\mathcal{M}$. For a measure μ on $\mathcal{M}$, the space L^p on $\mathcal{M}$ associated with the measure μ will be denoted $L^p(\mathcal{M}\,;\mathrm{d}\mu)$, $1 \leq p \leq +\infty$.

2 The $3+1$ decomposition of the geometry and the equation

A space-time containing nothing but a rotating uncharged black hole is described by the Kerr metric. In Boyer-Lindquist coordinates on $\mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, it takes the form

$$g = \left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4aMr\sin^2\theta}{\rho^2}\,\mathrm{d}t\mathrm{d}\varphi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 - \frac{\sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi^2, \tag{2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta\,,\ \Delta = r^2 - 2Mr + a^2\,,\ \sigma^2 = \left(r^2+a^2\right)\rho^2 + 2Mra^2\sin^2\theta\,,$$

where M is the mass of the black hole and a is its angular momentum per unit mass. The whole space-time has only one singularity : the set of points $\{\rho^2 = 0\}$ (that is to say the equatorial ring of the $\{r=0\}$ sphere : $\{r=0\,,\ \theta=\pi/2\}$), where the curvature blows up. The spheres where Δ vanishes, called horizons, are merely coordinate singularities. There are three types of Kerr space-times (only two of which contain a black hole) depending on the respective importance of M and a :

- Slow Kerr space-time for $0 < |a| < M$. Δ has two real roots

$$0 < r_- = M - \sqrt{M^2 - a^2} < M < r_+ = M + \sqrt{M^2 - a^2} < 2M, \tag{3}$$

  so the space-time has two horizons, the spheres $\{r = r_-\}$ and $\{r = r_+\}$, on either side of $\{r = M\}$. The case $a = 0$ reduces to Schwarzschild's space-time.

- Extreme Kerr space-time for $|a| = M$. M is then the double root of Δ and the sphere $\{r = M\}$ is the only horizon.

- Fast Kerr space-time for $|a| > M$. Δ has no real root and the space-time has no horizon. There is no black hole in this case ; the ring singularity is a naked singularity.

We only consider slow Kerr metrics ; they are usually considered as the generic description of a space-time containing simply a rotating uncharged black hole, since the extreme case is believed to be unstable. The two horizons separate space-time into three connected components called Boyer-Lindquist blocks: block I, denoted here $\mathcal{B}_I$, is the exterior of the black hole $\{r > r_+\}$; block II, $\{r_- < r < r_+\}$, is a dynamic region situated beyond the outer horizon and where the inertial frames are dragged towards the inner horizon ; block III, $\{r < r_-\}$, is the part of space-time located beyond the inner horizon, it contains the ring singularity and a time machine called Carter's time machine.

We study the propagation in block I, i.e. outside the black hole, of solutions to the non linear Klein-Gordon equation

$$\Box_g u + m^2 u + \lambda |u|^2 u = 0\,,\ \ \lambda \geq 0\,, m \geq 0\,. \tag{4}$$

We first give (in subsection 2.1) a brief description of block I and equation (4) using Boyer-Lindquist coordinates. Then, following [24], we perform (in subsection 2.2) a $3+1$ decomposition of the geometry and of the equation ; this gives us a natural Hilbert space framework for solving the global Cauchy problem for (4) in block I.

2.1 The ergosphere and super-radiance

The exterior of the black hole, like any Boyer-Lindquist block, is not stationary. The only (modulo multiplication by a constant) Killing vector field globally defined in block I and that is timelike for r large enough is $\partial/\partial t$. This vector field, however, is not timelike everywhere in block I. There is a toroidal region $\mathcal{E}$, called the ergosphere, surrounding the horizon, where $\partial/\partial t$ is spacelike ; $\mathcal{E}$ is defined by $g_{tt} < 0$ and $r > r_+$, i.e.

$$\mathcal{E} = \left\{ (t, r, \theta, \varphi)\ ;\ r_+ < r < M + \sqrt{M^2 - a^2 \cos^2\theta} \right\}.$$

Inside $\mathcal{E}$, the effects of rotation are extreme and the quantity $a\varphi$ is strictly increasing along any non spacelike future oriented curve.

Though $\partial/\partial t$ is not timelike everywhere in block I, the function t of the Boyer-Lindquist coordinates is indeed a time function globally defined outside the black hole : this means that the level hypersurfaces Σ_t of t are spacelike Cauchy hypersurfaces and $\nabla^a t$ is a timelike future oriented vector field in block I. Hence, it makes sense to study field equations outside the black hole as evolution equations on $\mathbb{R}_t \times \Sigma$, $\Sigma =]r_+, +\infty[\times S^2_\omega$, using t as time parameter.

The expression of the d'Alembertian $\Box_g$ in Boyer-Lindquist coordinates can be calculated using the formula

$$\Box_g = \frac{1}{|g|^{1/2}} \frac{\partial}{\partial x^{\mathbf{a}}} \left(|g|^{1/2} g^{\mathbf{ab}} \frac{\partial}{\partial x^{\mathbf{b}}} \right)$$

where $|g|$ denotes $|\det g|$. We obtain

$$\Box_g = \frac{\sigma^2}{\Delta\rho^2} \frac{\partial^2}{\partial t^2} + \frac{4aMr}{\Delta\rho^2} \frac{\partial^2}{\partial t \partial\varphi} - \frac{1}{\rho^2} \frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\rho^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} - \frac{\rho^2 - 2Mr}{\Delta\rho^2 \sin^2\theta} \frac{\partial^2}{\partial\varphi^2}\,.$$

Equation (4) thus takes the form

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \frac{4aMr}{\sigma^2} \frac{\partial^2 u}{\partial t \partial\varphi} - \frac{\Delta}{\sigma^2} \frac{\partial}{\partial r} \left(\Delta \frac{\partial u}{\partial r} \right) - \frac{\Delta}{\sigma^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) \\ &\qquad - \frac{\rho^2 - 2Mr}{\sigma^2 \sin^2\theta} \frac{\partial^2 u}{\partial\varphi^2} + \frac{\Delta\rho^2}{\sigma^2} m^2 u + \lambda \frac{\Delta\rho^2}{\sigma^2} |u|^2 u = 0\,. \end{aligned} \tag{5}$$

This can be written in a more synthetic manner as

$$\partial_t^2 u - 2ik\partial_t u + hu + \lambda \frac{\Delta\rho^2}{\sigma^2} |u|^2 u = 0\,, \tag{6}$$

$$k = \frac{2iaMr}{\sigma^2}\,\partial_\varphi\,,\ \ h = -\frac{\Delta}{\sigma^2}\,\partial_r\,\Delta\,\partial_r - \frac{\Delta}{\sigma^2 \sin\theta} \partial_\theta \sin\theta\,\partial_\theta - \frac{\rho^2 - 2Mr}{\sigma^2 \sin^2\theta}\,\partial^2_\varphi + \frac{\Delta\rho^2}{\sigma^2} m^2\,.$$

Both operators h and k are symmetric on $L^2(\Sigma;\sigma^2\Delta^{-1}\mathrm{d}r\mathrm{d}\omega)$, where $\mathrm{d}\omega=\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$ is the euclidian measure on S^2. Hence, there is a natural conserved quantity for equation (6), denoted E_{BL}, as "energy with respect to Boyer-Lindquist coordinates", given by

$$E_{BL}(u)=\|\partial_t u(t)\|^2+(u(t),hu(t))+\frac{1}{2}\left(u(t),\lambda\frac{\Delta\rho^2}{\sigma^2}|u(t)|^2u(t)\right)\,,\ \forall t\in\mathbb{R}\,,\qquad(7)$$

where $\|.\|$ and $(.,.)$ denote the $L^2(\Sigma;\sigma^2\Delta^{-1}\mathrm{d}r\mathrm{d}\omega)$ norm and inner product. This conserved quantity is not positive because the operator h is not positive. Indeed,

$$(u,hu)=\int_\Sigma\left\{\frac{\Delta^2}{\sigma^2}|\partial_r u|^2+\frac{\Delta}{\sigma^2}|\partial_\theta u|^2+\frac{\rho^2-2Mr}{\sigma^2\sin^2\theta}|\partial_\varphi u|^2+\frac{\Delta\rho^2m^2}{\sigma^2}|u|^2\right\}\frac{\sigma^2}{\Delta}\mathrm{d}r\mathrm{d}\omega$$

and the factor ρ^2-2Mr which multiplies $|\partial_\varphi u|^2$ is positive outside the ergosphere and negative inside $\mathcal{E}$.

Remark 2.1 *There is another, perhaps more physical (or geometrical) way of obtaining the expression of the conserved quantity. The stress-energy-momentum tensor of the scalar field u is given by*

$$8\pi T_{\mathbf{ab}}=2\frac{\partial u}{\partial x^{\mathbf{a}}}\frac{\partial\bar{u}}{\partial x^{\mathbf{b}}}-g_{\mathbf{ab}}\left(g^{\mathbf{cd}}\frac{\partial u}{\partial x^{\mathbf{c}}}\frac{\partial\bar{u}}{\partial x^{\mathbf{d}}}-m^2|u|^2-\frac{\lambda}{2}|u|^4\right)$$

and satisfies the conservation law

$$\nabla^a T_{ab}=0\,.\qquad(8)$$

This together with the fact that $\partial/\partial t$ is a Killing vector implies that the 1-form $T_{a0}\mathrm{d}x^a$ is closed. Hence, denoting by $\mathbf{T}^a$ the unit future oriented vector field normal to the hypersurfaces Σ_t, given by (see for example [24])

$$\mathbf{T}^a=\frac{1}{(g_{bc}\nabla^b t\nabla^c t)^{1/2}}\nabla^a t\,,\ \mathbf{T}^a\frac{\partial}{\partial x^a}=\sqrt{\frac{\sigma^2}{\Delta\rho^2}}\left(\frac{\partial}{\partial t}+\frac{2aMr}{\sigma^2}\frac{\partial}{\partial\varphi}\right)\,,$$

we obtain that the energy of the field, as measured by an observer[1] *whose* 4*-velocity vector is $\partial/\partial t$, is conserved by the evolution and given by*

$$E_{BL}(u,t)=\int_{\Sigma_t}\mathbf{T}^a T_{a0}\mathrm{dVol}\,,\qquad(9)$$

where $\mathrm{dVol}=\sqrt{\rho^2\sigma^2\Delta^{-1}}\,\mathrm{d}r\mathrm{d}\omega$ *is the measure on Σ induced by g. An explicit calculation of (9) gives* $1/(8\pi)$ *times the expression (7).* **The quantity E_{BL} is only positive definite outside the ergosphere since $\partial/\partial t$ is only timelike outside the ergosphere. Moreover, the fact that there exists no globally defined timelike Killing vector field in block I implies that there is no positive definite conserved energy.**

This lack of positivity of the conserved quantity is what allows super-radiance to take place. Super-radiance is the analogue, at the level of fields of integral spin, of the Penrose process (see for example [8] or [29]), a mechanism by which particles can extract energy from the ergosphere.

