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Non linear Klein-Gordon equation on Schwarzschild-like metrics

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Résumé: On résout le problème de Cauchy global pour une equation non linéaire de Klein-Gordon à l'extérieur d'un trou noir sphérique. On établit l'existence à l'horizon d'un champ de radiation rentrante (condition d'impédance de T.DAMOUR). Dans le cas d'un espace-temps asymptotiquement plat, les champs sans masse vérifient à l'infini la condition de radiation sortante de Sommerfeld.

Abstract: We solve the global Cauchy problem for a non linear Klein-Gordon equation outside a spherical Black-Hole. On the horizon of the Black-Hole, the fields satisfy T.DAMOUR's impedance condition. When space-time is asymptotically flat, massless fields satisfy Sommerfeld's outgoing condition at infinity.

Introduction

In Minkowski space-time, global solutions of non-linear Klein-Gordon equation $\Box u + m^2 u + \lambda |u|^2 u = 0$, $\lambda \geq 0$ have a well known asymptotic behavior (see for example J. Ginibre and G. Velo [10] or W. Strauss [13]). Inflationary cosmological scenarii of quantum cosmology with symetry break of Higgs's fields make this equation very interesting to study in a curved space-time. The Cauchy problem on a globally hyperbolic regular manifold $\mathbb{R}_t \times V_x$ has been solved by F.Cagnac and Y.Choquet-Bruhat [4]. The purpose of this paper is to prove similar results in space-times defined by a spherical black-hole, i.e. the curves $x = cst$ are not uniformly time-like near a horizon. The Cauchy problem for the Yang-Mills system in the Schwarzschild metric has been worked out by W.T.Shu [11] using very delicate methods developped by D.Christodoulou and S.Klainerman. These methods require a massless field as well as small and very regular initial data. We deal here with the simpler case of Klein-Gordon equation. We adopt an approach already used in [4][5][6]. A Sobolev embedding result on a Riemannian manifold and a conserved energy allow us to establish the global existence for arbitrarily big solutions with the least possible regularity. Then we use Kruskal coordinates to study the asymptotic behavior at the horizon and Penrose coordinates for the behavior at infinity.

Let us consider the manifold $\mathbb{R}_t \times]0, +\infty[_r \times S^2_{\theta,\phi}$ endowed with the pseudo-riemannian metric

$$g_{\mu\nu} dx^\mu dx^\nu = F(r) e^{2\delta(r)} dt^2 - [F(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2] \tag{1}$$

where $F, \delta \in C^\infty(]0, +\infty[_r)$. We assume the existence of three values r_ν of r, $0 \leq r_- < r_0 < r_+ \leq +\infty$, the only possible zeros of F, such that

$$F(r_\nu) = 0 \quad F'(r_\nu) = 2\kappa_\nu \quad \kappa_\nu \neq 0 \quad if\ 0 < r_\nu < +\infty$$
$$F(r) > 0\ for\ r \in]r_0, r_+[\quad F(r) < 0\ for\ r \in]r_-, r_0[$$

When they are finite and non zero, r_-, r_0 and r_+ are the radii of the spheres called: horizon of the black hole (r_0), Cauchy horizon (r_-) and cosmological horizon (r_+). These horizons are fictitious singularities which can be removed by Kruskal-Szekeres transformations. κ_ν is the surface gravity at the horizon $\{r = r_\nu\}$. If r_+ is infinite, we assume moreover that

$$F(r) = 1 - \frac{r_1}{r} + O\left(r^{-2}\right) \quad , \ r_1 > 0 \ , \quad \delta(r) = \delta(+\infty) + o(r^{-1}) \qquad r \to +\infty$$
$$k = 1, 2, 3 \ , \quad \partial_r^k F(r) \ , \ \partial_r^k \delta(r) \quad = O(r^{-k-1}) \ , \quad r \to +\infty$$

All these properties are satisfied by usual spherical black holes, described by functions F and δ of the following type

$$\delta = 0,\ F(r) = 1 - \frac{2M}{r} + \gamma_{pq} Q^p Q^q r^{-2} + \frac{\Lambda r^2}{3} \tag{2}$$

In other words, a convenient choice of the coefficients in the previous expression gives any usual solution like Schwarzschild, Reissner-Nordstrøm, Yasskin, asymptotically flat ($\Lambda = 0$) or DeSitter ($\Lambda \neq 0$). M, Q^p, γ_{pq} and Λ must be held to be the mass of the black-hole, the gauge charge, the invariant metric on the Lie group and the cosmological constant. The function δ allows us to include in our study the case of non-abelian chromatic black-holes associated with the SU(2) gauge group. These black-holes were discovered numerically by P.Bizon [3] and their existence has been proved by J.A. Smoller, A.G. Wasserman and T.S.Yau [12].

We study the nonlinear equation of scalar fields with mass $m \geq 0$ and spin zero

$$\Box_g u + m^2 u + \xi R u + \lambda |u|^2 u = 0\ ,\ \ \lambda \geq 0\ ,\ \ \xi \in \mathbb{R} \tag{3}$$

where $\Box_g$ and R are the wave operator and the scalar curvature associated with metric (1)

$$\Box_g = g^{-\frac{1}{2}} \frac{\partial}{\partial x^\alpha} \left(g^{\frac{1}{2}} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right)\ \ ,\ \ g = |det(g_{\alpha\beta})| = e^{2\delta} r^4 sin^2\theta \tag{4}$$

and

$$R = F'' + 3F'\delta' + \frac{4F'}{r} + F\left[2\delta'' + 2\delta' + \frac{4\delta'}{r} + \frac{2}{r^2}\right] - \frac{2}{r^2} \tag{5}$$

In order to make long range interactions disappear in the case of massless fields and to straighten radial null geodesics, we introduce the Regge-Wheeler coordinate r_* such that

$$\frac{dr}{dr_*} = Fe^\delta \tag{6}$$

An elementary calculation shows that function $f = ru$ satisfies

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r_*^2} + Fe^{2\delta}\left[\frac{-\Delta_{S^2}}{r^2} + m^2 + r^{-1}e^{-\delta}\frac{d}{dr}(Fe^\delta) + \xi R + \frac{\lambda}{r^2}|f|^2\right] f = 0 \tag{7}$$

where Δ_{S^2} is the Laplace-Beltrami operator on the sphere S^2. Since we want the energy to be positive outside the black-hole, we make the hypothesis that the linear potential in equation (3) is positive between r_0 and r_+

$$m^2 + r^{-1}e^{-\delta}\tfrac{d}{dr}(Fe^\delta) + \xi R > 0 \text{ for } r \in [r_0, r_+] \tag{H1}$$

This condition is satisfied by any kind of black hole given by (2) for $m = 0$ and $\xi = -1/6$ if $\Lambda \neq 0$. In the case of coloured black holes ([3][12]), we have checked $(H1)$ using numerical experiments.

Notations: Let (M, g) be a Riemannian manifold, $\mathcal{C}_0^\infty(M)$ denotes the set of $\mathcal{C}^\infty$functions with compact support in M, $H^k(M,g)$, $k \in \mathbb{N}$ is the Sobolev space, completion of $\mathcal{C}_0^\infty(M)$ for the norm

$$\|f\|^2_{H^k(M)} = \sum_{j=0}^{k} \int_M \left\langle \nabla^j f, \nabla^j f \right\rangle d\mu$$

where ∇^j, $d\mu$ and $<,>$ are respectively the covariant derivatives, the measure of volume and the scalar product associated with metric g. We write $L^2(M,g) = H^0(M,g)$.

If E is a distribution space on M, E_{comp} represents the subspace of elements of E with compact support in M.

The 2-dimensional euclidian sphere S^2_ω is endowed with its usual metric

$$d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2\ ,\ \ 0 \leq \theta \leq \pi\ ,\ \ 0 \leq \varphi < 2\pi$$

1 Linear Klein-Gordon equation

Through a method similar to J.Dimock's [8], we study the linear version of equation (7)

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r_*^2} + Fe^{2\delta}\left[\frac{-\Delta_{S^2}}{r^2} + m^2 + r^{-1}e^{-\delta}\frac{d}{dr}(Fe^{\delta}) + \xi R\right] f = 0 \tag{8}$$

its hamiltonian form is

$$\frac{\partial U}{\partial t} = -HU \tag{9}$$

where

$$U = \begin{pmatrix} f \\ \partial_t f \end{pmatrix} \quad , \quad H = \begin{pmatrix} 0 & -1 \\ h & 0 \end{pmatrix} \quad , \quad h = -\frac{\partial^2}{\partial r_*^2} + Fe^{2\delta}\left[\frac{-\Delta_{S^2}}{r^2} + m^2 + r^{-1}e^{-\delta}\frac{d}{dr}(Fe^{\delta}) + \xi R\right] \tag{10}$$

We prove that H is skew adjoint on a Hilbert space. Let us define the Hilbert space $\mathcal{K}$ by

$$\mathcal{K} = L^2(\Pi; dr_*^2 + d\omega^2) \simeq L^2(\mathbb{R}; dr_*^2) \otimes L^2(S^2; d\omega^2) \quad , \quad \Pi = \mathbb{R}_{r_*} \times S^2_\omega \tag{11}$$

