Global results for the Rarita-Schwinger equations and Einstein vacuum equations.

L.J.Mason

The Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, England

and

J.P.Nicolas

Centre de Mathématiques, UMR CNRS 7640, Ecole Polytechnique,

91128 Palaiseau Cedex, France.

Abstract

We prove global existence and uniqueness of solutions to the Rarita-Schwinger evolution equations compatible with the constraints. We use a gauge fixing for the Rarita-Schwinger equations for helicity 3/2 fields in curved space that leads to a straightforward Hilbert space framework for their study. We explain how these results might be applied to the global analysis of the full Einstein vacuum equations and provide a complete analysis as a basis for such applications. These and a programme for developing an scattering/inverse scattering transform for the full Einstein equations are discussed.

1   Introduction

In this article we give a detailed analysis of the Rarita-Schwinger equations for helicity 3/2 fields in a general curved vacuum Lorentzian space-time. Existence and uniqueness theorems for linear symmetric hyperbolic equations are well understood according to general schemes, see for example Jerome, 1983, chapters 6 and 7, [5]. However a number of equations in mathematical physics, in particular the Rarita-Schwinger equations, do not quite fit into these schemes because they have both gauge freedom and constraints and one must analyse these aspects of the equations on a piecemeal basis. In this article we introduce a gauge fixing based on that used by Witten in his proof of the positive energy theorem, [15]. This leads to a natural Hilbert space framework for the analysis of the equations and we prove existence and uniqueness compatible with the constraints in this framework.

Our interest in the Rarita-Schwinger equations arises for three reasons:

1. perhaps the most important reason in the short term, is the fact that the initial data sets for general relativity can naturally be expressed as a pair of helicity 3/2 fields via the spin connection (these are formally gauge equivalent to zero, but have non-zero norm in the Hilbert space). This is completely natural in the sense that it is compatible with the evolution and constraint equations and the Hilbert space inner product gives precisely the ADM energy which is conserved. Thus the analysis of the helicity 3/2 equations has direct applications to the analysis of the full vacuum equations: estimates for the helicity 3/2 equations give a priori estimates for the spin connection for solutions of the full vacuum equations. This identification is given in section 7.2.1.

2. Penrose's proposal that helicity 3/2 fields should provide a vehicle for a definition of a twistor in vacuum space-times, [10].

3. The third is the proposal, made by a number of authors over the years, that the helicity 3/2 equations provide a type of Lax pair for the full vacuum equations, raising the

possibility that one might be able to develop an inverse scattering transform (although this cannot be such as to lead to complete integrability of the equations). These applications are discussed briefly in the last section but will be followed up in detail only in subsequent papers.

2   Notation and definitions

We work on a 4-dimensional Lorentzian space-time (M, g). We assume, for consistency of the helicity 3/2 equations, that the space-time satisfies the Einstein vacuum equations. For the initial value problem we assume that the space-time is globally hyperbolic with a foliation by orientable Cauchy surfaces Σ_t, $t \in \mathbb{R}$. With these assumptions, (M, g) admits a spin structure and we choose one. We use the two-component spinor notation and conventions of Penrose and Rindler [11]. Thus the bundle of positive or self-dual spinors is denoted $\mathbb{S}^{A'}$ and negative or anti-self-dual spinors is denoted $\mathbb{S}^A$ and the tangent bundle $T^a M = \mathbb{S}^A \otimes \mathbb{S}^{A'}$.

Many formulae are simplified by the use of the notation of spinor-indexed differential forms (using a form of Cartan's calculus). In this notation, a spinor or tensor expression taking values in the tensor product of a bundle of forms with a bundle of spin tensors is written with the differential form indices suppressed but the other indices explicit. The indexed 1-form $\mathrm{d}x^{AA'} \in \Omega^1 \otimes \mathbb{S}^A \otimes \mathbb{S}^{A'}$ is the Kronecker-delta that maps the 1-form, $\nu_{AA'}$ as an indexed quantity to $\nu_{AA'}\mathrm{d}x^{AA'}$ the 1-form with its indices suppressed. We use d to denote the covariant exterior derivative, $\mathrm{d} = \mathrm{d}x^{AA'}\nabla_{AA'}$ where ∇_a acts as the standard spin connection on spinor indexed quantities and the suppressed form index is understood to be skew-symmetrized over the form indices in the quantity on which d acts. Thus d is the standard exterior derivative augmented so as to act on spinor indexed quantities using the Levi-Civita spin connection.[1]

We introduce the future pointing time-like normal to Σ_t normalized, for convenience, to have squared length 2,

$$T_a \mathrm{d}x^a = N\mathrm{d}t\,, \quad T_a T^a = 2\,.$$

This leads to a $3 + 1$ decomposition of the metric

$$g = \tfrac{1}{2}N^2 \mathrm{d}t^2 - h$$

where h is the part of g orthogonal to T^a.

The spinor form $T^{AA'}$ of T^a can be used to convert primed indices into unprimed indices. The normalization implies that

$$T_A^{A'} T_{B'}^A = -\varepsilon_{B'}{}^{A'}\,,$$

so that if one performs the conversion twice, one obtains minus the identity. We can express a 4-vector orthogonal to T^a as a symmetric spinor (of either type) as follows. If $T^{AA'}V_{AA'} = 0$ then $V_{A'B'} = T_{A'}{}^A V_{B'A}$ and $V_{AB} = T_A{}^{A'} V_{BA'}$ are both symmetric and determine $V^{AA'}$ by $V_{AA'} = -T_A{}^{B'} V_{A'B'} = -T_{A'}{}^B V_{AB}$.