[1]Such observers can only exist outside the ergosphere where $\partial/\partial t$ is timelike, hence, (9) is to be understood as the energy of the field measured by distant stationary observers, the typical example being an observer static at infinity.

If we consider the linear equation ($\lambda = 0$), the absence of positive conserved energy has two immediate consequences : first, the conserved quantity does not define a natural Hilbert space framework in which to study the evolution of solutions ; second, for any Hilbert space framework that we may choose, the evolution will not be unitary. This is disastrous for the development of a scattering theory, but not for solving the Cauchy problem. The strategy used in [15] to prove the existence and uniqueness of solutions to the linear equation was to define a new energy norm as the square root of

$$\|\partial_t u(t)\|^2 + \big(u(t), (h + k^2)u(t)\big)$$

which is easily seen to be a positive definite quadratic form, and then to control the growth of this norm by means of an energy estimate. We choose here to use a more geometrical method. We perform a $3+1$ decomposition of the geometry of block I and of the equation. This gives us a natural Hilbert space framework for the linear equation inherited from the positive definite energy on each spacelike slice. The growth of the Hilbert norm is then also controlled by an energy estimate.

2.2 The $3 + 1$ decomposition

The description of block I in terms of Boyer-Lindquist coordinates is based on two quantities :

- the time function t which induces a foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ by its level hypersurfaces ;
- K^a, the only Killing vector field globally defined on $\mathcal{B}_I$ to be timelike near spacelike infinity[2].

The vector field K^a fixes the product structure $\mathcal{B}_I = \mathbb{R}_t \times \Sigma$, i.e. the points on different hypersurfaces Σ_t are identified along the integral lines of K^a. This choice of product structure is characterized by the property that the coordinate vector field $\partial/\partial t$ is equal to (or is a constant multiple of) the Killing vector field K^a.

The principle of the $3+1$ decomposition is to base the whole structure on the function t alone. The choice of product structure $\mathcal{B}_I = \mathbb{R}_t \times \Sigma$ is induced in a natural manner by the foliation itself : instead of K^a, we use $\mathbf{T}^a$, the unit future pointing vector field normal to the hypersurfaces Σ_t (or equivalently $\nabla^a t$). This choice can be made explicit by introducing the new coordinates τ, R, Θ, Φ, defined in terms of Boyer-Lindquist coordinates as

$$\tau = t\,,\ R = r\,,\ \Theta = \theta\,,\ \Phi = \varphi - t\alpha\,,\ \alpha = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2aMr}{\sigma^2}\,. \tag{10}$$

We obtain a description of block I that is sometimes referred to as the point of view of locally non rotating observers. The metric now appears as the sum of its orthogonal projections along T^a and the hypersurfaces Σ_τ and no longer contains terms coupling time and space variables :

$$g = N^2 \mathrm{d}\tau^2 - h(\tau) \tag{11}$$

where

$$N = \left(g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}} \right)^{\frac{1}{2}} = \left(\frac{\Delta\rho^2}{\sigma^2} \right)^{\frac{1}{2}}$$

[2]Note that t and K^a are not as independent as they might appear. The function t is in some sense subordinate to K^a : it is such that all hypersurfaces Σ_t, $t \in \mathbb{R}$, are obtained by pushing an initial hypersurface, say Σ_0, along the flow of K^a.

and

$$\begin{aligned}
h(\tau) &= -g_{rr}\mathrm{d}R^2 - g_{\theta\theta}\mathrm{d}\Theta^2 - g_{\varphi\varphi}\left(\mathrm{d}\Phi + \tau\frac{\partial\alpha}{\partial R}\mathrm{d}R + \tau\frac{\partial\alpha}{\partial\Theta}\mathrm{d}\Theta\right)^2 \\
&= -\left(g_{rr} + \tau^2\left(\frac{\partial\alpha}{\partial R}\right)^2 g_{\varphi\varphi}\right)\mathrm{d}R^2 - \left(g_{\theta\theta} + \tau^2\left(\frac{\partial\alpha}{\partial\Theta}\right)^2 g_{\varphi\varphi}\right)\mathrm{d}\Theta^2 \\
&\quad -g_{\varphi\varphi}\mathrm{d}\Phi^2 - 2\tau^2\frac{\partial\alpha}{\partial R}\frac{\partial\alpha}{\partial\Theta}g_{\varphi\varphi}\mathrm{d}R\mathrm{d}\Theta - 2\tau\frac{\partial\alpha}{\partial R}g_{\varphi\varphi}\mathrm{d}R\mathrm{d}\Phi - 2\tau\frac{\partial\alpha}{\partial\Theta}g_{\varphi\varphi}\mathrm{d}\Theta\mathrm{d}\Phi\,.
\end{aligned}$$

Note that

$$\mathbf{T}_a\mathrm{d}x^a = N\mathrm{d}\tau\,,\ \mathbf{T}^a\frac{\partial}{\partial x^a} = \frac{1}{N}\frac{\partial}{\partial\tau}\,.$$

The metric g in its decomposed form (11) is now time dependent since $\partial/\partial\tau$ is proportional to $\mathbf{T}^a$ and is not a Killing vector field (otherwise block I would be static). It satisfies the following properties :

($\mathcal{P}_1$) for each $\tau\in\mathbb{R}$, $\Sigma_\tau = (\Sigma, h(\tau))$ is a $\mathcal{C}^\infty$ Riemannian manifold with smooth boundary $\partial\Sigma = \{r_+\}\times S^2_{\theta\varphi}$;

($\mathcal{P}_2$) N is strictly positive on Σ and vanishes on $\partial\Sigma$; it is independent of τ, and it is $\mathcal{C}^\infty$ and uniformly bounded on $\overline{\Sigma}$ as well as all its derivatives ;

($\mathcal{P}_3$) $h_{ab}\in\mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}^\infty_b\left(\bar{\Sigma}\,;T_{ab}\mathcal{M}\right)\right)$; $h^{ab}\in\mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}^\infty_b\left(\bar{\Sigma}\,;T^{ab}\mathcal{M}\right)\right)$; if we introduce the radial variable

$$u(R) := \int_{r_+}^R F^{-1/2}(s)\mathrm{d}s\,;\ F(R) = \frac{\Delta}{R^2} = \frac{(R-r_+)(R-r_-)}{R^2}\,;$$

we have

$$\begin{aligned}
h(0) &= \frac{\rho^2}{R^2}\mathrm{d}u^2 + \frac{\rho^2}{(1+u)^2}(1+u)^2\mathrm{d}\Theta^2 \\
&\quad + \left[\frac{(R^2+a^2)\rho^2 + 2MRa^2\sin^2\Theta}{\rho^2(1+u)^2}\right](1+u)^2\sin^2\Theta\,\mathrm{d}\Phi^2\,;
\end{aligned}$$

this shows that $h(\tau)$ is (locally uniformly in time and uniformly on Σ) equivalent to the euclidian metric on $\mathbb{R}^3\setminus\bar{B}(0,1)$

$$\mathrm{d}u^2 + (1+u)^2\mathrm{d}\Theta^2 + (1+u)^2\sin^2\Theta\,\mathrm{d}\Phi^2\,; \tag{12}$$

we also have that $h(\tau)$ is asymptotically flat (i.e. $h(\tau)$ tends to the metric (12) as $R\to\infty$) ;

($\mathcal{P}_4$) the determinent of $h(\tau)$ is independent of τ, we denote it $|h|$; the determinent of g is also independent of τ since $\det g = -N^2|h|$; we denote $|g| = |\det g|$.

We use (11) and the fact that $|g|$ is independent of τ to express $\Box_g$ in terms of coordinates τ, R, Θ, Φ :

$$\begin{aligned}
\Box_g &= \frac{1}{|g|^{\frac12}}\frac{\partial}{\partial x^{\mathbf{a}}}\left(|g|^{\frac12}g^{\mathbf{ab}}\frac{\partial}{\partial x^{\mathbf{b}}}\right) = \frac{1}{N^2}\frac{\partial^2}{\partial\tau^2} - \frac{1}{N|h|^{\frac12}}\frac{\partial}{\partial x^\alpha}\left(N|h|^{\frac12}h^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right) \\
&= \frac{1}{N^2}\left(\frac{\partial^2}{\partial\tau^2} - N^4\hat{\Delta}\right)
\end{aligned}$$

where $\hat{\Delta}$ is the Laplace-Beltrami operator associated with the metric $\hat{h} = N^2 h$

$$\hat{\Delta} = \Delta_{\hat{h}} = \frac{1}{|\hat{h}|^{\frac{1}{2}}} \frac{\partial}{\partial x^\alpha} \left(|\hat{h}|^{\frac{1}{2}} \hat{h}^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right) = \frac{1}{N^3 |h|^{\frac{1}{2}}} \frac{\partial}{\partial x^\alpha} \left(N |h|^{\frac{1}{2}} h^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right) . \tag{13}$$

This gives us the new expression of equation (4)

$$\frac{\partial^2 u}{\partial \tau^2} - N^4 \hat{\Delta} u + N^2 m^2 u + \lambda N^2 |u|^2 u = 0 \, . \tag{14}$$

3 The global Cauchy problem outside the black hole

The Cauchy problem for equation (4) has been solved in Sobolev spaces in [7], on general smooth globally hyperbolic space-times without boundary satisfying the essential property that, in a $3+1$ decomposed form, the timelike coordinate vector field is uniformly timelike on the whole space-time. This property is equivalent to the lapse function being uniformly bounded and bounded away from zero. Here, not only do we have a boundary, but the lapse function vanishes there. However, an important property of the geometry of block I will allow us to use the results of [7] "away from the horizon" and to prove the existence of smooth solutions to (14). This combined with an energy estimate will solve the Cauchy problem for the linear equation. For the non linear equation, we use a Duhamel formula and the classic Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ to obtain uniqueness of minimum regularity solutions. An energy estimate valid for smooth solutions is extended to solutions of lower regularity and gives global existence.