Let Y_{lm} be the basis of spherical harmonics for $L^2(S^2; d\omega^2)$ satisfying

$$-\Delta_{S^2} Y_{lm} = l(l+1) Y_{lm} \ for \ |m| \leq l \ , \ \|Y_{lm}\|_{L^2} = 1$$

We write $\mathcal{K}$ as

$$\mathcal{K} = \underset{l,m}{\oplus} \mathcal{K}_{lm} \quad , \quad \mathcal{K}_{lm} = L^2(\mathbb{R}) \otimes Y_{lm} \tag{12}$$

and for each element of $\mathcal{K}$, the following decomposition is unique

$$f = \sum_{l,m} f_{lm} \otimes Y_{lm} \quad , \quad f_{lm} \in L^2(\mathbb{R}; dr_*^2) \tag{13}$$

On each $\mathcal{C}_0^\infty(\mathbb{R}) \otimes Y_{lm} = \mathcal{K}_{lm} \cap \mathcal{C}_0^\infty(\Pi)$

$$h = h_{lm} \otimes 1 \quad , \quad h_{lm} = -\frac{d^2}{dr_*^2} + V_l \quad , \quad V_l = Fe^{2\delta}\left[\frac{l(l+1)}{r^2} + m^2 + r^{-1}e^{-\delta}\frac{d}{dr}(Fe^{\delta}) + \xi R\right] \tag{14}$$

h is easily prolonged to

$$D(h) = \left\{ f = \sum_{l,m} f_{lm} \otimes Y_{lm} \in \mathcal{K};\ \partial^2_{r_*} f_{lm} \in L^2(\mathbb{R}; dr_*^2); \sum_{lm} \|f_{lm}\|^2_{L^2(\mathbb{R}; dr_*^2)} < \infty \right\} \tag{15}$$

as

$$h = \sum_{l,m} h_{lm} \otimes 1 \tag{16}$$

and is self-adjoint on $D(h)$ by Kato's theorem. The V_l being positive by $(H1)$, h is strictly positive on $D(h)$. h is therefore inversible with dense domain and so is $\mu = h^{1/2}$. Let $D(\mu)$ be the domain of μ in $\mathcal{K}$

$$D(\mu) = \{f \in \mathcal{K}; \mu f \in \mathcal{K}\} \tag{17}$$

and $[D(\mu)]$ its completion for the inner product associated with the norm $\|\mu f\|_{\mathcal{K}}$. We define the Hilbert space

$$\mathcal{H}_0 = [D(\mu)] \oplus \mathcal{K} \tag{18}$$

completion of $[\mathcal{C}_0^\infty(\Pi)]^2$ for the norm

$$\| \,^t(f,p)\|^2_{\mathcal{H}_0} = \int_\Pi \left\{ \left|\frac{\partial f}{\partial r_*}\right|^2 + Fe^{2\delta}\left[\frac{|\nabla_{S^2} f|^2}{r^2} + \left(m^2 + r^{-1}e^{-\delta}\frac{d}{dr}(Fe^{\delta}) + \xi R\right)|f|^2\right] + |p|^2 \right\} dr_* d\omega \tag{19}$$

Let us consider

$$[D(\mu^2)] = \{f \in [D(\mu)] \ ; \ \mu f \in D(\mu)\} \tag{20}$$

and the dense subspace of $\mathcal{H}_0$

$$D(H) = [D(\mu^2)] \oplus D(\mu) \tag{21}$$

Then operator H with domain $D(H)$ est skew adjoint on $\mathcal{H}_0$. We apply Stone's theorem and obtain the following result

Proposition 1.1. *Given* ${}^t(\varphi, \psi) \in \mathcal{H}_0$*, equation (8) has a unique solution* f *such that*

$${}^t(f, \partial_t f) \in \mathcal{C}(\mathbb{R}_t, \mathcal{H}_0) \qquad {}^t(f, \partial_t f) \mid_{t=0} \ = \ {}^t(\varphi, \psi)$$

defined by

$${}^t(f, \partial_t f) = e^{-Ht} \left[\ {}^t(\varphi, \psi) \right]$$

Moreover, the linear energy of f *is conserved*

$$\left\| \ {}^t(f, \partial_t f)\big|_{t=T} \right\|_{\mathcal{H}_0} = \left\| \ {}^t(\varphi, \psi) \right\|_{\mathcal{H}_0} \quad \forall T \in \mathbb{R}$$

2 Non linear Klein-Gordon equation

The first task is to define a functional framework suitable to our non linear problem; we observe the existence of a conserved positive energy: if f is a regular solution of equation (7) with compact support in $[0,T]_t \times \mathbb{R}_{r_*} \times S^2_\omega$, we multiply (7) by $\partial_t f$, integrate by parts and we obtain

$$\mathcal{E}(U(T)) = \|U(T)\|^2_{\mathcal{H}_0} + \int_\Pi F e^{2\delta} \frac{\lambda}{2r^2} |f(T)|^4 dr_* d\omega = \mathcal{E}(U(0)) \tag{22}$$

where $U = \ {}^t(f, \partial_t f)$. Although we only need to control the solution in $\mathcal{H}_0$ and L^4 to define $\mathcal{E}$, we want to estimate the L^2 norm of the non linear term, i.e. the L^6 norm of f. This is merely to be able to use Picard's method which will naturally assure unicity and regularity. It is therefore convenient to introduce a complete subspace of $\mathcal{H}_0$ for which the required Sobolev embedding result holds. We define the Hilbert space $\mathcal{H}$, completion of $\left[\mathcal{C}^\infty_0(]r_0, r_+[_r \times S^2_\omega)\right]^2$ for the norm

$$\left\| \ {}^t(\varphi, \psi) \right\|^2_{\mathcal{H}} = \int_\Pi \left\{ \left| \frac{\partial \varphi}{\partial r_*} \right|^2 + F \frac{|\nabla_{S^2} \varphi|^2}{r^2} + |\varphi|^2 + |\psi|^2 \right\} dr_* d\omega \tag{23}$$

The hamiltonian form of equation (7) is

$$\frac{\partial U}{\partial t} = -HU - J(U) \tag{24}$$

where H and h are defined in (10) and

$$J(U) = \left[\begin{array}{c} 0 \\ \lambda \frac{F e^{2\delta}}{r^2} |f|^2 f \end{array} \right] \tag{25}$$

We now give the main theorem of this paragraph

Theorem 2.1. *Given* ${}^t(\varphi, \psi)$ *in* $\mathcal{H}$*, equation (7) has a unique solution such that*

$${}^t(f, \partial_t f) \in \mathcal{C}(\mathbb{R}_t, \mathcal{H}) \tag{26}$$

$${}^t(f, \partial_t f) \mid_{t=0} \ = \ {}^t(\varphi, \psi) \tag{27}$$

Furthermore, for any $T \in \mathbb{R}$

$$\mathcal{E} \left(\ {}^t(f, \partial_t f)\big|_{t=T} \right) = \mathcal{E}(\ {}^t(\varphi, \psi)) \tag{28}$$

Remark 2.1. *The study of asymptotic behavior at the horizon (part 4) will show that for initial data* φ *and* ψ *in* $\mathcal{C}^\infty(\mathbb{R}_{r_*}, \times S^2_\omega)$*, equation (7) has a unique solution* $f \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega)$ *satisfying (27).*

Proof: We define a transformation S on $\mathcal{C}(\mathbb{R}_t;\mathcal{H})$ by

$$SU(t) = e^{-Ht}U_0 - \int_{(0,t)} e^{-H(t-s)}J(U(s))ds;\ U \in \mathcal{C}(\mathbb{R}_t;\mathcal{H}),\ t \in \mathbb{R} \tag{29}$$

where $U_0 = {}^t(\varphi, \psi)$. We wish to find $U \in \mathcal{C}(\mathbb{R}_t;\mathcal{H})$ such that

$$U = SU \ in\ \mathcal{C}(\mathbb{R}_t;\mathcal{H}) \tag{30}$$

In order to prove that (30) is equivalent to the Cauchy problem for equation (24) and to obtain the existence and unicity of its solutions, we realize e^{-Ht} as a strongly continuous group of bounded operators on $\mathcal{H}$ and give a Sobolev embedding result.