We also decompose the covariant derivative into its time-like and space-like parts

$$\nabla_a = \mathrm{D}_a + \tfrac{1}{2}T_a \nabla_T\,, \quad T^a \mathrm{D}_a = 0\,.$$

---

[1]This notation only makes good sense in the context of Penrose's abstract index convention, Penrose (1986). Those who are uncomfortable with this convention should replace $\mathrm{d}x^{AA'}$ by a Van der Waerden symbol $\theta^{AA'}$ and d with D to make clear the connection dependence of d when acting on indexed forms.

By conversion of primed to unprimed indices we introduce the following slightly different forms of the space-like covariant derivative

$$\mathrm{D}_{AB} = T_A{}^{A'}\mathrm{D}_{BA'}, \quad \text{and } \mathrm{D}_{A'B'} = T_{A'}{}^{A}\mathrm{D}_{B'A}\,.$$

Note that D_a is the four-dimensional covariant derivative restricted to act tangent to Σ_t and is distinct from the intrinsic Levi-Civita connection on Σ_t. In particular we have

$$\mathrm{D}_a T_b = K_{ab}$$

where $T^a K_{ab} = 0$ and $K_{ab} = K_{(ab)}$ is $\sqrt{2}$ times the extrinsic curvature. Thus the conversion of primed spinor indices into unprimed ones does not commute with this restriction of the 4-dimensional covariant derivative (although it does with the Levi-Civita spin connection of h).

It will be convenient also to introduce the quantity

$$N_{AA'} = \mathrm{D}_{AA'}\log N = -\frac{1}{2}\nabla_T T_{AA'}$$

where the last equality follows from $T^b\nabla_b T_a = 2T^b\nabla_{[b}T_{a]} = 2T^b(\mathrm{D}_{[b}N)(\nabla_{a]}t) = -2\mathrm{D}_a\log N$. We also define $N_{AB} = T_A{}^{A'}N_{BA'}$.

3 The Rarita-Schwinger equations

3.1 The equations in space-time

The Rarita-Schwinger equations have their simplest expression in terms of spinor valued differential forms. The basic variable is a spinor indexed $1-$form $\sigma_A = \sigma_{BB'A}\mathrm{d}x^{BB'}$. The field equations are

$$\mathrm{d}\sigma_A \wedge \mathrm{d}x^{AA'} = 0\,. \tag{1}$$

In flat space-time, the form σ_A is a potential for the field ψ_{ABC} defined by

$$\mathrm{d}\sigma_A = \psi_{ABC}\mathrm{d}x^B_{B'} \wedge \mathrm{d}x^{CB'}\,.$$

The field determines σ_A up to the gauge freedom

$$\sigma_A \mapsto \sigma_A + \mathrm{d}\nu_A \tag{2}$$

where ν_A is an arbitrary spinor field. In curved space, the field is no longer invariant under these gauge transformations, and we take the true degrees of freedom to be defined as potentials σ_A satisfying equation (1) modulo the gauge freedom (2).

Making explicit all the indices, equation (1) gives

$$\nabla^{A'}_{(A}\sigma_{B)A'}{}^{B} = 0\,, \qquad \nabla^{A}_{(A'}\sigma_{B')AB} = 0\,,$$

with gauge freedom $\sigma_{AA'B} \mapsto \sigma_{AA'B} + \nabla_{AA'}\nu_B$.

There are two consistency requirements for the field equations:

1. pure gauge potentials should be solutions to the equations

2. we must have two differential identities among the field equations to compensate for the fact that they are over-determined. (The 8 unknown dependent variables can be reduced to 6 by means of a gauge transformation; for example one can use a gauge transformation to set $V^a\sigma_{aB}$ to zero by defining ν_B to be the integral along the integral curves of the vector V^a of $V^a\sigma_{aB}$. However, there are then 8 field equations on 6 unknowns so that the equations are over-determined. In the context of a $3+1$ decomposition there will be two constraint equations on initial data. We require two identities which will imply that the evolution equations will preserve the constraints.)

Both consistency requirements are met iff the Einstein vacuum equations are satisfied. They follow from the identity

$$\mathrm{d}x^{AA'} \wedge \mathrm{d}^2\alpha_A = \frac{i}{2}\alpha_A G_b^{AA'*} X^b \qquad (3)$$

where G_{ab} is the Einstein tensor, α_A is a spinor indexed form and $X_a = \frac{1}{6}\varepsilon_{abcd}\mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d$ (see pp434-5 of [11]). When α_A is a spinor-indexed 0-form this implies that pure gauge potentials satisfy the field equations. When α_A is a spinor-indexed 1-form, this yields the desired two differential identities between the field equations.

There is a conserved current associated to solutions of the Rarita-Schwinger equations whose integral over a hypersurface yields a conserved positive definite inner product on solutions modulo gauge. The differential form version of the field equations yields immediately that the 3-form

$$i\sigma_A \wedge \bar{\sigma}_{A'} \wedge \mathrm{d}x^{AA'} \qquad (4)$$

is closed when σ_A satisfies the field equations and is exact when σ_A is pure gauge. Thus it must vanish on $\sigma_A = \mathrm{d}\nu_A$ for a compactly supported ν_A.