The geometry of block I described using the time variable t of the Boyer-Lindquist coordinates has the following property : light rays (null geodesics) only reach the horizon for infinite values of t. The typical example is given by principal null geodesics. The incoming ones describe the movement of a massless particle aimed directly at the centre of the black hole from infinity. They are defined as the integral lines of the vector field (expressed in Boyer-Lindquist coordinates)

$$N^a \frac{\partial}{\partial x^a} = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} - \frac{\partial}{\partial r} + \frac{a}{\Delta} \frac{\partial}{\partial \varphi} \, .$$

The outgoing principal null geodesics describe the trajectory of a massless particle going away from the horizon so that, when it reaches infinity, it is aimed in the direction opposite to that of the centre of the black hole ; they are the integral lines of the vector field

$$L^a \frac{\partial}{\partial x^a} = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{a}{\Delta} \frac{\partial}{\partial \varphi} \, .$$

Principal null geodesics can be understood as the straightest routes to or from the horizon. If we introduce a new radial variable r_* such that

$$\frac{\mathrm{d} r_*}{\mathrm{d} r} = \frac{r^2 + a^2}{\Delta} \text{ (which is } > 0 \text{ on }]r_+, +\infty[\text{)} \tag{15}$$

the horizon $\{r = r_+\}$ corresponds to $r_* \to -\infty$ and on principal null geodesics, we have $\dot{r_*} = \pm \dot{t}$. Therefore, the integral lines of N^a (resp. L^a) only reach the horizon as $t \to +\infty$ (resp. $t \to -\infty$).

The consequence for the propagation of fields is that, if we consider some initial data living away from the horizon, say

$$u_0\,,\ u_1\ \in \mathcal{C}_0^\infty\left(\Sigma\right)\,,$$

the support of a corresponding solution to (14) will only reach the horizon for infinite values of τ. More precisely, if we limit ourselves to a compact time interval $[-T,T]_\tau$, the solution will remain at a uniformly positive spatial distance from the horizon. Hence, on the support of the solution and for finite values of τ, we do not see the boundary nor the fact that the lapse function is not uniformly bounded away from zero. The results of [7] can therefore be applied to obtain the existence of smooth solutions of (14) associated with smooth initial data supported away from the horizon. Note that in the case $\lambda = 0$, we can use, instead of [7], the classic results of F.G. Friedlander [14] or J. Leray [18]. We obtain the following lemma

Lemma 3.1 *For initial data $u_0\,,\ u_1\ \in \mathcal{C}_0^\infty\left(\Sigma\right)$, and for any initial time $s\in\mathbb{R}$, equation (14) has a solution u in $\mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}_0^\infty(\Sigma)\right)$ satisfying*

$$u(s) = u_0 \text{ and } \partial_\tau u(s) = u_1\,.$$

Definition 3.1 *The energy of such solutions as measured by an observer whose four-velocity vector is $\partial/\partial\tau$ is expressed as*

$$E_{3+1}(u,\tau) = \int_{\Sigma_\tau} \mathbf{T}^a T_{a0}\mathrm{dVol}\,,\ T_{a0} = T_{ab}\left(\frac{\partial}{\partial\tau}\right)^a = T_{ab}N\mathbf{T}^a\,,$$

that is to say

$$E_{3+1}(u,\tau) = \int_{\Sigma_\tau}\frac{1}{N}T_{00}\mathrm{dVol}\,.$$

Putting $E(u,\tau) = 8\pi E_{3+1}(u,\tau)$, we obtain

$$E(u,\tau) = \int_{\Sigma_\tau}\left(|\partial_\tau u|^2 + N^2h^{\alpha\beta}\partial_\alpha u\partial_\beta\bar{u} + N^2m^2\,|u|^2 + \frac{1}{2}\lambda N^2\,|u|^4\right)\frac{1}{N}\mathrm{dVol}\,, \qquad (16)$$

dVol *being the volume form defined by h on Σ,*

$$\mathrm{dVol} = \gamma\mathrm{d}x\,,\ \gamma = |h|^{\frac{1}{2}}\,.$$

Of course, the expression (16) can be obtained in a more functional analytic manner : at each time τ, the operator $\hat{\Delta}$ is essentially self-adjoint on $L^2(\Sigma\,;|\hat{h}|^{1/2}\mathrm{d}x)$, $|\hat{h}|^{1/2} = N^3\gamma$; hence $N^4\hat{\Delta}$ is essentially self-adjoint on $L^2(\Sigma\,;N^{-1}\gamma\mathrm{d}x)$; this yields at each time a natural energy norm for solutions of (14) that is exactly given by (16).

Solutions of (14) in $\mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}_0^\infty(\Sigma)\right)$ satisfy the following energy estimate. The proof can be obtained in the usual way by multiplying the equation by $\partial_\tau u$ and integrating by parts on $]s,\tau[\times\Sigma$ (one can also take the time derivative of the energy and estimate the terms which do not cancel each other out by virtue of the equation).

Proposition 3.1 *There exists a continuous, strictly positive function K_λ on $\mathbb{R}^2$ such that $K_\lambda(\tau,\tau) = 1$ and for each $u\in\mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}_0^\infty(\Sigma)\right)$ solution of (14), we have for any $s,\tau\in\mathbb{R}$,*

$$E(u,\tau) \le K_\lambda(s,\tau)E(u,s)\,. \qquad (17)$$

We can now prove the existence and uniqueness of finite energy solutions for the linear equation

$$\frac{\partial^2 u}{\partial \tau^2} - N^4 \hat{\Delta} u + N^2 m^2 u = 0\,. \tag{18}$$

Definition 3.2 *We consider on $\overline{\Sigma}$ a fixed smooth Riemannian metric η, for example $\eta = h(0)$, or more simply the euclidian metric (12). We define the function space $\mathcal{H}$ as the completion of $\mathcal{C}_0^\infty(\Sigma) \oplus \mathcal{C}_0^\infty(\Sigma)$ in the norm*

$$\left\| {}^t(\phi\,,\ \psi) \right\|_{\mathcal{H}}^2 = \int_\Sigma \left(|\psi|^2 + N^2 \left|\nabla\phi\right|^2 + N^2 m^2 \left|\phi\right|^2 \right) \frac{1}{N} \mathrm{dVol}$$

where

$$|\nabla\phi|^2 = \eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\overline{\phi}\,.$$

Finite energy solutions of (18) are naturally defined as the functions u on $\mathbb{R}\times\Sigma$ satisfying (18) in the sense of distributions on $\mathbb{R}\times\Sigma$ and such that ${}^t(u\,,\ \partial_\tau u) \in \mathcal{C}\left(\mathbb{R}_\tau\,;\mathcal{H}\right)$[3].

Theorem 1 *For any initial data ${}^t(u_0\,,\ u_1) \in \mathcal{H}$, for any initial time $s\in\mathbb{R}$, equation (18) has a unique solution u such that*

$${}^t(u\,,\ \partial_\tau u) \in \mathcal{C}\left(\mathbb{R}_\tau\,;\mathcal{H}\right)\,,\ u(s) = u_0\,,\ \partial_\tau u(s) = u_1\,.$$

Proof of theorem 1 : the uniqueness of finite energy solutions can be proved locally, using the same arguments that led to the result of lemma 3.1. We consider on the "initial" hypersurface Σ_s a compact set K and we denote Ω its domain of influence (both in the future and in the past). Ω is a compact subset of $\mathbb{R}\times\Sigma$ on which N and h are uniformly bounded and bounded away from zero. The uniqueness of finite energy solutions in Ω is therefore a consequence of the standard theory of linear hyperbolic operators with smooth coefficients (see [16]). One can also use in Ω the uniqueness results of [7] in the linear case. Note that the uniqueness in Ω can be proved directly by means of local energy estimates : a usual way of obtaining such estimates for finite energy solutions is to regularize the solution u by convolution and to perform energy estimates for the regularized functions (there are slightly technical aspects because the regularized functions do not satisfy the same equation, their d'Alembertian appears on the right hand-side of the estimates and must be seen, using the properties of convolution, to converge to zero at least weakly in L^2_{loc} instead of merely in H^{-1}, allowing to extend the estimate to the solution u). Now, Σ_s being a Cauchy hypersurface in block I, its domain of influence is the whole of $\mathcal{B}_I$. Hence, if we consider larger and larger compact sets K, their domain of influence Ω will cover $\mathcal{B}_I$ completely. This guarantees the uniqueness of finite energy solutions of (18).

Using this uniqueness and the existence of smooth solutions given by lemma 3.1, we can define on $\mathcal{C}_0^\infty(\Sigma)\oplus\mathcal{C}_0^\infty(\Sigma)$ the propagator for (18)

$$\mathcal{U}(\tau,s)\ :\ {}^t(u_0\,,\ u_1) \longmapsto\ {}^t(u(\tau)\,,\ \partial_\tau u(\tau)) \tag{19}$$

where $u \in \mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}_0^\infty(\Sigma)\right)$ is the finite energy solution of (18) such that $u(s) = u_0$ and $\partial_\tau u(s) = u_1$. The energy estimate (17) allows us to extend $\mathcal{U}$ as a propagator on $\mathcal{H}$, i.e. satisfying

[3] The equivalence, uniform in space and locally uniform in time, between $h(\tau)$ and η, implies that it is equivalent to control the energy of u and to control the $\mathcal{H}$-norm of $(u, \partial_\tau u)$

(i) $\mathcal{U}(\tau,\sigma) \in \mathcal{L}(\mathcal{H})$ for any $\tau, \sigma \in \mathbb{R}$; $\|\mathcal{U}(\tau, s)\|_{\mathcal{L}(\mathcal{H})} \leq K_0(s,\tau)$ (K_0 being the function K_λ of estimate (17) in the linear case, i.e. for $\lambda = 0$) ;

(ii) $\mathcal{U}(\tau,\tau) = \mathrm{Id}_{\mathcal{H}}$; $\mathcal{U}(\tau,\sigma)\mathcal{U}(\sigma,s) = \mathcal{U}(\tau,s)$;

(iii) for any $V \in \mathcal{H}$, $\mathcal{U}(\tau,\sigma)V \in \mathcal{C}\left(\mathbb{R}^2_{\tau,\sigma}\,;\mathcal{H}\right)$;

(iv) we denote $A(\tau)$ the time-dependent hamiltonian of equation (18)

$$A(\tau) = \begin{pmatrix} 0 & 1 \\ N^2\Delta_{\hat{h}} - N^2 m^2 & 0 \end{pmatrix} ;$$

we have, for any $V \in \mathcal{H}$,

$$\frac{\partial}{\partial \tau}\mathcal{U}(\tau,\sigma)V = A(\tau)\mathcal{U}(\tau,\sigma)V \text{ and } \frac{\partial}{\partial \sigma}\mathcal{U}(\tau,\sigma)V = -\mathcal{U}(\tau,\sigma)A(\sigma)V$$

in the sense of distributions on $\mathbb{R} \times \Sigma$.