Lemma 2.1. *$\mathcal{H}$ is invariant under e^{-Ht} and $e^{-Ht}\mid_{\mathcal{H}}$ is a strongly continuous group of bounded operators on $\mathcal{H}$ satisfying*

$$\left\|e^{-Ht}\right\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq C_0(1+|t|) \tag{31}$$

Proof of lemma 2.1: Since e^{-Ht} is a unitary group on $\mathcal{H}_0$, it suffices to prove that for U_0 in $\mathcal{H}$

$$\left(t \mapsto e^{-Ht}U_0\right) \in \mathcal{C}(\mathbb{R}_t, \mathcal{H}) \tag{32}$$

Let $V_0 \in [\mathcal{C}_0^\infty(]r_0, r_+[\times S^2)]^2$, and f be the first component of $e^{-Ht}V_0$

$$\left\|e^{-Ht}V_0\right\|_{\mathcal{H}}^2 \leq C\|V_0\|_{\mathcal{H}}^2 + \|f(t)\|_{\mathcal{K}}^2$$

We estimate $\|\varphi(t)\|_{\mathcal{K}}$ by

$$\|f(t)\|_{\mathcal{K}} \leq \|f(0)\|_{\mathcal{K}} + \int_0^t \|\partial_t f(\tau)\|_{\mathcal{K}}\, d\tau \leq \|V_0\|_{\mathcal{H}} + |t|\|V_0\|_{\mathcal{H}_0}$$

which gives

$$\left\|e^{-Ht}V_0\right\|_{\mathcal{H}} \leq C_0(1+|t|)\|V_0\|_{\mathcal{H}} \tag{33}$$

By density, (33) yields the stability of $\mathcal{H}$ by e^{-Ht} and (31). There remains to prove the continuity of $e^{-Ht}U_0$ at $t=0$. We approach U_0 by a sequence $(U_0^n)_{n\in\mathbb{N}}$ in $\left[\mathcal{C}_0^\infty\left(]r_0, r_+[\times S^2\right)\right]^2$

$$\left\|e^{-Ht}U_0 - U_0\right\|_{\mathcal{H}} \leq \left(1 + C_0(1+|t|)\right)\|U_0 - U_0^n\|_{\mathcal{H}} + \left\|U_0^n - e^{-Ht}U_0^n\right\|_{\mathcal{H}} \quad \square$$

Proposition 2.1. *Given the Riemannian manifold*

$$V =]r_0, r_+[\times S_\omega^2\ ,\ ds^2 = F(r)^{-1}dr^2 + r^2 d\omega^2 \tag{34}$$

$H^1(V)$ and $L^6(V)$ being respectively the completions of $\mathcal{C}_0^\infty\left(]r_0, r_+[\times S_\omega^2\right)$ for the norms

$$\|\varphi\|_{H^1(V)}^2 = \int_\Pi \left\{F^{-1}\left|\frac{\partial\varphi}{\partial r_*}\right|^2 + \left|\frac{\nabla_{S^2}\varphi}{r}\right|^2 + |\varphi|^2\right\} F^{1/2}r^2 dr_* d\omega \tag{35}$$

$$\|\varphi\|_{L^6(V)}^6 = \int_\Pi |\varphi|^6 F^{1/2} r^2 dr_* d\omega \tag{36}$$

we have

$$H^1(V) \hookrightarrow L^6(V) \tag{37}$$

Proof of proposition 2.1: According to T.Aubin [1], we show that in the domain $r \in]r_0, r_1]$, $r_0 < r_1 < r_+$, the sectionnal curvatures of V are bounded (see Th. Frankel [9]) and that its injectivity radius is uniformly bounded by below. In the domain $r \geq r_1$, norms (35) and (36) are equivalent to the usual H^1 and L^6 norms on $\mathbb{R}^3$ for which the embedding result is well known. $\square$

From the previous proposition and lemma we deduce

Lemma 2.2. *J is continuous from $\mathcal{H}$ to $\mathcal{H}$ and satisfies*

$$\exists C_1 > 0\ ;\ \forall U, V \in \mathcal{H} \quad \|J(U) - J(V)\|_{\mathcal{H}} \leq C_1(\|U\|_{\mathcal{H}}^2 + \|V\|_{\mathcal{H}}^2)\|U - V\|_{\mathcal{H}} \tag{38}$$

Proof of lemma 2.2: Let $U = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}$, proposition 2.1 gives

$$\|J(U)\|_{\mathcal{H}}^2 \leq C \left\| F^{\frac{1}{4}} \frac{f}{r} \right\|_{H^1(V)}^6$$

A simple calculation shows

$$\left\| F^{\frac{1}{4}} \frac{f}{r} \right\|_{H^1(V)}^2 = \int_\Pi \left\{ F^{-1} \left| \frac{\partial}{\partial r_*} \left(F^{\frac{1}{4}} \frac{f}{r} \right) \right|^2 + \left| \frac{1}{r} \nabla_{S^2} \left(F^{\frac{1}{4}} \frac{f}{r} \right) \right|^2 + \left| F^{\frac{1}{4}} \frac{f}{r} \right|^2 \right\} F^{\frac{1}{2}} r^2 dr_* d\omega \leq C \|U\|_{\mathcal{H}}^2$$

and similarly for $U, V \in \mathcal{H}$, we get (38). □

We need an analogous lemma on functionnal spaces which will guarantee enough regularity to justify energy estimates. Let $D(H)_{\mathcal{H}}$ be the complete subspace of $\mathcal{H}$ defined by

$$D(H)_{\mathcal{H}} = \{U \in \mathcal{H}; HU \in \mathcal{H}\} \tag{39}$$

with norm

$$\|U\|_{D(H)_{\mathcal{H}}}^2 = \|U\|_{\mathcal{H}}^2 + \|HU\|_{\mathcal{H}}^2 \tag{40}$$

Lemma 2.3. *There exists a function C_∞, continuous and positive on $]r_0, r_+[^2$ such that*

$$\begin{gathered} \forall f \in \mathcal{C}_0^\infty\left(\mathbb{R}_{r_*} \times S_\omega^2\right); Supp(f) \ \subset\ [R_1, R_2]_r \times S_\omega^2 \ \subset\ ]r_0, r_+[_r \times S_\omega^2 \\ \|f\|_{L^\infty} \leq C_\infty(R_1, R_2)\left[\|f\|_{H^1(V)} + \|hf\|_{\mathcal{K}}\right] \end{gathered} \tag{41}$$

Proof of lemma 2.3: The compact support of the functions allow us to work in Minkowski space where the usual Sobolev embedding result

$$H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$$

holds. □

An immediate consequence is

Lemma 2.4. *J is continuous from $D(H)_{\mathcal{H}-comp}$ to itself and there exists a function C_2, continuous positive on $]r_0, r_+[^2$ such that*

$$\forall U, V \in D(H)_{\mathcal{H}-comp}, Supp(U) \cup Supp(V) \ \subset\ [R_1, R_2]_r \times S_\omega^2 \ \subset\ ]r_0, r_+[_r \times S_\omega^2$$

$$\|J(U)\|_{D(H)_{\mathcal{H}}} \leq C_2(R_1, R_2)[\|U\|_{\mathcal{H}} + \|HU\|_{\mathcal{H}}]\|U\|_{\mathcal{H}}^2 \tag{42}$$

$$\|J(U) - J(V)\|_{D(H)_{\mathcal{H}}} \leq C_2(R_1, R_2)[\|U\|_{D(H)_{\mathcal{H}}}^2 + \|V\|_{D(H)_{\mathcal{H}}}^2]\|U - V\|_{D(H)_{\mathcal{H}}} \tag{43}$$

Local solutions of (30) in $D(H)_{\mathcal{H}-comp}$ are regular enough $\left(H_{comp}^2\left([0, T]_t \times \mathbb{R}_{r_*} \times S_\omega^2\right)\right.$ if they exist on $[0, T]$) to justify the integrations by parts necessary to obtain an energy estimate. Theorem 2.1 is then a standard consequence of (22) and lemma 2.2 and 2.4. □

3 Asymptotic behavior at the horizon

We study the asymptotic behavior of regular solutions of equation (7) at the horizon of the black hole, i.e. in the neighbourhood of $\{r = r_0\}$. Even if it means rescaling the time variable, we suppose that $\delta(r_0) = 0$. We need to define new coordinates which enable us to cross the horizon. We choose KRUSKAL-SZEKERES variables

$$X = \frac{1}{2} e^{\kappa_0 r_*}\left(e^{\eta \kappa_0 t} + \eta e^{-\eta \kappa_0 t}\right) \quad , \quad T = \frac{1}{2} e^{\kappa_0 r_*}\left(e^{\eta \kappa_0 t} - \eta e^{-\eta \kappa_0 t}\right) \quad , \quad \eta = \frac{r - r_0}{|r - r_0|} \quad , \quad \kappa_0 = \frac{1}{2} F'(r_0) \tag{44}$$

where

$$r_* = \frac{1}{2\kappa_0}\left\{ Log|r-r_0| - \int_{r_0}^{r}\left[\frac{1}{r-r_0} - \frac{2\kappa_0}{Fe^{\delta}}\right]dr\right\} \tag{45}$$