3.2 The $3+1$ decomposition and a gauge choice

We now fix the gauge freedom in such a way that the integral of the 3-form, (4) above becomes positive definite on solutions to the equations. The gauge we choose is analogous to the Coulomb gauge in electrodynamics and is based on that used by Witten [15].

We first perform a 3+1 splitting on σ_A:

$$\sigma_{bA}\mathrm{d}x^b = \alpha_A\mathrm{d}t + \phi_{bA}\mathrm{d}x^b, \text{ where } \quad T^b\phi_{bA} = 0.$$

We impose the gauge condition

$$\phi_{AA'}{}^A = 0.$$

In this gauge the field $\phi_{BB'A}$ has just four independent components as the spinor $\phi_{BB'A}T_C^{B'}$ is now totally symmetric over ABC. In order to transform from a general gauge to this gauge, we must invert the Witten operator $\nu^A \mapsto D_{AA'}\nu^A$. The invertibility in the asymptotically flat framework considered subsequently follows from the work of Parker and Taubes, [9]. (These arguments extend readily to the compact case also.) The gauge transformations that preserve this condition must satisfy the Witten equation, $\mathrm{D}_{AA'}\nu^A = 0$ and the invertibility of the Witten operator implies that there is no residual gauge freedom with $\nu^A \to 0$ at ∞.

3.2.1 Inner product

With this reduction, the integral of the form $-i\sigma_A \wedge \bar{\sigma}_{A'} \wedge \mathrm{d}x^{AA'}$ gives the positive definite inner product

$$\langle \phi_1, \phi_2 \rangle = \int_{\Sigma_t} i\sigma_A \wedge \bar{\sigma}_{A'} \wedge \mathrm{d}x^{AA'} = \int_{\Sigma_t} -\phi_{aB}\bar{\phi}_{B'}^a T^{BB'}\mathrm{d}\,\mathrm{Vol}_{\Sigma_t}$$

where $\mathrm{d}\,\mathrm{Vol}_{\Sigma_t} = \frac{1}{\sqrt{2}}T_d X^d$, $X_d = -\frac{1}{6}\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \varepsilon_{abcd}$ and the minus sign arises from our conventions in which the Lorentzian metric is negative definite on space-like vectors.

3.2.2 The constraint and evolution equations

With this decomposition of σ_A the full equations become

$$X^{BB'}\left(\nabla_{B'}^A \phi_{ABA'} + N(\varepsilon_{A'B'}\mathrm{D}_B^A\alpha_A - \mathrm{D}_{A'B'}\alpha_B)\right) = 0 \qquad (5)$$

These equations have three irreducible parts: two parts have two components each and only involve derivatives tangential to Σ_t, and the third has four components and determines the evolution of ϕ_{aB}. The first spatial equation involves ϕ_{aB} alone and is treated separately from

4

the others as a constraint equation. It arises from $\mathrm{d}\sigma_B \wedge \mathrm{d}x^{BB'}|_{\Sigma_t} = 0$ (i.e. the contraction of $T^{BB'}$ in place of $X^{AA'}$ in equation (5)). This gives

$$\mathrm{D}^{AB}\phi_{A'AB} = 0\,. \tag{6}$$

The second spatial equation arises by contraction of $\varepsilon^{A'B'}$ into (5):

$$2\mathrm{D}_{AB}\alpha^B = N\nabla^{BB'}\phi_{ABB'} = N\left(T_A^{C'}K_{CC'BB'}\phi^{BB'C} + N^b\phi_{bA}\right) \tag{7}$$

where the second equality follows by use of (6). This is an elliptic equation for α^A in terms of $\phi_{AA'B}$. The evolution equations are an irreducible part of

$$T_{(A'}^A \nabla_T \phi_{B')AB} = -2\mathrm{D}_{(A'}^A \phi_{B')AB} - \frac{2}{N}\mathrm{D}_{A'B'}\alpha_B\,.$$

These are 6 equations and are a combination of (7) and the four evolution equations for $\phi_{AA'B}$. These can be separated out by writing $\phi_{ABC} = T_A^{A'}\phi_{BA'C}$ and eliminating all primed spinor indices using $T^{AA'}$. This yields

$$\nabla_T \phi_{ABC} = \mathrm{D}_{(A}^D \phi_{BC)D} + N_{(A}^D \phi_{BC)D} - K_{(AB}{}^{EF}\phi_{C)EF} + \frac{1}{6}K\phi_{ABC} - \frac{2}{N}\mathrm{D}_{(AB}\alpha_{C)}\,. \tag{8}$$

When the slices Σ_t are asymptotically flat or compact, we can use the inverse $(\mathrm{D}^{-1})_A^B$ of the Witten operator $\alpha_A \mapsto \mathrm{D}_A^B \alpha_B$ to eliminate α_A using equation (7) to obtain the nonlocal evolution equation

$$\begin{aligned}
\nabla_T \phi_{ABC} &= \mathrm{D}_{(A}^D \phi_{BC)D} + N_{(A}^D \phi_{BC)D} - K_{(AB}{}^{EF}\phi_{C)EF} + \frac{1}{6}K\phi_{ABC} \\
&\quad - \frac{2}{N}\mathrm{D}_{(AB}(\mathrm{D}^{-1})_{C)}^D\{NK_{DEFG}\phi^{EFG} + N^b\phi_{bD}\}\,.
\end{aligned} \tag{9}$$

where $\nu_A = (\mathrm{D}^{-1})_A^B \mu_B$ is equivalent to $\mu_A = \mathrm{D}_{AB}\nu^B$ with $\nu^A \to 0$ at ∞.