This concludes the proof of theorem 1. □

We now proceed to solving the Cauchy problem for the non linear equation. We do not work on the energy space $\mathcal{H}$ but on a slightly smaller function space that will allow us to use the flat Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ to control the non linear term.

Definition 3.3 *We introduce the function space $\mathbb{H}^1$, completion of $\mathcal{C}_0^\infty(\Sigma)$ in the norm*

$$\|\phi\|^2_{\mathbb{H}^1} = \int_\Sigma \left(N\,|\nabla\phi|^2 + \frac{1}{N}\,|\phi|^2 \right) \mathrm{dVol}$$

and the space $\mathbb{H} = \mathbb{H}^1 \oplus L^2(\Sigma\,; N^{-1}\mathrm{dVol})$, i.e. the norm on $\mathbb{H}$ is given by

$$\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|^2_{\mathbb{H}} = \int_\Sigma \left\{ N\,|\nabla\phi|^2 + \frac{1}{N}\,|\phi|^2 + \frac{1}{N}\,|\psi|^2 \right\} \mathrm{dVol}$$

$$= \int_\Sigma \left\{ |\psi|^2 + N^2\,|\nabla\phi|^2 + |\phi|^2 \right\} \frac{1}{N}\mathrm{dVol}\,.$$

We clearly have $\mathbb{H} \hookrightarrow \mathcal{H}$.

Theorem 2 *For any initial data $(u_0, u_1) \in \mathbb{H}$, for any initial time $s \in \mathbb{R}$, equation (14) admits a unique solution u such that*

$$\begin{pmatrix} u \\ \partial_\tau u \end{pmatrix} \in \mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})\,,\ u(s) = u_0\,,\ \partial_\tau u(s) = u_1.$$

Moreover, this solution satisfies energy estimate (17).

Proof of theorem 2 : our first task is to check that the space $\mathbb{H}$ is stable under the linear evolution. We write (18) in its hamiltonian form :

$$\frac{\partial V}{\partial \tau} = A(\tau)V \tag{20}$$

$$V = \begin{pmatrix} u \\ \partial_\tau u \end{pmatrix}\,,\ A(\tau) = \begin{pmatrix} 0 & 1 \\ N^2\Delta_{\hat{h}} - N^2 m^2 & 0 \end{pmatrix}.$$

We consider some initial data $V_0 = {}^t(\phi,\psi) \in \{\mathcal{C}_0^\infty(\Sigma)\}^2$, some initial time $s \in \mathbb{R}$, and $V \in \mathcal{C}^\infty(\mathbb{R}_\tau\,;\{\mathcal{C}_0^\infty(\Sigma)\}^2)$ the solution of (20) such that $V(s) = V_0$. For each τ, we can estimate the norm of $V(\tau)$ in $\mathbb{H}$ as follows :

$$\|V(\tau)\|_{\mathbb{H}} \leq \|V(\tau)\|_{\mathcal{H}} + \|u(\tau)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} \leq K_0(s,\tau)\,\|V(s)\|_{\mathcal{H}} + \|u(\tau)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})}$$

and

$$\begin{aligned}\|u(\tau)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} &\leq \|u(s)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} + \int_{]s,\tau[} \|\partial_\tau u(\sigma)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})}\,\mathrm{d}\sigma \\ &\leq \|V(s)\|_{\mathbb{H}} + \int_{]s,\tau[} \|V(\sigma)\|_{\mathcal{H}}\,\mathrm{d}\sigma\,.\end{aligned}$$

Since we have as well $\mathbb{H} \hookrightarrow \mathcal{H}$, we conclude that there exists a continuous, strictly positive function $\tilde{K}$ on $\mathbb{R}^2$, $\tilde{K}(\tau,\tau) = 1$, such that, for any solution u of (18) in $\mathcal{C}^\infty(\mathbb{R}_\tau\,;\mathcal{C}_0^\infty(\Sigma))$,

$$\|V(\tau)\|_{\mathbb{H}} \leq \tilde{K}(s,\tau)\,\|V(s)\|_{\mathbb{H}}\,.$$

This entails by density that $\mathbb{H}$ is stable under the propagator $\mathcal{U}$ and $\mathcal{U}$ satisfies the following properties on $\mathbb{H}$:

(a) $\mathcal{U}(\tau,\sigma) \in \mathcal{L}(\mathbb{H})$ for any $\tau,\sigma \in \mathbb{R}$; $\|\mathcal{U}(\tau,s)\|_{\mathcal{L}(\mathbb{H})} \leq \tilde{K}(s,\tau)$;

(b) $\mathcal{U}(\tau,\tau) = \mathrm{Id}_{\mathbb{H}}$; $\mathcal{U}(\tau,\sigma)\,\mathcal{U}(\sigma,s) = \mathcal{U}(\tau,s)$;

(c) for any $V \in \mathbb{H}$, $\mathcal{U}(\tau,\sigma)V \in \mathcal{C}\left(\mathbb{R}^2_{\tau,\sigma}\,;\mathbb{H}\right)$.

The next step is to study the continuity of the non linear term on $\mathbb{H}$. The Hamiltonian form of (14) is :

$$\frac{\partial V}{\partial \tau} = A(\tau)V + J(V)\,,\ J\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ -\lambda N^2\,|\phi|^2\,\phi\end{pmatrix}. \tag{21}$$

Lemma 3.2 *There exists a constant $C_J > 0$ such that, for each F, G in $\mathbb{H}$,*

$$\|J(F)\|_{\mathbb{H}} \leq C_J\,\|F\|^3_{\mathbb{H}}\,, \tag{22}$$

$$\|J(F) - J(G)\|_{\mathbb{H}} \leq C_J\left(\|F\|^2_{\mathbb{H}} + \|G\|^2_{\mathbb{H}}\right)\|F - G\|_{\mathbb{H}}\,. \tag{23}$$

Proof of lemma 3.2 : we only need to establish (23) to prove both inequalities. Let

$$F = \begin{pmatrix}\phi\\ \psi\end{pmatrix},\ G = \begin{pmatrix}f\\ g\end{pmatrix},$$

we have

$$J(F) - J(G) = \begin{pmatrix}0\\ -\lambda N^2\left(|\phi|^2\,\phi - |f|^2\,f\right)\end{pmatrix}.$$

Writing the second component as

$$-\lambda N^2\left\{\phi^2\left(\bar{\phi} - \bar{f}\right) + f^2\left(\bar{\phi} - \bar{f}\right) + \phi\bar{f}\,(\phi - f) - \phi f\left(\bar{\phi} - \bar{f}\right) + f\bar{\phi}\,(\phi - f)\right\},$$

we obtain

$$\left|\lambda N^2\left(|\phi|^2\,\phi - |f|^2\,f\right)\right| \leq CN^2\,|\phi - f|\left(|\phi|^2 + |f|^2\right).$$

We can therefore estimate the norm in $\mathbb{H}$ of $J(F)-J(G)$ as follows :

$$\begin{aligned}
\|J(F)-J(G)\|_{\mathbb{H}} &\leq C\left\|N^2\left(|\phi-f|\right)\left(|\phi|^2+|f|^2\right)\right\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} \\
&\leq C\left(\left\|N^{3/2}\left|\phi-f\right|\left|\phi\right|^2\right\|_{L^2(\Sigma\,;\mathrm{dVol})}+\left\|N^{3/2}\left|\phi-f\right|\left|f\right|^2\right\|_{L^2(\Sigma\,;\mathrm{dVol})}\right) \\
&\leq C'\left[\left\|N^{1/2}\left(\phi-f\right)\right\|_{L^6(\Sigma)}\left(\left\|N^{1/2}\phi\right\|^2_{L^6(\Sigma)}+\left\|N^{1/2}f\right\|^2_{L^6(\Sigma)}\right)\right]\,, \quad (24)
\end{aligned}$$

$L^6(\Sigma)$ denoting $L^6(\Sigma\,;\mathrm{dVol})$. Now the flat Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ immediately entails

$$H^1_0\left(\Sigma\right)\hookrightarrow L^6\left(\Sigma\,;\mathrm{dVol}\right) \quad (25)$$

where we define $H^1_0\left(\Sigma\right)$ as the completion of $\mathcal{C}^\infty_0(\Sigma)$ in the norm

$$\|u\|^2_{H^1(\Sigma)}=\int_\Sigma\left\{|u|^2+|\nabla u|^2\right\}\mathrm{dVol}\,.$$

For u in $\mathcal{C}^\infty_0(\Sigma)$, we have

$$\left\|N^{1/2}u\right\|^2_{H^1(\Sigma)}=\int_\Sigma\left\{N|u|^2+N\left|\nabla u\right|^2+\left|\frac{N^{-1/2}}{2}\nabla N\right|^2|u|^2\right\}\mathrm{dVol}\leq C\left\|u\right\|^2_{\mathbb{H}^1} \quad (26)$$

using the uniform boundedness of ∇N on $\overline{\Sigma}$. This inequality is still valid for $u\in\mathbb{H}^1$ by density. Putting together (24), (25) and (26) gives (23) and concludes the proof of lemma 3.2. $\Box$

We can now solve the global Cauchy problem for (21) in $\mathbb{H}$:

$$\left\{\begin{array}{l}\partial_\tau V=A(\tau)V+J(V)\,;\\ V(s)=V_0\in\mathbb{H}\,;\;\;V\in\mathcal{C}\left(\mathbb{R}_\tau\,;\mathbb{H}\right)\,;\end{array}\right. \quad (27)$$

by studying its Duhamel formulation :

$$V(\tau)=\mathcal{U}(\tau,s)V_0+\int_s^\tau\mathcal{U}(\tau,\sigma)J(V(\sigma))\mathrm{d}\sigma\,;\;\;V\in\mathcal{C}\left(\mathbb{R}_\tau\,;\mathbb{H}\right)\,. \quad (28)$$

Proposition 3.2 *Problems (27) and (28) are equivalent. Moreover, the solutions of (28) are unique.*

Proof of proposition 3.2.
First step : equivalence between (27) and (28). A remark essential for the proof is that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{U}(\tau,s)=A(\tau)\mathcal{U}(\tau,s)$$

in the strong sense on $\mathcal{L}(\mathbb{H}\,;L^2_{\mathrm{loc}}(\Sigma)\oplus H^{-1}_{\mathrm{loc}}(\Sigma))$. If we consider V a solution of (28), differentiating the integral equation[4], we get the equality in $\mathcal{C}\left(\mathbb{R}_\tau\,;L^2_{\mathrm{loc}}(\Sigma)\oplus H^{-1}_{\mathrm{loc}}(\Sigma)\right)$:

$$\frac{\partial V}{\partial\tau}(\tau)=A(\tau)\mathcal{U}(\tau,s)V_0+A(\tau)\int_s^\tau\mathcal{U}(\tau,\sigma)J(V(\sigma))\mathrm{d}\sigma+J(V(\tau))=A(\tau)V(\tau)+J(V(\tau))\,.$$

[4]Recall that by lemma 3.2, $V\in\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$ entails $J(V)\in\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$.