Schwarzschild coordinates give two local maps with domains $(\mathbb{R}_t \times]r_-, r_0[_r \times S^2_\omega)$ and $(\mathbb{R}_t \times]r_0, r_+[_r \times S^2_\omega)$, but fail to represent the horizon $\{r = r_0\}$. Kruskal-Szekeres coordinates define an atlas with a single map $\{T + X > 0\} \times S^2_\omega$ and the horizon appears as the caracteristic submanifold $\{T = X > 0\} \times S^2_\omega$. A straightforward calculation shows that equation (7) on $\mathbb{R}_t \times V$ is equivalent to

$$\frac{\partial^2 f}{\partial T^2} - \frac{\partial^2 f}{\partial X^2} - A(T,X)\Delta_{S^2} f + B(T,X) f + \lambda A(T,X)|f|^2 f = 0 \tag{46}$$

with

$$A(T,X) = \kappa_0^{-2} r^{-2}|F| e^{-2\kappa_0 r_* + 2\delta} \quad , \quad B(T,X) = \kappa_0^{-2}|F| e^{-2\kappa_0 r_* + 2\delta}\left(m^2 + r^{-1}e^{-\delta}\frac{\partial}{\partial r}(Fe^{\delta}) + \xi R\right) \tag{47}$$

in the domain $\Omega = \{(T,X,\omega), X > |T|, \omega \in S^2\}$ which represents the exterior of the black hole in Kruskal-Szekeres variables. This kind of hyperbolic equation has already been studied by W.Von Wahl [14]. We solve the Cauchy problem associated to (46) in the neighbourhood of $\overline{\Omega} = \{(T,X,\omega), X \geq |T|, \omega \in S^2\}$ for very regular initial data. We obtain the asymptotic behavior of the field when $r \longrightarrow r_0, t \longrightarrow +\infty$ showing the regularity of the solution f of (46) at the horizon $X = |T|$.

Theorem 3.1. *Given φ, ψ in $\mathcal{C}_0^\infty(]r_0, r_+[_r \times S^2_\omega)$, there exists a unique $\hat{f}$ in $\mathcal{C}^\infty(\mathbb{R}_s \times S^2_\omega)$ such that, for any asymptotic direction (s,ω) in $\mathbb{R}_s \times S^2_\omega$, the solution f of (7) associated with the initial data ${}^t(\varphi,\psi)$ satisfies*

$$\lim_{t\longrightarrow+\infty} f(t, r_* = -t+s, \omega) = \hat{f}(s,\omega) \quad , \quad \lim_{t\longrightarrow+\infty} (\partial_t - \partial_{r_*}) f(t, r_* = -t+s, \omega) = 0 \tag{48}$$

Proof: (H1) and the hypothesis on F and δ yield that A and B can be prolonged to $\mathbb{R}_T \times \mathbb{R}_X$ as infinitely differentiable, strictly positive functions which we will still denote by A and B. We solve a Cauchy problem for an equation of type (46) prolonged to $\mathbb{R}_T \times \mathbb{R}_X \times S^2_\omega$.

Proposition 3.1. *Let A, B and Q be three real functions of two real variables t and x satisfying*

$$A, B, Q \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}_x) \quad , \quad A > 0 \; , \; B \geq 0 \; , \; Q \geq 0$$

and

$$\exists D \in \mathcal{C}(\mathbb{R}_t \times \mathbb{R}_x) \text{ such that } \forall t, x \in \mathbb{R}$$
$$\partial_t Q(t,x) \leq D(t,x) Q(t,x) \tag{49}$$

Then for each $\varphi \in H^1_{comp}(\mathbb{R}_x \times S^2_\omega; dx^2 + d\omega^2)$, $\psi \in L^2_{comp}(\mathbb{R}_x \times S^2_\omega; dx^2 + d\omega^2)$ and $s \in \mathbb{R}$, equation

$$\partial_t^2 f - \partial_x^2 f - A(t,x)\Delta_{S^2} f + B(t,x) f + Q(t,x)|f|^2 f = 0 \tag{50}$$

has one and only one solution f satisfying

$$f\,|_{t=s} = \varphi \; ; \; \partial_t f\,|_{t=s} = \psi$$
$$f \in \mathcal{C}(\mathbb{R}_t, H^1(\mathbb{R}_x \times S^2_\omega; dx^2 + d\omega^2)) \cap \mathcal{C}^1(\mathbb{R}_t, L^2(\mathbb{R}_x \times S^2_\omega; dx^2 + d\omega^2))$$

Furthermore, if φ and ψ belong to $\mathcal{C}_0^\infty\left(\mathbb{R}_x \times S^2_\omega\right)$, the solution f of (50) associated to the initial data φ, ψ and to an initial time $s \in \mathbb{R}$ belongs to $\mathcal{C}_0^\infty\left(\mathbb{R}_t \times \mathbb{R}_x \times S^2_\omega\right)$.

Proof of proposition 3.1: We study the linear equation associated to (50)

$$\partial_t^2 f - \partial_x^2 f - A(t,x)\Delta_{S^2} f + B(t,x) f = 0 \tag{51}$$

its hamiltonian form is

$$\frac{\partial V}{\partial t} = -\hat{H}V - P(t)V \tag{52}$$

where

$$V = \begin{pmatrix} f \\ \partial_t f \end{pmatrix} \quad \hat{H} = \begin{pmatrix} 0 & -1 \\ -\partial_x^2 & 0 \end{pmatrix} \quad P(t) = \begin{pmatrix} 0 & 0 \\ -A(t,.)\Delta_{S^2} + B(t,.) & 0 \end{pmatrix} \tag{53}$$

We introduice the following Hilbert spaces

$$k \in \mathbb{N}^* \quad , \quad \mathcal{H}^k = H^k(\mathbb{R}_x \times S^2_\omega; dx^2 + d\omega^2) \oplus H^{k-1}(\mathbb{R}_x \times S^2_\omega; dx^2 + d\omega^2) \tag{54}$$

$$H_1^x = BL^1(\mathbb{R}_x) \oplus L^2(\mathbb{R}_x) \tag{55}$$

$$D(\hat{H}) = \{\hat{V} \in H_1^x; \hat{H}\hat{V} \in H_1^x\} \tag{56}$$

$BL^1(\mathbb{R}_x)$ being the completion of $\mathcal{C}_0^\infty(\mathbb{R}_x)$ for the L^2 norm of the first derivative; for $R > 0$

$$k \in \mathbb{N}^* \quad \mathcal{H}^k_R = \{V \in \mathcal{H}^k; Supp(V) \subset [-R, R]_x \times S^2_\omega\} \tag{57}$$

$$k \in \mathbb{N}^* \quad H^k_R\left(\mathbb{R} \times S^2; dx^2 + d\omega^2\right) = \left\{f \in H^k\left(\mathbb{R} \times S^2; dx^2 + d\omega^2\right) \ ; \ Supp(f) \in [-R, R] \times S^2\right\} \tag{58}$$

$$1 \le p \le \infty \quad L^p_R\left(\mathbb{R} \times S^2; dx^2 + d\omega^2\right) = \left\{f \in L^p\left(\mathbb{R} \times S^2; dx^2 + d\omega^2\right) \ ; \ Supp(f) \in [-R, R] \times S^2\right\} \tag{59}$$

$$D(\hat{H})_R = \{\hat{V} \in D(\hat{H}); Supp(\hat{V}) \subset [-R, R]_x\} \tag{60}$$

and eventually

$$k \in \mathbb{N}^* \quad \mathcal{H}^k_{comp} = \bigcup_{R>0} \mathcal{H}^k_R \ ; \ D(\hat{H})_{comp} = \bigcup_{R>0} D(\hat{H})_R \tag{61}$$

Spaces (57) to (60) are all distribution spaces, thank to the compact support. We establish

Lemma 3.1. *There exists a family of bounded operators $(\mathcal{U}(t,s))_{t,s\in\mathbf{R}}$ defined on $\mathcal{H}^1_{comp}$ for $R > 0$ such that*

$$\forall t, s \in \mathbb{R} \ \ \mathcal{U}(t,s) \in \mathcal{L}\left(\mathcal{H}^1_R; \mathcal{H}^1_{R+|t-s|}\right) \cap \mathcal{L}\left(\mathcal{H}^2_R; \mathcal{H}^2_{R+|t-s|}\right) \tag{62}$$

$$\forall r, s, t \in \mathbb{R} \quad \mathcal{U}(r,s)\mathcal{U}(s,t) = \mathcal{U}(r,t) \tag{63}$$

$$\forall t \in \mathbb{R} \quad \mathcal{U}(t,t) = Id_{\mathcal{H}^1_{comp}} \tag{64}$$

Furthermore, given $U_0 \in \mathcal{H}^k_{comp}$, $k = 1, 2$

$$\mathcal{U}(t,s)U_0 \in \mathcal{C}\left(\mathbb{R}_t \times \mathbb{R}_s; \mathcal{H}^k\right) \tag{65}$$

and for $s \in \mathbb{R}$, $U_s : t \mapsto \mathcal{U}(t,s)U_0$ is the only solution of (52) in $\mathcal{C}(\mathbb{R}_t; \mathcal{H}^1_{comp})$ satisfying $U_s(s) = U_0$.