3.2.3   Preservation of constraints

Our strategy will be to solve the evolution equation independently of the constraints (6) and then to solve the constraints (6) on an initial surface and deduce from the consistency condition (3) that they will be satisfied at all subsequent times. This follows from the two differential identities between the field equations as follows.

If we are granted equations (7) and (9), then, in the differential form notation of the previous section,

$$\mathrm{d}x^{AA'} \wedge \mathrm{d}\sigma_A = C^{A'}T_b X^b$$

where $C_{A'}$ is proportional to the constraints $\mathrm{D}^{BC}\phi_{BCA'}$ by a numerical factor and $X^a = \frac{1}{4}\varepsilon^a{}_{bcd}\mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d$. The consistency condition (3) in vacuum gives that the left hand side of the above equation is covariantly closed identically. Thus

$$\nabla_b(T^b NC^{A'}) = \nabla_T(NC^{A'}) + \sqrt{2}KNC^{A'} = 0\,. \tag{10}$$

Thus, if $C^{A'}$ vanishes initially, it will remain zero.

Note that the evolution is unitary when the constraints are imposed (as follows directly from the closure of the form $-i\sigma_A \wedge \bar{\sigma}_{A'} \wedge \mathrm{d}x^{AA'}$).

4   Function spaces and the hypotheses on the metric

We have assumed our space-time to be globally hyperbolic, so the Cauchy hypersurfaces Σ_t of our foliation are homeomorphic to a given 3-manifold Σ (Geroch [3]). We work on

the smooth 4-manifold $\mathbb{R}_t \times \Sigma$ endowed with a Lorentzian metric g. We will specify our assumptions on g by requiring that it is continuously differentiable in time (up to a certain order) with values in some function space of Sobolev type on Σ.

In order to define Sobolev spaces on Σ, we equip it with a smooth Riemannian 3-metric $\tilde{h}$ that is Euclidean outside a compact set K. The metric space $(\Sigma, \tilde{h})$ is necessarily complete. Let $\tilde{D}$ and $\mathrm{dVol}_{\tilde{h}}$ be respectively the covariant derivative and the metric volume element on Σ associated with the metric $\tilde{h}$.

We will be concerned with spinors and tensors on Σ. We assume that we have a smooth spin frame (and hence orthonormal frame) that is global and constant where $\tilde{h}$ is flat near infinity. This identifies spinors and tensors with their components. In the following we will use $\langle , \rangle$ to denote the natural positive definite inner product induced by $\tilde{h}$ on these quantities. The standard function spaces with spinor and tensor values on Σ, suppressing indices, are:

- $\mathcal{C}^k(\Sigma)$, $k \in \mathbb{N} \cup \{\infty\}$, is the space of k times continuously differentiable functions on Σ. The subspace of compactly supported functions is denoted $\mathcal{C}_0^k(\Sigma)$ and of functions uniformly bounded on Σ together with their derivatives is denoted $\mathcal{C}_b^k(\Sigma)$.

- $H^s(\Sigma)$, $s \in \mathbb{N}$, is the Sobolev space given by completing $\mathcal{C}_0^\infty(\Sigma)$ in the norm

$$\|f\|_{H^s(\Sigma)} = \left\{ \sum_{p=0}^{s} \int_{\Sigma} \left\langle \tilde{D}^p f, \tilde{D}^p f \right\rangle \mathrm{dVol}_{\tilde{h}} \right\}^{1/2}, \tag{11}$$

(this agrees with the standard definition for all Riemannian manifolds which are compact or Euclidean at infinity, Hebey [4]). The space $H^0(\Sigma)$ will be denoted $L^2(\Sigma)$.

- $H_\delta^s(\Sigma)$ for $s \in \mathbb{N}$ and $\delta \in \mathbb{R}$, is the weighted Sobolev space given by completing $\mathcal{C}_0^\infty(\Sigma)$ in the norm

$$\|f\|_{H_\delta^s(\Sigma)} = \left\{ \sum_{p=0}^{s} \int_{\Sigma} \left(1 + r^2\right)^{\delta+p} \left\langle \tilde{D}^p f, \tilde{D}^p f \right\rangle \mathrm{dVol}_{\tilde{h}} \right\}^{1/2}, \tag{12}$$

where $r(x)$ is the $\tilde{h}$-distance from x to a fixed point $O \in \Sigma$. (The function space is independent of the choice of O.) The space $H_\delta^0(\Sigma)$ will be denoted $L_\delta^2(\Sigma)$.