Hence, V satisfies (27). Conversely, if V is a solution of (27), putting

$$\Phi(\tau) = J(V(\tau)) \in \mathcal{C}\left(\mathbb{R}_\tau\,;\mathbb{H}\right)\,,$$

we see that V is a solution of

$$\partial_\tau \Psi(\tau) = A(\tau)\Psi(\tau) + \Phi(\tau)\,,\ \Psi(s) = V_0\,,\ \Psi \in \mathcal{C}\left(\mathbb{R}_\tau\,;\mathbb{H}\right)\,. \tag{29}$$

The uniqueness of solutions of (29) follows directly from the uniqueness for (20) and the following expression defines a solution :

$$\Psi(\tau) = \mathcal{U}(\tau, s)V_0 + \int_s^\tau \mathcal{U}(\tau,\sigma)\Phi(\sigma)\mathrm{d}\sigma\,.$$

It follows that V satisfies (28).

Second step : local uniqueness of solutions of (28). This is a straightforward consequence of lemma 3.2. We consider $T > 0$ and V, W two solutions of (28) on $[s, s+T]$ (the same thing can of course be done on $[s-T, s]$), that is to say, V and W are solutions of :

$$\Psi(\tau) = \mathcal{U}(\tau, s)V_0 + \int_s^\tau \mathcal{U}(\tau,\sigma)J(\Psi(\sigma))\mathrm{d}\sigma \ \text{ for } s \leq \tau \leq s+T\,;\ \Psi \in \mathcal{C}\left([s,s+T]\,;\mathbb{H}\right)\,. \tag{30}$$

We have for $\tau \in [s, s+T]$,

$$V(\tau) - W(\tau) = \int_0^\tau \mathcal{U}(\tau,\sigma)\left(J(V(\sigma) - J(W(\sigma))\right)\mathrm{d}\sigma\,.$$

Hence, putting $C = C_J \max\left\{\tilde{K}(\sigma,\tau)\,;\ \sigma,\tau \in [s,s+T]\right\}$,

$$\begin{aligned}
\|V(\tau) - W(\tau)\|_{\mathbb{H}} &\leq C \int_s^\tau \left(\|V(\sigma)\|_{\mathbb{H}}^2 + \|W(\sigma)\|_{\mathbb{H}}^2\right)\|V(\sigma) - W(\sigma)\|_{\mathbb{H}}\,\mathrm{d}\sigma \\
&\leq C\left(\sup_{\sigma\in[s,s+T]}\left(\|V(\sigma)\|_{\mathbb{H}}^2 + \|W(\sigma)\|_{\mathbb{H}}^2\right)\right)\int_s^\tau \|V(\sigma) - W(\sigma)\|_{\mathbb{H}}\,\mathrm{d}\sigma
\end{aligned}$$

which, by Gronwall's lemma, entails that $V \equiv W$. This proves the uniqueness of solutions of (28) and thence the uniqueness of solutions of (27). The proof of proposition 3.2 is complete. $\square$

Our last task is to prove the global existence of solutions of (27). The result of lemma 3.1 allows us to do this without having first to prove local existence using a fixed point theorem. Let $V_0 \in \mathbb{H}$, $s \in \mathbb{R}$, we consider a sequence $\left\{V_0^n = {}^t\left(u_0^n, u_1^n\right)\right\}_n$ in $\left(\mathcal{C}_0^\infty(\Sigma)\right)^2$ such that

$$V_0^n \longrightarrow V_0 \text{ in } \mathbb{H}$$

and we denote by $V^n = {}^t(u^n, \partial_\tau u^n)$ the solution of (21) in $\left(\mathcal{C}^\infty\left(\mathbb{R}_\tau\,;\mathcal{C}_0^\infty(\Sigma)\right)\right)^2$ such that $V^n(s) = V_0^n$. The existence of V^n is guaranteed by lemma 3.1 and its uniqueness by proposition 3.2. We show that $\{V^n\}_n$ is a Cauchy sequence in $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$. For this purpose, we first need to check that $\{V^n\}_n$ is bounded on $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$. Let $T > 0$, we denote

$$\kappa(s,T) = \max_{\tau\in[s-T,s+T]} K_\lambda(s,\tau)\,.$$

For $\tau \in [s-T, s+T]$ and n in $\mathbb{N}$, we have the energy estimate

$$E(u^n,\tau) \leq \kappa(s,T)E(u^n,s)\,.$$

Now

$$\begin{aligned} E(u^n,s) &= \int_{\Sigma_s}\left(|\partial_\tau u^n|^2 + N^2 h^{\alpha\beta}\partial_\alpha u^n \partial_\beta \overline{u^n} + N^2 m^2 |u^n|^2 + \frac{1}{2}\lambda N^2 |u^n|^4\right) N^{-1}\mathrm{dVol} \\ &\leq C\left(\|V_0^n\|_{\mathcal{H}}^2 + \frac{1}{2}\left\|\lambda N^2 (u_0^n)^3\right\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} \|u_0^n\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})}\right) \end{aligned}$$

and we have

$$\left\|\lambda N^2 (u_0^n)^3\right\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} = \|J(V_0^n)\|_{\mathbb{H}} \leq C_J \|V_0^n\|_{\mathbb{H}}^3\,,\ \ \|u_0^n\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} \leq \|V_0^n\|_{\mathbb{H}}\,.$$

Hence, since $\{V_0^n\}_n$ converges in $\mathbb{H}$, the sequence $\{E(u^n,\tau)\}_n$ is bounded uniformly in $\tau \in [s-T, s+T]$. Besides, the quadratic part of the energy, i.e. the energy E without the quartic term, is (locally uniformly in time) equivalent to the $\mathcal{H}$ norm. We infer that the sequence $\{\|V^n(\tau)\|_{\mathcal{H}}\}_n$ is bounded uniformly in $\tau \in [s-T, s+T]$. In order to control the norm of $V^n(\tau)$ in $\mathbb{H}$, we have yet to control the norm of $u^n(\tau)$ in $L^2(\Sigma\,;N^{-1}\mathrm{dVol})$; we do this exactly as in the proof of the stability of $\mathbb{H}$ under $\mathcal{U}$:

$$\begin{aligned} \|u^n(\tau)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} &\leq \|u^n(s)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})} + \int_{]s,\tau[} \|\partial_\tau u^n(\sigma)\|_{L^2(\Sigma\,;N^{-1}\mathrm{dVol})}\,\mathrm{d}\sigma \\ &\leq \|V_0^n\|_{\mathbb{H}} + \int_{]s,\tau[} \|V^n(\sigma)\|_{\mathcal{H}}\,\mathrm{d}\sigma\,. \end{aligned}$$

It follows that $\{\|V^n(\tau)\|_{\mathbb{H}}\}_n$ is bounded uniformly in $\tau \in [s-T, s+T]$. Let

$$\tilde{C}(s,T) = \sup\left\{\|V^n(\tau)\|_{\mathbb{H}}\,;\ \tau \in [s-T, s+T]\,,\ n \in \mathbb{N}\right\}\,.$$

We then prove that $\{V^n\}_n$ is a Cauchy sequence in $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$. Let $T > 0$, we consider $\tau \in [s-T, s+T]$, n and m in $\mathbb{N}$:

$$\|V^m(\tau) - V^n(\tau)\|_{\mathbb{H}} = \left\|\mathcal{U}(\tau,s)\,(V_0^m - V_0^n) + \int_s^\tau \mathcal{U}(\tau,\sigma)\,(J(V^m(\sigma)) - J(V^n(\sigma)))\,\mathrm{d}\sigma\right\|_{\mathbb{H}}\,.$$

Denoting

$$\tilde{\kappa}(s,T) = \max_{\tau,\sigma\in[s-T,s+T]} \tilde{K}(\sigma,\tau)\,,$$

using lemma 3.2 and the bound on $\{\|V^n(\tau)\|_{\mathbb{H}}\}_n$, we have

$$\begin{aligned} &\|V^m(\tau) - V^n(\tau)\|_{\mathbb{H}} \\ &\quad\leq \tilde{\kappa}(s,T)\left\{\|V_0^m - V_0^n\|_{\mathbb{H}} + 2C_J\left(\tilde{C}(s,T)\right)^2 \int_{]s,\tau[} \|V^m(\sigma) - V^n(\sigma)\|_{\mathbb{H}}\,\mathrm{d}\sigma\right\}\,. \end{aligned}$$

The convergence of $\{V_0^n\}_n$ in $\mathbb{H}$ together with Gronwall's lemma imply that $\{V^n\}_n$ is a Cauchy sequence in $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$. Denoting V the limit of $\{V^n\}_n$ in $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$, we have $J(V^n) \to J(V)$ in $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$ by continuity of J on $\mathbb{H}$. Hence V satisfies (21) in the sense of distributions on $\mathbb{R}_\tau \times \Sigma$, in addition to which $V(s) = V_0$ and $V \in \mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$. Thus, V is a global solution of (27). Note also that V satisfies energy estimate (17) since, by lemma 3.2, the energy E is continuous on $\mathcal{C}(\mathbb{R}_\tau\,;\mathbb{H})$. This concludes the proof of theorem 2. □

4 Asymptotic behaviour of smooth massless fields at the horizon and at infinity

We use Roger Penrose's technique of conformal compactification to prove, for smooth solutions compactly supported in space, the existence of smooth asymptotic profiles at the horizon and at null infinity. This technique requires the equation to be invariant under conformal transformations. Hence, we only consider in this chapter equation (4) for $m = 0$, i.e. the non linear wave equation :

$$\Box_g u + \lambda |u|^2 u = 0\,,\ \lambda \geq 0\,. \tag{31}$$

4.1 Penrose compactification of the exterior of the black hole

The compactification of block I is based on the structure given by the two principal null geodetic congruences. We recall that the outgoing and incoming principal null geodesics are respectively the integral lines of the principal null vector fields, expressed in Boyer-Lindquist coordinates as :

$$L^a \frac{\partial}{\partial x^a} = \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi}\,;$$

$$N^a \frac{\partial}{\partial x^a} = \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi}\,.$$

Two coordinate systems, globally defined on block I using either outgoing or incoming principal null geodesics as coordinate lines, allow us to show the regularity of the metric g across the horizon and to construct a metric $\hat{g}$, conformally equivalent to g, which is smooth on future and past null infinities.