Proof of lemma 3.1: We decompose equation (51) using (13). We get

$$\partial_t^2 f_{lm} - \partial_x^2 f_{lm} + C_l(t,x) f_{lm} = 0 \tag{66}$$

where

$$C_l(t,x) = l(l+1)A(t,x) + B(t,x) \tag{67}$$

The hamiltonian form of (66) is

$$\frac{\partial V_{lm}}{\partial t} = -\hat{H}V_{lm} - P_l(t)V_{lm} \quad , \quad V_{lm} = \begin{pmatrix} f_{lm} \\ \partial_t f_{lm} \end{pmatrix} \quad , \quad P_l(t) = \begin{pmatrix} 0 & 0 \\ C_l(t,.) & 0 \end{pmatrix} \tag{68}$$

$\hat{H}$ is a skew adjoint operator on H_1^x with dense domain $D(\hat{H})$. $e^{-\hat{H}t}$ is therefore a strongly continuous one parameter unitary group on H_1^x as well as on the successive domains of $\hat{H}$ on H_1^x, such as $D(\hat{H})$ for instance. Poincaré's theorem yields for $R > 0$

$$D(\hat{H})_R \simeq H^2(\mathbb{R}_x)_R \oplus H^1(\mathbb{R}_x)_R$$

where

$$H^k(\mathbb{R}_x)_R = \left\{U \in H^i(\mathbb{R}_x); Supp(U) \subset [-R,R]\right\} \quad , \quad k \in \mathbb{N}$$

Since $C_l \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}_x) \hookrightarrow \mathcal{C}(\mathbb{R}_t; H^1_{loc}(\mathbb{R}_x))$, we get

$$\forall t \in \mathbb{R} \quad P_l(t) \in \mathcal{L}(D(\hat{H})_R; D(\hat{H})_R) \;\; and \quad \lim_{t-s\to 0} \quad \|P_l(t) - P_l(s)\|_{\mathcal{L}(D(\hat{H})_R; D(\hat{H})_R)} = 0$$

The Cauchy problem for (68) on $D(\hat{H})_{comp}$ is therefore well posed. Let us now introduice the dense subspace of $\mathcal{H}^1_R$ for $R > 0$

$$D(\hat{H})_{R,finite} = \left\{ U = \sum_{finite} U_{lm} \otimes Y_{lm}; \;\; U_{lm} \in D(\hat{H})_R \right\} \tag{69}$$

We can define a propagator $\mathcal{U}$ on $D(\hat{H})_{R,finite}$ by: for $U_0 \in D(\hat{H})_{R,finite}$, $R > 0$ and for $s \in \mathbb{R}$, $t \mapsto \mathcal{U}(t,s)U_0$ is the only solution of (52) associated with the initial data U_0 and the initial time s. $\mathcal{U}$ satisfies

$$\mathcal{U}(t,s) \in \mathcal{L}\left(D(\hat{H})_{R,finite}; D(\hat{H})_{R+|t-s|,finite}\right) \quad \forall R > 0 \; \forall t,s \in \mathbb{R} \tag{70}$$

$$\mathcal{U}(t,s)U_0 \in \mathcal{C}\left(\mathbb{R}_t \times \mathbb{R}_s; D(\hat{H})_{R+|t-s|,finite}\right) \quad \forall R > 0 \; \forall U_0 \in D(\hat{H})_{R,finite} \tag{71}$$

$$\forall t,s,r \in \mathbb{R} \;\; \mathcal{U}(t,s)\mathcal{U}(s,r) = \mathcal{U}(t,r) \tag{72}$$

$$\forall t \in \mathbb{R} \;\; \forall R > 0 \quad \mathcal{U}(t,t) = Id_{D(\hat{H})_{R,finite}} \tag{73}$$

We need to prolong $\mathcal{U}$ to $\mathcal{H}^1_{comp}$. Let $U_0 \in D(\hat{H})_{R,finite}$, $R > 0$ and $s \in \mathbb{R}$, we denote $U_s : t \mapsto \mathcal{U}(t,s)U_0$. If f is the first component of U_s, then

$$f \in H^2_{loc}\left(\mathbb{R}_t \times \mathbb{R}_x \times S^2_\omega; dt^2 + dx^2 + d\omega^2\right)$$

We can multiply (51) by $\partial_t f$ and integrate by parts on $\Omega_{s,T} =]s,T[_t \times \mathbb{R}_x \times S^2_\omega$ for any T in $\mathbb{R}$. Using the properties of A, B, Q and Poincaré's inequality, we obtain the existence of a fonction $\mathbb{K}$, continuous on $\mathbb{R}^2 \times \mathbb{R}^+$, strictly positive, such that for $T \in \mathbb{R}$

$$\|\mathcal{U}(t,s)U_0\|_{\mathcal{H}^1} \leq \mathbb{K}(s,T,R)\|U_0\|_{\mathcal{H}^1} \quad \forall t \in (s,T) \tag{74}$$

Thus, for any $R > 0$, $t,s \in \mathbb{R}$

$$\mathcal{U}(t,s) \in \mathcal{L}\left(\mathcal{H}^1_R; \mathcal{H}^1_{R+|t-s|}\right)$$

Furthermore, if $U_0 \in \mathcal{H}^1_R$, $R > 0$

$$\mathcal{U}(t,s)U_0 \in \mathcal{C}\left(\mathbb{R}_t \times \mathbb{R}_s; \mathcal{H}^1_{R+|t-s|}\right)$$

and $t \mapsto \mathcal{U}(t,s)U_0$ is the only solution of (52) associated with the initial data U_0 and the initial time s. For initial data in $\mathcal{H}^2_R$, $R > 0$, applying ∂_x and the generators of the rotation group

$$L_1 = sin\varphi\frac{\partial}{\partial\theta} + \frac{cos\varphi}{tg\theta}\frac{\partial}{\partial\varphi} \quad L_2 = -cos\varphi\frac{\partial}{\partial\theta} + \frac{sin\varphi}{tg\theta}\frac{\partial}{\partial\varphi} \quad L_3 = \frac{\partial}{\partial\varphi} \tag{75}$$

to equation (52) yields easily (62) and (65) for $k = 2$. □

As for the non linear equation (50), its hamiltonian form is

$$\frac{\partial V}{\partial t} = -\hat{H}V - P(t)V - K(t,V) \tag{76}$$

where $V, \hat{H}$ and $P(t)$ have already been defined for the linear problem and

$$K\left(t, \begin{pmatrix} f \\ g \end{pmatrix}\right) = \begin{pmatrix} 0 \\ Q(t,.)|f|^2 f \end{pmatrix} \tag{77}$$

Let us now study the continuity of K.

Lemma 3.2. *Let* $U \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{H}^k_{R+|t|}\right)$*,* $R>0$*,* $k=1,2$*, then*

$$K(t,U(t)) \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{H}^k_{R+|t|}\right)$$

Furthermore, there exists some functions C_1*,* C_2 *and* C_3*, continuous on* $\mathbb{R}\times\mathbb{R}^+$ *such that*

$$\forall U,V \in \mathcal{H}^1_R \ , \ \forall t \in \mathbb{R}$$
$$\|K(t,U)-K(t,V)\|_{\mathcal{H}^1} \le C_1(t,R)\left[\|U\|^2_{\mathcal{H}^1}+\|V\|^2_{\mathcal{H}^1}\right]\|U-V\|_{\mathcal{H}^1} \tag{78}$$

$$\forall U,V \in \mathcal{H}^2_R \ , \ \forall t \in \mathbb{R}$$
$$\|K(t,U)-K(t,V)\|_{\mathcal{H}^2} \le C_2(t,R)\left[\|U\|^2_{\mathcal{H}^2}+\|V\|^2_{\mathcal{H}^2}\right]\|U-V\|_{\mathcal{H}^2} \tag{79}$$
$$\|K(t,U)\|_{\mathcal{H}^2} \le C_3(t,R)\|U\|^2_{\mathcal{H}^1}\|U\|_{\mathcal{H}^2} \tag{80}$$

The proof of lemma 3.3 is very simple. We merely use the fact that the functions are compactly supported in order to make all the calculations in Minkowski space. Then, we apply the usual Sobolev embedding

$$H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \quad \square$$

The regularity of local $\mathcal{H}^2$ solutions allow us to perform an energy estimate. Using Poincaré's inequality, $(H1)$ and the hypothesis on A, B and Q, we obtain the existence of a function $\mathbb{K}_1$, continuous and strictly positive on $\mathbb{R}^2\times\mathbb{R}^+$ such that, for $U_0 \in \mathcal{H}^2_R$, $R>0$, for $s\in\mathbb{R}$, if the solution U_s of (76) associated to U_0 and s exists on $[s,T]$, $T\in\mathbb{R}$, then

$$\mathbb{E}(U_s(t),t) \le \mathbb{K}_1(s,T,R)\mathbb{E}(U_0,s) \quad \forall t\in(s,T) \tag{81}$$

where

$$\mathbb{E}\left(\begin{pmatrix} f \\ g \end{pmatrix}, t\right) = \left\|\begin{pmatrix} f \\ g \end{pmatrix}\right\|^2_{\mathcal{H}^1} + \frac{1}{2}\int_{\mathbb{R}\times S^2} Q(t,.)|f|^4 dx d\omega \tag{82}$$