- $\mathcal{C}_\delta^k(\Sigma)$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, is the weighted Hölder space of functions in $\mathcal{C}^k(\Sigma)$ for which the norm

$$\|f\|_{\mathcal{C}_\delta^k(\Sigma)} = \sup_{x \in \Sigma} \sum_{l=0}^{k} \left\{ \left(1 + r^2\right)^{\delta+l} \left\langle \tilde{D}^l f, \tilde{D}^l f \right\rangle \right\}^{1/2} \tag{13}$$

is finite.

We will use this notation when the spinor or tensor in question is unambiguous. Otherwise we shall use the notation $\mathcal{C}^k(\Sigma, \mathbb{S}_A)$, $H_\delta^k(\Sigma, \mathbb{S}_{A'(AB)})$, $\mathcal{C}_\delta^k(\Sigma, T_{ab})$ to denote the class of spinor or tensor fields that we are considering. We can now express our requirements for the metric g.

Definition 4.1 *We say that the metric g on $\mathbb{R}_t \times \Sigma$ belongs to the class (k, δ), k a positive integer, $\delta \in \mathbb{R}$, if*

$$g - \tilde{g} \in \mathcal{C}^l\left(\mathbb{R}_t; H_\delta^{k-l}(\Sigma)\right), 0 \le l \le k \tag{14}$$

where $\tilde{g} = dt \otimes dt - \left(1 + \frac{m}{r}\right) \tilde{h}$ is a background Lorentzian metric on $\mathbb{R} \times \Sigma$. We define the class (∞, δ) as the intersection of all classes (k, δ) for $k \in \mathbb{N}$.

Remark 4.1 *Weighted Sobolev spaces give, in addition to the local control on the derivatives given by the usual Sobolev spaces, more flexibility in the control on the rate at which quantities fall off at infinity than usual Sobolev spaces provide. We see, using the continuous embedding*

$$H_\delta^k(\Sigma) \hookrightarrow \mathcal{C}_{\delta'}^{k-2}(\Sigma), \quad \delta' < \delta + 3/2, k \ge 2, \tag{15}$$

6

(Choquet-Bruhat and Christodoulou [1]), that if g is of class (k, δ), $k \geq 2$, $\delta \in \mathbb{R}$, then

$$g - \tilde{g} \in \mathcal{C}^l \left(\mathbb{R}_t; \mathcal{C}^{k-l-2}_{\delta'}(\Sigma) \right), 0 \leq l \leq k - 2, \delta' < \delta + 3/2. \tag{16}$$

Thus, as $r \to +\infty$,

$$\tilde{D}^l(g - \tilde{g}) = O(r^{-\delta'-l}), \quad 0 \leq l \leq k - 2, \ \delta' < \delta + 3/2.$$

Note that, due to the presence of m/r in $\tilde{g}$, we can only guarantee the following fall-off for the derivatives of g:

$$\tilde{D}^l g \in \mathcal{C}^p \left(\mathbb{R}_t; \mathcal{C}^{k-l-p-2}_{\alpha+l}(\Sigma) \right), \ 0 \leq l \leq k - 2, \ 0 \leq p \leq k - l - 2, \tag{17}$$

where α satisfies $\alpha < \delta + 3/2$ and $\alpha \leq 1$, i.e. as $r \to +\infty$

$$\tilde{D}^l g = O \left(r^{-\alpha-l} \right), \ 0 \leq l \leq k - 2, \ \alpha < \delta + 3/2 \ and \ \alpha \leq 1. \tag{18}$$

The extrinsic curvature satisfies

$$K_{ab} = D_a T_b = O(r^{-\delta'-1}), \ r \to +\infty, \ \delta' < \delta + 3/2 \tag{19}$$

and

$$N_{AA'} = -\frac{1}{2} D_T T_{AA'} = O(r^{-\delta'}), \ r \to +\infty, \ \delta' < \delta + 3/2. \tag{20}$$

The Rarita-Schwinger equation is associated with the metric g and not $\tilde{g}$, so that the natural function spaces for the solutions are Sobolev spaces on Σ_t associated with the 3-metric $h(t)$ induced by g. For $t \in \mathbb{R}$, $s \in \mathbb{N}$, we define the Sobolev space $H^s(\Sigma_t)$ as the completion of $\mathcal{C}^\infty_0(\Sigma)$ for the norm

$$\|f\|_{H^s(\Sigma_t)} = \left\{ \sum_{p=0}^s \int_\Sigma \left\langle \overline{D}^p f, \overline{D}^p f \right\rangle \mathrm{dVol}_{\Sigma_t} \right\}^{1/2}, \tag{21}$$

where $\overline{D}$ and dVol_{Σ_t} are the covariant derivative and the volume element on Σ associated with $h(t)$. With one more natural assumption on g, this space can be identified with the Sobolev space $H^k(\Sigma)$ defined using $\tilde{h}$. We suppose the metric g satisfies

(H) *There exists two continuous functions $C_1, C_2 > 0$ of t such that for all $(t, x) \in \mathbb{R} \times \Sigma$, the lapse function N and the eigenvalues λ_i, $i = 1, 2, 3$ of $h(t, x)$ as a symmetric form relative to $\tilde{h}$ satisfy*

$$C_1(t) \leq \lambda_i(t, x) \leq C_2(t), \quad C_1(t) \leq N(t, x) \leq C_2(t).$$