The Kerr-star coordinate system is based on incoming principal null geodesics. The idea is to introduce new coordinates t^* and φ^* of the form

$$t^* = t + T(r)\,,\ \varphi^* = \varphi + \Lambda(r)\,,$$

with the functions T and Λ such that

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{r^2+a^2}{\Delta}\,,\ \frac{\mathrm{d}\Lambda}{\mathrm{d}r} = \frac{a}{\Delta}\,.$$

Kerr-star coordinates $(t^*, r, \theta, \varphi^*)$ are defined globally on block I[5]. The incoming principal null geodesics now appear as the r coordinate curves parametrized by $s = -r$ (or $-r+C$) :

$$\dot{r} = -1\,,\ \dot{\theta} = 0\,,\ \dot{t^*} = \dot{t} + \frac{\mathrm{d}T}{\mathrm{d}r}\dot{r} = 0\,,\ \dot{\varphi^*} = \dot{\varphi} + \frac{\mathrm{d}\Lambda}{\mathrm{d}r}\dot{r} = 0\,.$$

The Kerr metric in Kerr-star coordinates takes the form

$$g = g_{tt}\mathrm{d}t^{*2} + 2g_{t\varphi}\mathrm{d}t^*\mathrm{d}\varphi^* + g_{\varphi\varphi}\mathrm{d}\varphi^{*2} - \rho^2\mathrm{d}\theta^2 - 2\mathrm{d}t^*\mathrm{d}r + 2a\sin^2\theta\,\mathrm{d}\varphi^*\mathrm{d}r\,, \tag{32}$$

[5] With the exception of the axis ($\theta = 0$ and $\theta = \pi$) ; this coordinate singularity, similar to that of spherical coordinates on $\mathbb{R}^3$, can be dealt with simply (see [25] lemma 2.2.2), we shall systematically ignore it.

where g_{tt}, $2g_{t\varphi}$, $g_{\theta\theta}$ and $g_{\varphi\varphi}$ are the coefficients of $\mathrm{d}t^2$, $\mathrm{d}t\mathrm{d}\varphi$, $\mathrm{d}\theta^2$ and $\mathrm{d}\varphi^2$ in the expression (2) of g in Boyer-Lindquist coordinates :

$$g_{tt} = 1 - \frac{2Mr}{\rho^2}\,,\ g_{t\varphi} = \frac{2aMr\sin^2\theta}{\rho^2}\,,\ g_{\theta\theta} = -\rho^2\,,\ g_{\varphi\varphi} = -\frac{\sigma^2}{\rho^2}\sin^2\theta\,.$$

In Boyer-Lindquist coordinates, the only metric coefficient to be singular at the horizon was g_{rr} and it does not appear in (32). It follows that g can be extended smoothly across the horizon $\{r = r_+\}$. Besides, it does not degenerate there ; indeed a simple calculation shows that the determinent of g is given by

$$\det(g) = -\rho^4 \sin^2\theta\,.$$

and does not vanish for $r = r_+$. Thus, we can add the horizon to block I as a smooth boundary[6]. We must however be careful in the interpretation of this gluing. The horizon described in the Kerr-star coordinate system is reached along incoming null geodesics, hence as $t \to +\infty$; it is the horizon that is reached by light rays or even material bodies falling into the black hole and not the horizon seen as $\mathbb{R}\times\partial\Sigma$ in the $3+1$ decomposition of $\mathcal{B}_I$. The hypersurface

$$\mathfrak{H}^+ = \mathbb{R}_{t^*} \times \{r = r_+\} \times S^2_{\theta,\varphi^*}$$

shall be referred to as the future horizon and is a smooth null hypersurface in the space-time $(\mathcal{B}_I \cup \mathfrak{H}^+\,,\ g)$. The fact that $\mathfrak{H}^+$ is null is easily shown considering the metric induced by g on hypersurfaces of constant r :

$$g_r = g_{tt}\mathrm{d}t^{*2} + 2g_{t\varphi}\mathrm{d}t^*\mathrm{d}\varphi^* + g_{\varphi\varphi}\mathrm{d}\varphi^{*2} - \rho^2\mathrm{d}\theta^2\,.$$

This induced metric has determinent

$$\det(g_r) = -\rho^2\left(g_{tt}g_{\varphi\varphi} - (g_{t\varphi})^2\right) = \rho^2\Delta\sin^2\theta$$

and thus degenerates for $\Delta = 0$, i.e. at the horizon[7].

The Kerr-star coordinate system also allows us to construct past null infinity (denoted $\mathfrak{I}^-$, the set of limit points of incoming principal null geodesics as $r \to +\infty$) and to interpret it as a smooth null hypersurface in a conformally rescaled space-time. In the expression (32) of the metric, we change the null coordinate r to $w = 1/r$, then we introduce the rescaled metric

$$\hat{g} = \Omega^2 g\,,\ \Omega = w = \frac{1}{r}\,. \tag{33}$$

In the coordinate system $(t^*, w, \theta, \varphi^*)$, $\hat{g}$ takes the form

$$\begin{aligned}\hat{g} \quad = \quad & \left(w^2 - \frac{2Mw^3}{1+a^2w^2\cos^2\theta}\right)\mathrm{d}t^{*2} + \frac{4Maw^3\sin^2\theta}{1+a^2w^2\cos^2\theta}\,\mathrm{d}t^*\mathrm{d}\varphi^* \\ & - \left(1 + a^2w^2 + \frac{2Ma^2w^3\sin^2\theta}{1+a^2w^2\cos^2\theta}\right)\sin^2\theta\,\mathrm{d}\varphi^{*2} \\ & - \left(1 + a^2w^2\cos^2\theta\right)\mathrm{d}\theta^2 + 2\mathrm{d}t^*\mathrm{d}w - 2a\sin^2\theta\,\mathrm{d}\varphi^*\mathrm{d}w\,.\end{aligned}$$

[6] In fact, the Kerr-star coordinate system is used to glue block II to the future of block I along the outer horizon and also to glue block III to the future of block II along the inner horizon. Our purpose here is merely to study the behaviour of solutions at the horizon and at infinity ; this only requires to glue these hypersurfaces to block I as boundaries.

[7] We have shown that the metric g does not degenerate at points of the hypersurface $\{r = r_+\}$ but its restriction to this hypersurface is degenerate. This shows that the hypersurface is null, i.e. one of its generators is null.

The metric $\hat{g}$ is therefore smooth on the domain

$$\mathbb{R}_{t^*} \times \left[0, \frac{1}{r_+}\right]_w \times S^2_{\theta,\varphi^*}$$

and does not degenerate at $\{w = 0\}$, since

$$\det(\hat{g}) = -w^4\rho^4 \sin^2\theta = -\left(1 + a^2w^2\cos^2\theta\right)^2 \sin^2\theta \neq 0 \text{ for } w = 0\,.$$

Any point $x_0 = (t^*_0, w = 0, \theta_0, \varphi^*_0)$ of $\{w = 0\}$ can be reached along the line

$$\gamma(s) = (t^* = t^*_0, w = s, \theta = \theta_0, \varphi^* = \varphi^*_0)$$

as s decreases to 0 ; this line is the incoming principal null geodesic corresponding to $t^* = t^*_0$, $\theta = \theta_0$ and $\varphi^* = \varphi^*_0$ parametrized by $s = 1/r$. Hence, past null infinity is described in coordinates $(t, w, \theta, \varphi^*)$ as

$$\mathfrak{I}^- = \mathbb{R}_{t^*} \times \{w = 0\} \times S^2_{\theta,\varphi^*}$$

and can be added to our rescaled space-time as a smooth boundary. The restriction of $\hat{g}$ to the hypersurface $\{w = 0\}$

$$\hat{g}_{|w=0} = -\mathrm{d}\theta^2 - \sin^2\theta\,\mathrm{d}\varphi^{*2}$$

is simply the opposite of the euclidian metric on the 2-sphere and is consequently degenerate since $\mathfrak{I}^-$ is a 3-surface. Thus, in the space-time $(\mathcal{B}_I \cup \mathfrak{H}^+ \cup \mathfrak{I}^-\,,\ \hat{g})$, $\mathfrak{I}^-$ is a smooth null hypersurface.

Working now with outgoing instead of incoming principal null geodesics, we define star-Kerr coordinates $({}^*t, r, \theta, {}^*\varphi)$, where the new variables ${}^*t$ and ${}^*\varphi$ are defined by

$${}^*t = t - T(r)\,,\ {}^*\varphi = \varphi - \Lambda(r)\,,$$

T and Λ being the same functions as in the definition of t^* and φ^*. Analogous constructions can be followed through with this coordinate system in which the outgoing principal null geodesics appear as the r coordinate lines. We thus define the past horizon[8]

$$\mathfrak{H}^- = \mathbb{R}_{{}^*t} \times \{r = r_+\} \times S^2_{\theta,{}^*\varphi}$$

and future null infinity $\mathfrak{I}^+$, the set of limit points of outgoing principal null geodesics as $r \to +\infty$, also described as

$$\mathfrak{I}^+ = \mathbb{R}_{{}^*t} \times \{w = 0\} \times S^2_{\theta,{}^*\varphi}\,.$$

We obtain a conformally "compactified" space-time

$$\left(\mathcal{B}_I \cup \mathfrak{H}^+ \cup \mathfrak{H}^- \cup \mathfrak{I}^+ \cup \mathfrak{I}^-\,,\ \hat{g}\right)$$

that is a smooth Lorentzian manifold with boundary $\mathfrak{H}^+ \cup \mathfrak{H}^- \cup \mathfrak{I}^+ \cup \mathfrak{I}^-$, reunion of four smooth null hypersurfaces. This space-time is not compact however since the boundary is not complete ; four "points", or 2-spheres, have been missed out :

[8]The past horizon is the horizon from which light rays or material bodies can emerge, the horizon of a white hole rather than a black hole. White holes appear necessarily in the construction of maximal analytic extensions of eternal black hole space-times such as Schwarzschild or Kerr. See [24] or [25] for a more complete description of maximal Kerr space-time.