Global existence and unicity for $\mathcal{H}^1_{comp}$ initial data is a standard consequence of (81) and lemma 3.2 and 3.3. Furthermore, if $U_0\in\mathcal{H}^1_R$, $R>0$ and $s\in\mathbb{R}$, the only solution U_s of (76) associated to U_0 and s satisfies for any $T\in\mathbb{R}$

$$\|U_s(t)\|^2_{\mathcal{H}^1} \le \mathbb{K}_1(s,T,R)\mathbb{E}(U_0,s) \ \ \forall t\in[s,T] \tag{83}$$

It is a tedious but straightforward calculation to prove

Lemma 3.3. *Let* $k\in\mathbb{N}^*$

$$\forall t,s\in\mathbb{R} \quad \forall R>0 \quad \mathcal{U}(t,s)\in\mathcal{L}\left(\mathcal{H}^k_R;\mathcal{H}^k_{R+|t-s|}\right) \tag{84}$$

$$\forall U_0\in\mathcal{H}^k_R \quad , \quad R>0 \quad \mathcal{U}(t,s)U_0 \in \mathcal{C}\left(\mathbb{R}_t\times\mathbb{R}_s;\mathcal{H}^k\right) \tag{85}$$

There exists functions $C_{k,2}$ *and* $C_{k,3}$ *continuous and strictly positive on* $\mathbb{R}\times\mathbb{R}^+$ *such that, for* $U,V\in\mathcal{H}^k_R$*,* $R>0$ *and for* $t\in\mathbb{R}$

$$\|K(t,U)-K(t,V)\|_{\mathcal{H}^k} \le C_{k,2}(t,R)\left[\|U\|^2_{\mathcal{H}^k}+\|V\|^2_{\mathcal{H}^k}\right]\|U-V\|_{\mathcal{H}^k} \tag{86}$$

$$\|K(t,U)\|_{\mathcal{H}^k} \le C_{k,3}(t,R)\|U\|^2_{\mathcal{H}^{k-1}}\|U\|_{\mathcal{H}^k} \tag{87}$$

Thus, for initial data φ and ψ in $\mathcal{C}^\infty_0\left(\mathbb{R}_x\times S^2_\omega\right)$, the solution of (50) is itself very regular and proposition 3.1 is established. $\square$

Let us now consider equation (7) in(t,r_*,ω) variables with initial data φ and ψ in $\mathcal{C}^\infty\left(]r_0,r_1[_r\times S^2_\omega\right)$. This corresponds for equation (46) in Kruskal-Szekeres variables with an initial time $s=0$ and initial data φ_1 and ψ_1 in $\mathcal{C}^\infty\left(]0,+\infty[_X\times S^2_\omega\right)$ such that, for $X>0$

$$\varphi_1(X)=\varphi(r_*(X)) \quad , \quad \psi_1(X)=\psi(r_*(X))$$

with

$$r_*(X) = \frac{1}{\kappa_0} Log(X)$$

Let f_1 be the solution of (46) with initial time $s=0$ and initial data φ_1, ψ_1. f_1 can be prolonged as a $\mathcal{C}^\infty$ function on $\mathbb{R}_T \times \mathbb{R}_X \times S^2_\omega$ and the solution f of (7) associated with the initial data φ, ψ satisfies

$$f\left(t(T,X), r_*(T,X), \omega\right) = f_1(T,X,\omega) \quad \forall (T,X,\omega) \in \Omega$$

f is therefore $\mathcal{C}^\infty$ on $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega$ and we can define its asymptotic profile at the horizon: let $\hat{f} \in \mathcal{C}^\infty_0(\mathbb{R}_s \times S^2_\omega)$ be defined by

$$\hat{f}(s,\omega) = f_1(X_0, X_0, \omega) = \lim_{\substack{(T,X) \to (X_0,X_0) \\ (T,X,\omega)\in\Omega}} f_1(T,X,\omega) \quad where\ X_0 = \frac{1}{2} e^{\kappa_0 s} \tag{88}$$

then

$$\hat{f}(s,\omega) = \lim_{t \to +\infty} f(t, r_* = -t+s, \omega)$$

Now, for $(t, r_*, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2_\omega$

$$(\partial_t - \partial_{r_*}) f(t,r_*,\omega) = \kappa_0 \left[X(t,r_*) - T(t,r_*)\right] (\partial_T - \partial_X) f_1\left(T(t,r_*), X(t,r_*), \omega\right)$$

Putting $r_* = -t+s$ and allowing t to tend to $+\infty$ yields the radiation condition and concludes the proof of theorem 3.1. □

The same result holds for the Cosmological horizon (if r_+ is finite) with an outgoing radiation condition.

4 Asymptotic behavior at infinity

From now on, we assume that $r_+ = +\infty$. We study the asymptotic behavior at infinity of regular solutions of equation (7) in the case $m=0$. Even if it means rescaling the time variable, we assume that $\delta(+\infty) = 0$. We need to define a new set of variables in order to cross the spatial infinity. Applying Penrose's conform compactification method yields

$$\tau = Arctg(t+r_*) + Arctg(t-r_*) \quad , \quad \zeta = Arctg(t+r_*) - Arctg(t-r_*) \tag{89}$$

In Penrose variables, the whole space-time outside the black hole, including the horizon and the spatial infinity, is mapped into a compact set of $\mathbb{R}_\tau \times \mathbb{R}_\zeta \times S^2_\omega$. Equation (7) on $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega$ becomes

$$\partial^2_\tau f - \partial^2_\zeta f - \alpha(\tau,\zeta)\Delta_{S^2} f + \beta(\tau,\zeta) f + \lambda\alpha(\tau,\zeta)|f|^2 f = 0 \tag{90}$$

with

$$\alpha(\tau,\zeta) = Fe^{2\delta} r^{-2} (cos\tau + cos\zeta)^{-2} \quad , \quad \beta(\tau,\zeta) = Fe^{2\delta}(cos\tau + cos\zeta)^{-2}\left(r^{-1}e^{-\delta}\partial_r(Fe^\delta) + \xi R\right) \tag{91}$$

on domain $\Omega' = \left\{(\tau,\zeta,\omega); -\pi < \tau - \zeta < \pi; -\pi < \tau+\zeta < \pi; \omega \in S^2\right\}$. The only difference between equation (90) and (50) lies in the regularity of α and β. The spatial infinity is a true singularity of metric (1) and prevents us from prolonging α and β as regular functions on the whole (τ,ζ) plane. This lack of regularity appears in the asymptotic behavior

Theorem 4.1. *Given φ, ψ in $\mathcal{C}^\infty_0(]r_0, r_+[\times S^2_\omega)$, there exists a unique $\hat{f}_\infty$ in $\left(\mathcal{C}^0 \cap H^{1/2}_{loc}\right)(\mathbb{R}_s; L^2(S^2_\omega))$ such that, for any real number s, the solution f of (7) with $m=0$ associated with the initial data φ, ψ satisfies*

$$\lim_{t \longrightarrow +\infty} f(t, r_* = t-s, .) = \hat{f}_\infty(s,.) \quad and \quad \lim_{t\longrightarrow+\infty} (\partial_t + \partial_{r_*}) f(t, r_* = t-s, .) = 0 \quad in\ L^2(S^2_\omega) \tag{92}$$

Proof: Wishing to avoid pathological behaviors at the horizon or at the corners of the diamond, we limit our study to Ω'_ε defined for $\varepsilon > 0$ by

$$\Omega'_\varepsilon = \left\{(\tau,\zeta,\omega); -\pi < \tau-\zeta \leq \pi-\varepsilon \ ; -\pi+\varepsilon \leq \tau+\zeta < \pi \ ; Min\left(|\tau+\zeta|,|\tau-\zeta|\right) \leq \pi-\varepsilon \ ; \omega \in S^2\right\} \quad (93)$$

We define prolongations of α and β outside Ω'_ε as follows: for $(\tau,\zeta) \notin \Omega'_\varepsilon$

$$\gamma(\tau,\zeta) = \begin{cases} \gamma(\tau_1,\zeta) & if \quad -\pi+\varepsilon \leq \zeta \leq \pi - \frac{\varepsilon}{2} \\ \gamma(0,-\pi+\varepsilon) & if \quad \zeta \leq -\pi+\varepsilon \\ \gamma\left(\frac{\varepsilon}{2}, \pi-\frac{\varepsilon}{2}\right) & if \quad \zeta \geq \pi - \frac{\varepsilon}{2} \end{cases} \quad (94)$$

where γ is either α or β and (τ_1,ζ) is the projection parallel to $\mathbb{R}_\tau$ of (τ,ζ) onto the boundary of Ω'_ε. We denote by α_ε and β_ε the prolonged functions. It is a very tedious task but with no major difficulty to verify

$$\alpha_\varepsilon > 0 \ \ , \ \ \beta_\varepsilon \geq 0 \quad on\ \mathbb{R}_\tau \times \mathbb{R}_\zeta$$
$$\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{C}\left(\mathbb{R}_\tau; H^1_{loc}(\mathbb{R}_\zeta)\right) \cap \mathcal{C}\left(\mathbb{R}_\zeta; H^1_{loc}(\mathbb{R}_\tau)\right)$$
$$\left(\frac{\partial \alpha}{\partial \tau}\right)^+, \left(\frac{\partial \beta}{\partial \tau}\right)^+ \in L^\infty_{loc}\left(\left]\frac{-\varepsilon}{2}, +\infty\right[_\tau \times \mathbb{R}_\zeta\right)$$

where $\left(\frac{\partial \alpha_\varepsilon}{\partial \tau}\right)^+$ and $\left(\frac{\partial \beta_\varepsilon}{\partial \tau}\right)^+$ are the positive parts of $\frac{\partial \alpha_\varepsilon}{\partial \tau}$ and $\frac{\partial \beta_\varepsilon}{\partial \tau}$.