Lemma 4.1 *If the metric g is of class (k, δ), $k \geq 3$, $\delta > -3/2$, and satisfies hypothesis (H), we can define the spaces $H^l(\Sigma_t)$, $t \in \mathbb{R}$, for $0 \leq l \leq k - 2$ and the norms on these spaces are equivalent to the norms on $H^l(\Sigma)$, $0 \leq l \leq k - 2$, i.e. the identity map is an isomorphism from $H^l(\Sigma)$ onto $H^l(\Sigma_t)$, $t \in \mathbb{R}$, $0 \leq l \leq k - 2$. This norm equivalence is uniform on each compact time interval. At the level of minimum regularity spaces, the mapping*

$$f \in L^2(\Sigma) \longmapsto \frac{(det(h(t)))^{1/4}}{\left(det(\tilde{h}) \right)^{1/4}} f \in L^2(\Sigma_t) \tag{22}$$

is an isometry. This is a direct consequence of the definitions of $L^2(\Sigma)$ and $L^2(\Sigma_t)$. Note that the continuity of g and hypothesis (H) entail the completeness of $(\Sigma, h(t))$ for all $t \in \mathbb{R}$.

Proof: Assuming the metric satisfies (H), the measurable functions for $\mathrm{d\,Vol}_{\Sigma_t}$ are the same as those for $\mathrm{d\,Vol}_{\tilde{h}}$. Similarly the integrability of terms in $\left\langle \overline{D}^p f, \overline{D}^p f \right\rangle$ for which all the derivatives act on f is equivalent to that of the same terms in $\left\langle \tilde{D}^p f, \tilde{D}^p f \right\rangle$. More precisely,

these terms in $\left\langle \overline{D}^p f, \overline{D}^p f \right\rangle$ can be estimated by those in $\left\langle \tilde{D}^p f, \tilde{D}^p f \right\rangle$ and vice versa. Terms in which derivatives act on the connection coefficients and metric are controlled by lower order terms using the uniform bounds on the metric and its derivatives. We can only define Sobolev spaces of order lower than $k-2$ as the metric is not more than $\mathcal{C}_b^{k-2}$ and this prevents us from taking more than $k-2$ derivatives if we wish to have values in L^∞, as required for a continuous action on L^2 by multiplication. When applying $k-2$ covariant derivatives to $f \in \mathcal{C}_0^\infty(\Sigma)$, $k-3$ of them will act on the spin coefficients of the first covariant derivative and spin coefficients already are a first order derivative of the metric. The assumption $\delta > -3/2$ is the minimum requirement to ensure that $g - \tilde{g}$ tends to zero at spatial infinity. $\square$

We will therefore usually not distinguish $H^l(\Sigma)$ from $H^l(\Sigma_t)$; we will use the notation $H^l(\Sigma)$ unless the norm associated to a given hypersurface Σ_t is significant, in which case we shall write $H^l(\Sigma_t)$. When the precise value of the norm is not significant we will use $\mathrm{dVol}_{\tilde{h}}$ (writing simply $\mathrm{d\,Vol}$) and we shall write $\| \ \|_2$ instead of $\| \ \|_{L^2(\Sigma)}$.

Remark 4.2 (a) *Note that the conserved quantity is the $L^2(\Sigma_t)$ norm not the $L^2(\Sigma)$ norm. However, they are directly related by the isometry (22).*

(b) *If we choose a slicing by maximal Cauchy surfaces, the volume element dVol_{Σ_t} will be independent of time. We can then choose dVol_{Σ_t} as the measure of volume for all our function spaces $L^2(\Sigma)$, $H^k(\Sigma)$, etc.. With this choice, the norm of the solution in $L^2(\Sigma)$ will be preserved by the evolution.*

(c) *We can also define Sobolev and weighted Sobolev spaces for spinors using the space-like part D of the restriction of the 4-dimensional covariant derivative from the whole space-time (M, g) instead of the Levi-Civita spin connection $\overline{\mathrm{D}}$ of Σ_t. The operators D and $\overline{\mathrm{D}}$ differ by a zero order term which is a combination of the extrinsic curvature and therefore, the Sobolev and weighted Sobolev norms defined using D and $\overline{\mathrm{D}}$ respectively will be equivalent. This remark will play a role in the study of the non-local part of the equation in the proof of proposition 5.1.*

Definition 4.2 *We denote by $\mathcal{H}_t$ the space of helicity 3/2 fields for which the integral*

$$\langle \sigma, \sigma \rangle = -\int_{\Sigma_t} i\sigma_A \wedge \bar{\sigma}_{A'} \wedge \mathrm{d}x^{AA'} \tag{23}$$

is well defined. In the gauge introduced above, this reduces to

$$\langle \sigma, \sigma \rangle = -\frac{1}{\sqrt{2}} \int_{\Sigma_t} \phi_{aB'} \bar{\phi}_B^a T^{BB'} \mathrm{d\,Vol}_{\Sigma_t}$$

which is a manifestly positive definite inner product. (Note that with our conventions the metric on a space-like vector is negative definite.)