- the part of the horizon described in Boyer-Lindquist coordinates as

$$\mathfrak{H}_0 = \mathbb{R}_t \times \{r = r_+\} \times S^2_{\theta,\varphi}$$

can be understood as a single 2-sphere, called the crossing sphere, where the future and past horizons meet ; the metric g (and consequently also $\hat{g}$) is regular at this crossing sphere (see [25] for details) ;

- spacelike infinity, denoted i_0, is the set of limit points of spacelike lines (future or past oriented) going out to infinity at a speed uniformly greater than the speed of light, typical examples being the integral lines of the vector fields

$$\pm\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t} + C\frac{\partial}{\partial r} \pm \frac{a}{\Delta}\frac{\partial}{\partial \varphi}\,,\ C > 1\,.$$

In the rescaled space-time, i_0 is a singular point where $\mathfrak{I}^+$ and $\mathfrak{I}^-$ meet ; at best, a rescaled metric is continuous at i_0 and its derivative admits direction dependent limits there[9];

- future timelike infinity i_+ is the set of limit points of future oriented lines going out to infinity at a speed uniformly lower than the speed of light ; for example the integral lines of the vector fields

$$\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t} + C\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi}\,,\ 0 < C < 1\,;$$

in the rescaled space-time, i_+ is a singularity, much more serious than i_0, where two hypersurfaces of a very different nature ($\mathfrak{I}^+$ and $\mathfrak{H}^+$) meet ; past timelike infinity i_- plays a symmetric role in the infinite past.

We define the Penrose compactification of the exterior of a slow Kerr black hole as $\mathcal{B}_I$ to which have been glued the complete horizon $\mathfrak{H}^+ \cup \mathfrak{H}_0 \cup \mathfrak{H}^-$ as well as $\mathfrak{I}^+$ and $\mathfrak{I}^-$, equipped with the smooth metric $\hat{g}$. We denote it

$$\left(\overline{\mathcal{B}_I}\,,\ \hat{g}\right)\,,\ \overline{\mathcal{B}_I} = \mathcal{B}_I \cup \mathfrak{H}^+ \cup \mathfrak{H}_0 \cup \mathfrak{H}^- \cup \mathfrak{I}^+ \cup \mathfrak{I}^-\,.$$

It can be schematically represented as in figure 1.

4.2 Asymptotic profiles and radiation conditions for smooth solutions

Equation (31) is conformally invariant. This means that the two following propositions are equivalent :

1. $u \in \mathcal{C}^\infty(\mathcal{B}_I)$ is a solution of (31) ;
2. $\hat{u} = \Omega^{-1}u = ru \in \mathcal{C}^\infty(\mathcal{B}_I)$ is a solution of

$$\Box_{\hat{g}}\hat{u} + \frac{1}{6}R_{\hat{g}}\hat{u} + \lambda\,|\hat{u}|^2\,\hat{u} = 0 \tag{34}$$

where $R_{\hat{g}}$ is the scalar curvature associated with the metric $\hat{g}$ (recall that g being a solution to Einstein's vacuum equations, R_g is zero, which is the reason why it does not appear in (31)).

[9] Whenever a space-time contains energy, which is characterized by a non zero ADM mass, we cannot expect the rescaled metric to be more than Lipschitz at spacelike infinity (see [27] Vol. II or [29]).

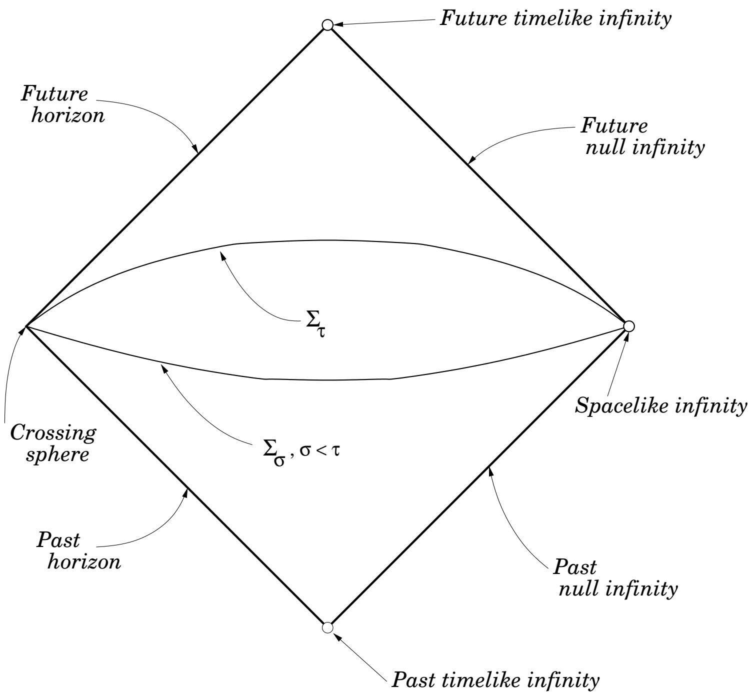

Figure 1: The Penrose diagram of block I

The metric $\hat{g}$ being smooth on $\overline{\mathcal{B}_I}$, so is $R_{\hat{g}}$, but both $\hat{g}$ and $R_{\hat{g}}$ become singular at i_0, i_+ and i_-. In order to work on a smooth space-time, we remove neighbourhoods of i_0, i_+ and i_-. The construction is based on the hypersurface Σ_0 ; this is in no way a special choice, the same construction can be performed based on any other hypersurface Σ_s. We consider in the physical space-time $(\mathcal{B}_I, g)$ a smooth inextendible timelike curve $\{\gamma(t)\}_{t\in\mathbb{R}}$ parametrized by the time variable of the Boyer-Lindquist coordinates. In the compactified space-time, the curve $\gamma(t)$ runs from i_- to i_+ as t varies from $-\infty$ to $+\infty$. For $\varepsilon > 0$, we remove from $\overline{\mathcal{B}_I}$ the future of $\gamma(1/\varepsilon)$ and the past of $\gamma(-1/\varepsilon)$. We also remove (assuming $\varepsilon < 1/r_+$) the complement of the domain of dependence of the compact subset of Σ_0 :

$$K_\varepsilon = \left\{(t, r, \theta, \varphi) \,; t = 0, r_+ \leq r \leq 1/\varepsilon, (\theta, \varphi) \in S^2\right\} .$$

We obtain a space-time $(\mathcal{M}_\varepsilon \, , \hat{g})$, shown in figure 2, that can be extended (see figure 3) as a smooth, globally hyperbolic, spatially compact space-time $\left(\mathbb{R}_\sigma \times S^3 \,; G_\varepsilon\right)$, $G_\varepsilon|_{\mathcal{M}_\varepsilon} = \hat{g}$, the time variable σ being chosen so that $K_\varepsilon \subset \{0\}_\sigma \times S^3$. We now consider u in $\mathcal{C}^\infty\left(\mathbb{R}_\tau \,; \mathcal{C}_0^\infty\left(\Sigma\right)\right)$ a solution of (31). Then $\hat{u} = ru$ is a smooth solution of (34) on $\mathcal{B}_I$. We choose $\varepsilon > 0$ small enough so that K_ε contains the supports of $u(0)$ and $\partial_\tau u(0)$, in this manner, the cut-off near i_0 does not alter the solution u. Then, for $\sigma_0 > 0$ not too large[10], the function ψ on $[-\sigma_0, \sigma_0] \times S^3$ obtained by extending $\hat{u}$ by zero in $\left([-\sigma_0, \sigma_0] \times S^3\right) \setminus \mathcal{M}_\varepsilon$ is a solution in $\mathcal{C}^\infty\left([-\sigma_0, \sigma_0] \,; \mathcal{C}^\infty(S^3)\right)$ of equation

$$\Box_{G_\varepsilon} \psi + \frac{1}{6} R_{G_\varepsilon} \psi + \lambda \left|\psi\right|^2 \psi = 0 \, . \tag{35}$$

The results of [7] guarantee that ψ can be extended in a unique manner as a solution of (35) in $\mathcal{C}^\infty\left(\mathbb{R}_\sigma \,; \mathcal{C}^\infty(S^3)\right)$. Moreover, by uniqueness of solutions of (31) in $\mathcal{B}_I$, ψ coincides

[10] We simply need to choose $\sigma_0 > 0$ small enough so that, for $\sigma \in [-\sigma_0, \sigma_0]$, the support of $\hat{u}$ does not touch the boundary of $\mathcal{M}_\varepsilon$.