Remark 4.1. *There obviously exists $D \in \mathcal{C}\left(\left]-\frac{\varepsilon}{2}, +\infty\right[_\tau \times \mathbb{R}_\zeta\right)$ such that almost everywhere on $\left]-\frac{\varepsilon}{2}, +\infty\right[_\tau \times \mathbb{R}_\zeta$*

$$\frac{\partial \alpha_\varepsilon}{\partial \tau}(\tau,\zeta) \leq D(\tau,\zeta)\alpha_\varepsilon(\tau,\zeta)$$

We wish to prove a general result similar to proposition 3.1 with functions

$$A, B, Q \in \mathcal{C}\left(\mathbb{R}_t; H^1_{loc}(\mathbb{R}_x)\right) \cap \mathcal{C}\left(\mathbb{R}_x; H^1_{loc}(\mathbb{R}_t)\right)$$

The energy estimates will require the time derivatives of these functions to be locally bounded by above, but it is only true for α_ε and β_ε on $\left]-\frac{\varepsilon}{2}, +\infty\right[_\tau \times \mathbb{R}_\zeta$. The domain $\left]-\infty, \frac{\varepsilon}{2}\right[_\tau \times \mathbb{R}_\zeta$ is dealt with by changing τ in $-\tau$ which changes the signs of $\frac{\partial \alpha_\varepsilon}{\partial \tau}$ and $\frac{\partial \beta_\varepsilon}{\partial \tau}$. We give a general theorem on $\mathbb{R}_t \times \mathbb{R}_x$ which can be applied to any open domain $]t_0, t_1[_t \times \mathbb{R}_x$, $-\infty \leq t_0 < t_1 \leq +\infty$.

Proposition 4.1. *Let A, B and Q be three real functions of two real variables t and x such that*

$$A, B, Q \in \mathcal{C}\left(\mathbb{R}_t; H^1_{loc}(\mathbb{R}_x)\right) \cap \mathcal{C}\left(\mathbb{R}_x; H^1_{loc}(\mathbb{R}_t)\right) \qquad A > 0 \quad B \geq 0 \quad Q \geq 0$$

and

$$\left(\frac{\partial A}{\partial t}\right)^+, \left(\frac{\partial B}{\partial t}\right)^+ \in L^\infty_{loc}\left(\mathbb{R}_t \times \mathbb{R}_x\right) \quad (95)$$

Assume moreover that there exists $D \in \mathcal{C}\left(\mathbb{R}_t \times \mathbb{R}_x\right)$ such that

$$\frac{\partial Q}{\partial t} \leq DQ \ \ almost\ everywhere\ on\ \mathbb{R}_t \times \mathbb{R}_x \quad (96)$$

then the global existence and unicity result of proposition 3.1 holds.

Proof of proposition 4.1: The proof of proposition 3.1 was made of four essential parts:
1. Unique global solutions for the Cauchy problem on one spherical harmonic.
2. Extension to $\mathcal{H}^1_{comp}$ of the propagator thus obtained.
3. Unique local solutions in $\mathcal{H}^1_{comp}$ for the non linear Cauchy problem.
4. Global solutions in $\mathcal{H}^1$.

Stages 1 and 3 only require $A, B \in \mathcal{C}\left(\mathbb{R}_t; H^1_{loc}(\mathbb{R}_x)\right)$ and $Q \in \mathcal{C}(\mathbb{R}_t \times \mathbb{R}_x)$. On the other hand, stages 2 and 4 consist merely of energy estimates but A, B and Q are not regular enough to justify the integrations by parts. Let

$$\gamma \in \mathcal{C}_0^\infty\left(\mathbb{R}_t \times \mathbb{R}_x\right) \ ; \ Supp(\gamma) \subset B(0,1) \ ; \ 0 \leq \gamma \leq 1 \ \ on \ \mathbb{R}_t \times \mathbb{R}_x \ ; \ \int_{\mathbb{R}^2} \gamma dt dx = 1$$

where $B(0,1)$ is the ball with centre 0 and radius 1 in $\mathbb{R}_t \times \mathbb{R}_x$. For $n \in \mathbb{N}^*$, we introduce

$$\gamma_n \in \mathcal{C}_0^\infty\left(\mathbb{R}_t \times \mathbb{R}_x\right) \quad , \quad \gamma_n(t,x) = n^2 \gamma(nt, nx)$$

and we define regularized functions

$$A_n = A * \gamma_n \ , \ B_n = B * \gamma_n \ , \ Q_n = Q * \gamma_n$$

They satisfy

$$A_n, B_n, Q_n \in \mathcal{C}^\infty\left(\mathbb{R}_t \times \mathbb{R}_x\right) \qquad A_n > 0 \ , \ B_n \geq 0 \ , \ Q_n \geq 0$$

and

$$\frac{\partial Q_n}{\partial t}(t,x) \leq D_n(t,x) Q_n(t,x) \quad \forall t, x \in \mathbb{R} \tag{97}$$

where D_n, defined by

$$D_n(t,x) = \underset{|(\tau,\zeta)| \leq \frac{1}{n}}{Max} \ |D(t+\tau, x+\zeta)|$$

converges uniformly to $|D|$ on any compact set of $\mathbb{R}_t \times \mathbb{R}_x$. With A_n, B_n, Q_n we associate equations

$$\frac{\partial U^n}{\partial t} = -\hat{H} U^n - P^n(t) U^n \quad , \quad P^n(t) = \begin{pmatrix} 0 & 0 \\ -A_n(t,.)\Delta_{S^2} + B_n(t,.) & 0 \end{pmatrix} \tag{98}$$

$$\frac{\partial U^n}{\partial t} = -\hat{H} U^n - P^n(t) U^n - K^n(t, U^n) \quad , \quad K^n\left(t, \begin{pmatrix} f \\ g \end{pmatrix}\right) = \begin{pmatrix} 0 \\ Q_n(t,.)|f|^2 f \end{pmatrix} \tag{99}$$

and with A, B, C equations

$$\frac{\partial U}{\partial t} = -\hat{H} U - P(t) U \tag{100}$$

$$\frac{\partial U}{\partial t} = -\hat{H} U - P(t) U - K(t, U) \tag{101}$$

where $\hat{H}, P$ and K follow the definitions of proposition 3.1. We first consider the linear problem. Given $U_0 \in D(\hat{H})_{R,finite}$, $R > 0$, we denote by U_s (resp. U_s^n) the solution of (98) (resp. 100) in $\mathcal{C}(\mathbb{R}_t; \mathcal{H}^1)$ associated with the initial data U_0 and initial time s. Proposition 3.1 gives for $n \in \mathbb{N}^*$

$$\forall T \in \mathbb{R} \ \ \forall t \in (s,T) \qquad \|U_s^n(t)\|_{\mathcal{H}^1} \leq \mathbb{K}^n(s,t,R) \|U_0\|_{\mathcal{H}^1} \tag{102}$$

with

$$\mathbb{K}^n \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}^+;]0 + \infty[)$$

We check very easily that

$$\forall t \in \mathbb{R} \qquad \lim_{n \to +\infty} \ U_s^n(t) = U_s(t) \quad in \ \mathcal{H}^1$$

and

$$\forall T \in \mathbb{R} \qquad \lim_{n \to +\infty} \ \mathbb{K}^n(s,T,R) = \mathbb{K}^\infty(s,T,R) < +\infty$$

thus, for $T \in \mathbb{R}$

$$\|U_s(t)\|_{\mathcal{H}^1} \leq \mathbb{K}^\infty(s,T,R) \|U_0\|_{\mathcal{H}^1} \quad \forall t \in (s,T) \tag{103}$$

As for the non linear problem, let $U_0 \in \mathcal{H}^1_R$, $R > 0$ and $s \in \mathbb{R}$. We denote by U_s a local solution in $\mathcal{C}\left([s,T];\mathcal{H}^1\right)$, $T \in \mathbb{R}$ of (101) associated with U_0 and s. Proposition 3.1 gives for $n \in \mathbb{N}^*$

$$\|U_s^n(t)\|_{\mathcal{H}^1} \leq \mathbb{K}_1^n(s,T,R)\mathbb{E}^n(U_0,s) \quad \forall t \in (s,T) \tag{104}$$

with

$$\mathbb{K}_1^n \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}^+;]0,+\infty[)$$

and

$$\mathbb{E}^n(U_0,s) = \|U_0\|^2_{\mathcal{H}^1} + \frac{1}{2}\int_{\mathbb{R}\times S^2} Q_n(s,x)|\varphi(x,\omega)|^4 dx d\omega$$