With a choice of unitary spin-frame (i.e. (o_A, ι_A) with $\iota^A = T^{AA'}\bar{o}_{A'}$), $\mathcal{H}_t \simeq L^2(\Sigma_t; \mathbb{C}^4)$ as ϕ_{bA} has four independent components and, with the unitary spin frame, the inner product is diagonalized. As seen above, each $L^2(\Sigma_t; \mathbb{C}^4)$ is isometric to the space $L^2(\Sigma; \mathbb{C}^4)$ constructed using the background metric $\tilde{h}$, that is each $\mathcal{H}_t$ is isometric to a fixed $\mathcal{H}$ defined as $L^2(\Sigma; \mathbb{C}^4)$. The isometry is not canonical as it relies on a choice of spin frame (and hence orthonormal tetrad) at each time, and an identification between the hypersurfaces at different times. With a maximal slicing, the isometry becomes constant.

Finally, we define a family of function spaces that will be used in the study of the non-local term.

Definition 4.3 *For $k \in \mathbb{N}$, $k \geq 1$, we define the space $\mathbb{H}^k(\Sigma)$ as the completion of $\mathcal{C}_0^\infty(\Sigma)$ for the norm $\|.\|_{\mathbb{H}^k}$:*

$$\|f\|_{\mathbb{H}^k(\Sigma)}^2 = \|f\|_{L_{-1}^2(\Sigma)}^2 + \|\tilde{\mathrm{D}}f\|_{H^{k-1}(\Sigma)}^2 = \int_\Sigma \left\{ (1+r^2)^{-1}|f|^2 + \sum_{p=1}^k \left\langle \tilde{\mathrm{D}}^p f, \tilde{\mathrm{D}}^p f \right\rangle \right\} \mathrm{dVol}. \tag{24}$$

5 The global Cauchy problem

In this section we prove existence and uniqueness of solutions to the evolution equation (9) independently of the constraint equation (6). The spatial equation (7) is solved by inverting the Witten operator to express α_A in terms of ϕ_{bA}. The elimination of α_A in the evolution equation leads to a non-local term. We shall see that this term is bounded and therefore does not obstruct the well-posedness of the Cauchy problem. In the next section, to find the physical solutions satisfying the constraints, we will show that the constrained subspace is preserved by the evolution.

If we express equation (9) in terms of the components of ϕ with respect to a unitary spin frame (i.e. a spinor dyad (o_A, ι_A) for which $\iota^A = T^{AA'} \bar{o}_{A'}$ and $\iota^A o_A = 1$) and in a local coordinate basis, we obtain a system of the form

$$\partial_t \phi = i \sum_{i=1}^{3} A^i(t,x) \frac{\partial}{\partial x^i} \phi + \text{ terms of order } 0$$

where the 4×4 matrices A^i are hermitian. Thus the system is a zero order perturbation of a first order symmetric hyperbolic system for which general existence results are known. This will enable us to prove global existence (independently of the constraints).

We recall equation (9)

$$\nabla_T \phi_{ABC} = \ \mathrm{D}^D_{(A} \phi_{BC)D} + N^D_{(A} \phi_{BC)D} - K_{(AB}{}^{EF} \phi_{C)EF} + \tfrac{1}{6} K \phi_{ABC}$$
$$- \tfrac{2}{N} \mathrm{D}_{(AB}(\mathrm{D}^{-1})^D_{C)} \{ N K_{DEFG} \phi^{EFG} + N^b \phi_{bD} \}.$$

We write this in the Hamiltonian form

$$\frac{\partial \phi}{\partial t} = iA(t)\phi + Q_1(t)\phi + Q_2(t)\phi \tag{25}$$

where $(iA(t)\phi)_{ABC}$ is the first order part in

$$N \mathrm{D}^D_{(A} \phi_{BC)D} \tag{26}$$

which in a local coordinate frame has the form

$$\sum_{i=1}^{3} A^i(t,x) \frac{\partial}{\partial x^i} \phi, \ A^i \text{ are hermitian matrices.}$$

The second term, Q_1, is the local potential given by

$$N \left\{ N^D_{(A} \phi_{BC)D} - K_{(AB}{}^{EF} \phi_{C)EF} + \frac{1}{6} K \phi_{ABC} + \text{ spin coefficients} \right\}, \tag{27}$$

where the spin coefficients arise from expressing both the spatial and temporal covariant derivatives in a spin frame. The last potential Q_2 is the non-local term

$$-2\mathrm{D}_{(AB}(\mathrm{D}^{-1})^D_{C)} \{ N K_{DEFG} \phi^{EFG} + N^b \phi_{bD} \}. \tag{28}$$

We first discuss the class of the background metric we require and then give the main theorems on the well-posedness of the global Cauchy problem for equation (25) for minimum and higher regularity solutions independently of the constraint (6).

Remark 5.1 *The metrics defined in the previous section of class (k, δ) with $\delta > 0$ give a standard definition of asymptotically flat space-times for general relativity; for these metrics, g and $\tilde{g}$ differ at infinity by terms smaller than $r^{-3/2}$ and the extrinsic curvature vanishes faster than $r^{-5/2}$. Occasionally weaker definitions of asymptotically flat space-times are considered, as for example in [9], and the only assumptions are that the metric tends towards*

the background metric faster that $1/r$ and the extrinsic curvature falls off faster than $1/r^2$. This is given by the class (k, δ), $\delta > -1/2$, and it is this class of metrics that we shall prove our results for here.