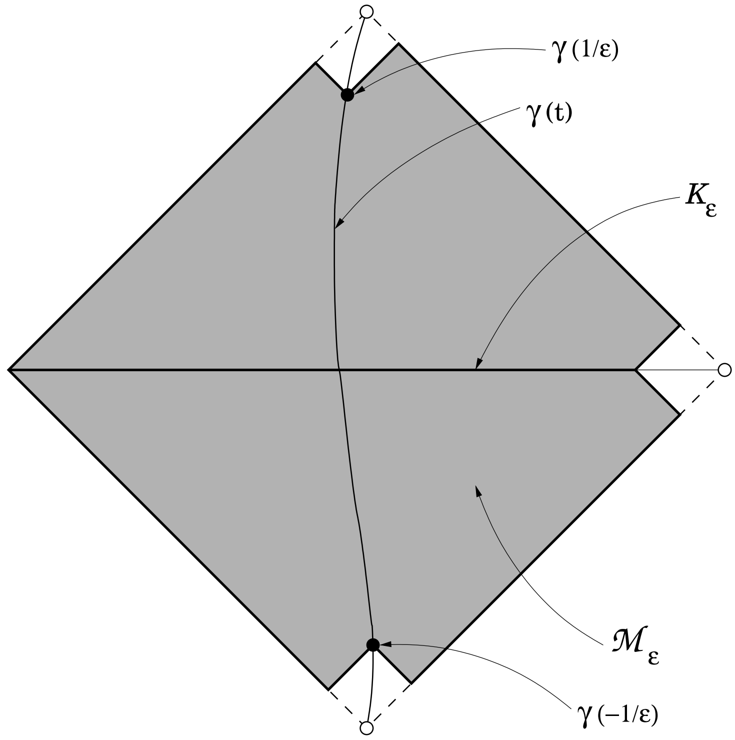

Figure 2: The construction of the space-time $\mathcal{M}_\varepsilon$

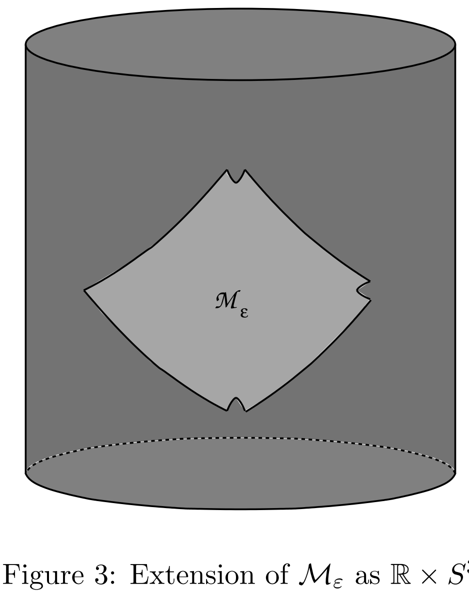

Figure 3: Extension of $\mathcal{M}_\varepsilon$ as $\mathbb{R} \times S^3$

with $\hat{u}$ on $\mathcal{M}_\varepsilon \cap \mathcal{B}_I$. This proves that $\hat{u}$ is smooth on $\mathcal{M}_\varepsilon$ for any $\varepsilon > 0$, whence the smoothness of $\hat{u}$ on $\overline{\mathcal{B}_I}$.

From this property of $\hat{u}$, we can infer two types of properties for u : the existence of asymptotic profiles at $\mathfrak{H}^\pm$ and $\mathfrak{I}^\pm$ and the fact that u satisfies Sommerfeld radiation conditions there. This is expressed more precisely in the following theorem :

Theorem 3 *Let $u \in \mathcal{C}^\infty(\mathbb{R}_t ; \mathcal{C}_0^\infty(\Sigma))$ be a solution of (31). There exists four smooth functions*

$$\Phi_{\mathfrak{H}^+}\,,\ \Phi_{\mathfrak{H}^-}\,,\ \Phi_{\mathfrak{I}^+}\,,\ \Phi_{\mathfrak{I}^-}\ \in \mathcal{C}^\infty\left(\mathbb{R}\times S^2\right)$$

defined simply as the traces of $\hat{u} = ru$ on $\mathfrak{H}^+$, $\mathfrak{H}^-$, $\mathfrak{I}^+$ and $\mathfrak{I}^-$, such that, if we denote $\gamma^-_{s,\omega}$, $s \in \mathbb{R}$, $\omega \in S^2$, the incoming principal null geodesic (parametrized by r) described in Kerr-star coordinates as

$$\gamma^-_{s,\omega}(r) = (t^* = s, r, (\theta, \varphi^*) = \omega)$$

and $\gamma^+_{s,\omega}$, $s \in \mathbb{R}$, $\omega \in S^2$, the outgoing principal null geodesic (parametrized by r) described in star-Kerr coordinates as

$$\gamma^+_{s,\omega}(r) = ({}^*t = s, r, (\theta, {}^*\varphi) = \omega)\ ,$$

we have for all $(s,\omega) \in \mathbb{R}\times S^2$

$$\lim_{r\to r_+} u\left(\gamma^-_{s,\omega}(r)\right) = \frac{1}{r_+}\Phi_{\mathfrak{H}^+}(s,\omega)\,, \tag{36}$$

$$\lim_{r\to +\infty} r\,u\left(\gamma^-_{s,\omega}(r)\right) = \Phi_{\mathfrak{I}^-}(s,\omega)\,, \tag{37}$$

$$\lim_{r\to r_+} u\left(\gamma^+_{s,\omega}(r)\right) = \frac{1}{r_+}\Phi_{\mathfrak{H}^-}(s,\omega)\,, \tag{38}$$

$$\lim_{r\to +\infty} r\,u\left(\gamma^+_{s,\omega}(r)\right) = \Phi_{\mathfrak{I}^+}(s,\omega)\,. \tag{39}$$

Moreover, u satisfies the radiation conditions, expressed in Boyer-Lindquist coordinates and using the Regge-Wheeler type coordinate r_* defined in (15) :

$$\lim_{r\to r_+}\left(\partial_t u + \partial_{r_*}u + \frac{a}{r^2+a^2}\partial_\varphi u\right)\left(\gamma^+_{s,\omega}(r)\right) = 0\,, \tag{40}$$

$$\lim_{r\to +\infty}\left(\partial_t u + \partial_{r_*}u\right)\left(\gamma^+_{s,\omega}(r)\right) = 0\,, \tag{41}$$

$$\lim_{r\to r_+}\left(\partial_t u - \partial_{r_*}u + \frac{a}{r^2+a^2}\partial_\varphi u\right)\left(\gamma^-_{s,\omega}(r)\right) = 0\,, \tag{42}$$

$$\lim_{r\to +\infty}\left(\partial_t u - \partial_{r_*}u\right)\left(\gamma^-_{s,\omega}(r)\right) = 0\,. \tag{43}$$

Remark 4.1 *Using the smoothness of $\frac{\partial u}{\partial \varphi}$ on $\overline{\mathcal{B}_I}$, established in the proof below, we can simplify (40) and (42) to*

$$\lim_{r\to r_+}\left(\partial_t u + \partial_{r_*}u + \frac{a}{r_+^2+a^2}\partial_\varphi u\right)\left(\gamma^+_{s,\omega}(r)\right) = 0\,,$$

$$\lim_{r\to r_+}\left(\partial_t u - \partial_{r_*}u + \frac{a}{r_+^2+a^2}\partial_\varphi u\right)\left(\gamma^-_{s,\omega}(r)\right) = 0\,.$$

This shows that the radiation conditions at the horizon differ from those at infinity by a rotation imposed on the field, the angular velocity of this rotation, $\frac{a}{r_+^2+a^2}$, being exactly the rotation speed of the horizon as seen by an observer static at infinity (see for example [29]).

Proof of theorem 3 : the part concerning the asymptotic profiles is simply a precise expression of the existence of smooth traces for $\hat{u}$ on $\mathfrak{H}^\pm$ and $\mathfrak{I}^\pm$; using the definitions of these hypersurfaces in terms of Kerr-star and star-Kerr corrdinates, the traces $\Phi_{\mathfrak{H}^+}$, $\Phi_{\mathfrak{H}^-}$, $\Phi_{\mathfrak{I}^+}$ and $\Phi_{\mathfrak{I}^-}$ are naturally defined as functions on $\mathbb{R} \times S^2$, pointwise limits of $\hat{u} = ru$ along principal null geodesics. The radiation conditions are consequences of the regularity of $\hat{u}$ on $\overline{\mathcal{B}_I}$. We prove conditions (40) and (41), the proof for (42) and (43) is similar using incoming instead of outgoing principal null geodesics.

Working in star-Kerr coordinates with $w = 1/r$, we know that $\frac{\partial \hat{u}}{\partial w}$ must be smooth on $\overline{\mathcal{B}_I}$ since $\hat{u}$ is. Now

$$w \frac{\partial \hat{u}}{\partial w} = -r^2 \frac{\partial u}{\partial r} - \hat{u} \,.$$

This implies that $r^2 \frac{\partial u}{\partial r}$ must be smooth on $\overline{\mathcal{B}_I}$ and hence

$$\frac{\partial u}{\partial r} = O\left(\frac{1}{r^2}\right) \text{ as } r \to +\infty \,.$$

Going back to Boyer-Lindquist coordinates, this gives for each $(s, \omega) \in \mathbb{R} \times S^2$:

$$\left(\frac{r^2 + a^2}{\Delta} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{a}{\Delta} \frac{\partial u}{\partial \varphi}\right) \left(\gamma^+_{s,\omega}(r)\right) = O\left(\frac{1}{r^2}\right) \text{ as } r \to +\infty \,.$$

We can multiply the whole quantity by $\frac{\Delta}{r^2+a^2}$ without changing the rate of fall-off. Since $u = w\hat{u}$ is smooth on $\overline{\mathcal{B}_I}$ and since $\frac{\partial}{\partial^* \varphi}$ is a smooth vector field on $\overline{\mathcal{B}_I}$ (essentially because in standard spherical coordinates $\frac{\partial}{\partial \varphi}$ is a smooth vector field on S^2), $\frac{\partial u}{\partial^* \varphi}$ is also smooth on $\overline{\mathcal{B}_I}$. In terms of Boyer-Lindquist coordinates, this simply means that $\frac{\partial u}{\partial \varphi}$ is smooth on $\overline{\mathcal{B}_I}$ and is therefore uniformly bounded on each principal null geodesic. Hence, we obtain

$$\left(\frac{\partial u}{\partial t} + \frac{\Delta}{r^2 + a^2} \frac{\partial u}{\partial r}\right) \left(\gamma^+_{s,\omega}(r)\right) = O\left(\frac{1}{r^2}\right) \text{ as } r \to +\infty \,,$$

which in turn implies (41).

The proof of (40) is even simpler. Since $\hat{u}$ is smooth on $\overline{\mathcal{B}_I}$, we have in star-Kerr coordinates that $\frac{\partial \hat{u}}{\partial w}$ is smooth on $\overline{\mathcal{B}_I}$, whence $\frac{\partial u}{\partial r}$ is smooth at $\mathfrak{H}^-$. This can be expressed in terms of Boyer-Lindquist coordinates as the property that, for each $(s, \omega) \in \mathbb{R} \times S^2$, the quantity

$$\left(\frac{r^2 + a^2}{\Delta} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{a}{\Delta} \frac{\partial u}{\partial \varphi}\right) \left(\gamma^+_{s,\omega}(r)\right)$$

has a finite limit as r tends to r_+. Multiplying the previous expression by $\frac{\Delta}{r^2+a^2}$, which tends to 0 as r tends to r_+, we obtain (40). This concludes the proof of theorem 3. $\square$

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