φ being the first component of U_0. When n tends to $+\infty$, we have

$$\lim_{n\to+\infty} \mathbb{E}^n(U_0,s) = \mathbb{E}(U_0,s) \quad , \quad \lim_{n\to+\infty} \mathbb{K}_1^n(s,T,R) = \mathbb{K}_1^\infty(s,T,R) < +\infty$$

using the continuity of the non linear term of both equations (99) and (101) in $\mathcal{H}^1$ we get

$$\forall t \in (s,T) \qquad \lim_{n\to+\infty} U_s^n(t) = U_s(t) \;\; in\; \mathcal{H}^1$$

This yields for $t \in (s,T)$

$$\forall t \in (s,T) \quad \|U_s(t)\|_{\mathcal{H}^1} \leq \mathbb{K}_1^\infty(s,T,R)\mathbb{E}(U_0,s)$$

and concludes the proof of proposition 4.1. □

Let us now consider for equation (7) in (t,r_*,ω) variables, initial data φ,ψ in $\mathcal{C}_0^\infty\left(]r_0,r_+[_r \times S^2_\omega\right)$. This means for equation (90) in Penrose variables an initial time $s = 0$ and initial data φ_1,ψ_1 dans $\mathcal{C}_0^\infty\left(]-\pi,\pi[_\zeta \times S^2_\omega\right)$ defined by

$$\varphi_1(\zeta) = \varphi(r_*(\zeta)) \quad \psi_1(\zeta) = \psi(r_*(\zeta)) \quad r_*(\zeta) = 2tg\left(\frac{\zeta}{2}\right) \quad \zeta \in \mathbb{R}$$

We know from proposition 4.1 that for any $\varepsilon > 0$, (90) has in Ω'_ε a unique solution f_ε which can be prolonged to $\mathbb{R}_\tau \times \mathbb{R}_\zeta \times S^2_\omega$ as an element of

$$\mathcal{C}\left(\mathbb{R}_\tau; H^1_{loc}\left(\mathbb{R}_\zeta \times S^2_\omega; d\zeta^2 + d\omega^2\right)\right) \cap \mathcal{C}^1\left(\mathbb{R}_\tau; L^2_{loc}\left(\mathbb{R}_\zeta \times S^2_\omega; d\zeta^2 + d\omega^2\right)\right)$$

such that

$$f_\varepsilon|_{\tau=0} = \varphi_1 \qquad \left.\frac{\partial f_\varepsilon}{\partial \tau}\right|_{\tau=0} = \psi_1$$

Thus, if for $\varepsilon > 0$ we consider

$$\Sigma_\varepsilon = \left\{(\tau,\zeta); \tau = \pi - \zeta, \frac{\varepsilon}{2} \leq \zeta \leq \pi - \frac{\varepsilon}{2}\right\}$$

endowed with the metric induced by the carthesian metric on $\mathbb{R}_\tau \times \mathbb{R}_\zeta$

$$ds^2_{\Sigma_\varepsilon} = 2d\zeta^2$$

the trace of f_ε on $\Sigma_\varepsilon \times S^2_\omega$ satisfies

$$f_\varepsilon|_{\Sigma_\varepsilon \times S^2_\omega} \in \left(\mathcal{C}^0 \cap H^{1/2}\right)\left(\Sigma_\varepsilon, L^2(S^2_\omega)\right)$$

and by unicity, if $\varepsilon_1 > \varepsilon$

$$f_\varepsilon|_{\Sigma_{\varepsilon_1} \times S^2_\omega} = f_{\varepsilon_1}|_{\Sigma_{\varepsilon_1} \times S^2_\omega}$$

Putting for $s \in \mathbb{R}$ and $\varepsilon > 0$ small enough

$$\hat{f}_\infty(s,.) = f_\varepsilon|_{\Sigma_\varepsilon \times S^2_\omega}\left(\frac{\pi}{2} + Arctg(s), \frac{\pi}{2} - Arctg(s), .\right)$$

we define in a unique way from the initial data φ, ψ a function

$$\hat{f}_\infty \in \left(\mathcal{C}^0 \cap H^{1/2}_{loc}\right)(\mathbb{R}_s, L^2(S^2_\omega))$$

and for $s \in \mathbb{R}$, $\hat{f}_\infty$ satisfies

$$\hat{f}_\infty(s,.) = \lim_{t \to +\infty} f(t, r_* = t - s, .) \quad in\ L^2(S^2_\omega)$$

where f is the solution of (7) associated to φ, ψ.

Then, we have to prove some regularity results before verifying the radiation condition. For $\varepsilon > 0$, f_ε satisfies

$$\frac{\partial^2 f_\varepsilon}{\partial \tau^2} - \frac{\partial^2 f_\varepsilon}{\partial \zeta^2} - \alpha_\varepsilon(\tau,\zeta)\Delta_{S^2} f_\varepsilon + \beta_\varepsilon(\tau,\zeta) f_\varepsilon + \lambda \alpha_\varepsilon(\tau,\zeta)|f_\varepsilon|^2 f_\varepsilon = 0 \tag{105}$$

Applying the generators of the rotation group to (105) yields

$$\Delta_{S^2} f_\varepsilon \in \mathcal{C}\left(\mathbb{R}_\tau; L^2\left(\mathbb{R}_\zeta \times S^2_\omega\right)\right) \tag{106}$$

We apply $cos^2\left(\frac{\tau+\zeta}{2}\right)\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta}\right)$ to (105). (106) and the asymptotic behaviors of α_ε and β_ε induced by the hypothesis on F and δ give

$$cos^2\left(\frac{\tau+\zeta}{2}\right)\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta}\right) f_\varepsilon \in \mathcal{C}\left(\left[-\pi + \frac{\varepsilon}{2}, \pi - \frac{\varepsilon}{2}\right]_\tau; H^1(\mathbb{R}_\zeta \times S^2_\omega)\right) \tag{107}$$

The limitation of the temporal domain is due to the choice of prolongation for α and β but is of no influence whatsoever on the final result. (107) yields the existence of a unique function

$$\hat{g}_\infty \in \left(\mathcal{C}^0 \cap H^{1/2}_{loc}\right)(\mathbb{R}_s; L^2(S^2_\omega))$$

such that, for $s \in \mathbb{R}$

$$\lim_{t \to +\infty} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}\right) f(t, r_* = t - s, .) = \hat{g}_\infty(s,.) \quad in\ L^2(S^2_\omega)$$

The last task is to prove that $\hat{g}_\infty = 0$. We know that

$$cos^2\left(\frac{\tau+\zeta}{2}\right)\left\|\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta}\right) f_\varepsilon\right\|_{L^2(S^2_\omega)} \in \mathcal{C}\left(\left]-\pi + \frac{\varepsilon}{2}, \pi - \frac{\varepsilon}{2}\right[\times \mathbb{R}_\zeta\right)$$

and

$$\left\|\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta}\right) f_\varepsilon\right\|_{L^2(S^2_\omega)} \in \mathcal{C}\left(\left[-\pi + \frac{\varepsilon}{2}, \pi - \frac{\varepsilon}{2}\right]_\tau; L^2(\mathbb{R}_\zeta)\right)$$

Furthermore, f_ε is $\mathcal{C}^\infty$ in the interior of Ω'_ε so that for $\tau \in \left]-\pi + \frac{\varepsilon}{2}, \pi - \frac{\varepsilon}{2}\right[$, there exists $\zeta_\tau < \pi - \tau$ such that

$$\left\|\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta}\right) f_\varepsilon(\tau,.)\right\|_{L^2(S^2_\omega)} \in \mathcal{C}\left([\zeta_\tau, \pi - \tau[\right)$$

We remark that if $u \in L^2(\mathbb{R}) \cap \mathcal{C}([x_0, \pi[)$, $x_0 < \pi$ and $cos^2\left(\frac{x}{2}\right)u(x)$ is continuous in π, then

$$\left.\left(cos^2\left(\frac{x}{2}\right)u(x)\right)\right|_{x=\pi} = 0$$

Thus

$$\left\| cos^2\left(\frac{\tau+\zeta}{2}\right)\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)f_\varepsilon\right\|_{L^2(S^2_\omega)}=0 \quad for \;\; \zeta=\pi-\tau \quad -\pi+\frac{\varepsilon}{2}<\tau<\pi-\frac{\varepsilon}{2}$$

This yields $\hat{g}_\infty \equiv 0$ and concludes the proof of theorem 4.1. □

Conclusion

The results described in this paper are a first step to take up difficult open problems such as: asymptotic behavior at the Cauchy horizon, asymptotic behavior and existence of wave operators at infinity for massive fields. They require a very sharp analysis of the linear propagator which remains to be done.

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