Our results are true for a larger class, namely (k, δ) with $\delta > -3/2$ (note that, for this class, the presence of m/r in $\tilde{g}$ is meaningless). However, the physical relevance is not clear and so we express our theorems for $\delta > -1/2$. The less physical metrics with $\delta > -3/2$ tend towards $\tilde{g}$ as $r \to +\infty$, are continuous, bounded in space and their first order derivatives and extrinsic curvature fall off at infinity faster than $1/r$. These are the only conditions we use in our proofs of the theorems. Note that for the injectivity of the Witten operator in the proof of proposition 5.1, we will only just be able to guarantee $\varepsilon > 0$ in inequality (60). If we were to weaken the fall-off assumptions on the metric further, we could no longer choose $\varepsilon > 0$ and this part of the proof would break down; we would then also have $A_1(t) - B \in H_\delta^k$ with a value of δ that would no longer allow us to apply the essential theorems from [1].

Theorem 1 *Let the metric g be of class $(4, \delta)$, $\delta > -1/2$ and satisfy hypothesis (H), then for $s \in \mathbb{R}$, given $\phi_0 \in \mathcal{H}_s$, equation (25) has a unique solution $\phi_s(t) \in \mathcal{H}_t$ for each t, such that, using the isometry $\mathcal{H}_t \simeq \mathcal{H}$,*

$$\phi_s \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{H}\right), \ \phi_s|_{t=s} = \phi_0.$$

The propagator

$$\mathcal{V}(t, u): \ \phi_s(u) \longmapsto \phi_s(t)$$

is a continuous semi-group of operators on $\mathcal{H}$ satisfying the properties

(i) $\mathcal{V}$ is strongly continuous on $\mathbb{R}^2$ to $\mathcal{L}(\mathcal{H})$ with $\mathcal{V}(t, t) = Id$.

(ii) $\mathcal{V}(t, s)\mathcal{V}(s, r) = \mathcal{V}(t, r)$.

The solutions described in theorem 1 are solutions in the sense of distributions on $\mathbb{R} \times \Sigma$. For smoother solutions we have

Theorem 2 *If the metric g is of class (k, δ), $k \geq 4$, $\delta > -1/2$, and satisfies hypothesis (H), then for $\phi_0 \in H^m(\Sigma; \mathbb{C}^4)$, $1 \leq m \leq k - 3$, the associated solution $\phi_s(t)$ of equation (25) satisfies*

$$\phi \in \mathcal{C}^l\left(\mathbb{R}_t; H^{m-l}(\Sigma; \mathbb{C}^4)\right), \ 0 \leq l \leq m.$$

The propagator $\mathcal{V}$ satisfies the additional properties

(iii) $\mathcal{V}(t, s): H^m(\Sigma; \mathbb{C}^4) \hookrightarrow H^m(\Sigma; \mathbb{C}^4)$, $\mathcal{V}$ is strongly continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}(H^m(\Sigma; \mathbb{C}^4))$.

(iv) $\frac{\partial}{\partial t}\mathcal{V}(t, s) = (iA(t) + Q_1(t) + Q_2(t))\mathcal{V}(t, s)$, $\frac{\partial}{\partial s}\mathcal{V}(t, s) = -\mathcal{V}(t, s)(iA(s) + Q_1(s) + Q_2(s))$. Both derivatives exist in the strong sense in $\mathcal{L}\left(H^m(\Sigma; \mathbb{C}^4), H^{m-1}(\Sigma; \mathbb{C}^4)\right)$ and are strongly continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}\left(H^m(\Sigma; \mathbb{C}^4), H^{m-1}(\Sigma; \mathbb{C}^4)\right)$.

Before we prove theorems 1 and 2, we state a general result on the non-local part of the equation which will be used in the proofs of theorems 1 and 2.

Proposition 5.1 *If the metric is of class (k, δ), $k \geq 4$, $\delta > -1/2$, and satisfies hypothesis (H), the two operators defining the non-local potential satisfy the following properties:*

1) For $1 \leq p \leq k - 2$, $0 \leq l \leq p - 1$, $l, p \in \mathbb{Z}$,

$$\left(\psi_C \longmapsto \mathrm{D}_{(AB}\psi_{C)}\right) \in \mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(\mathbb{H}^{p-l}(\Sigma; \mathbb{C}^2); H^{p-l-1}(\Sigma; \mathbb{C}^4)\right)\right).$$

2) For $1 \leq p \leq k - 2$, $0 \leq l \leq p - 1$, $l, p \in \mathbb{Z}$,

$$\left(\psi_D \longmapsto \mathrm{D}_C^D\psi_D\right) \in \mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(\mathbb{H}^{p-l}(\Sigma; \mathbb{C}^2); H^{p-l-1}(\Sigma; \mathbb{C}^2)\right)\right)$$

and this operator (the Witten operator) is an isomorphism from $\mathbb{H}^p(\Sigma; \mathbb{C}^2)$ onto $H^{p-1}(\Sigma; \mathbb{C}^2)$, $1 \leq p \leq k - 2$, at each time $t \in \mathbb{R}$.

Consequently, the non-local operator $\psi_D \longmapsto \mathrm{D}_{(AB}(\mathrm{D}^{-1})_{C)}^D\psi_D$ involved in the definition of the non-local potential Q_2 belongs to the following spaces for $1 \leq p \leq k - 2$, $0 \leq l \leq p - 1$, $l, p \in \mathbb{Z}$:

$$\mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(H^{p-l-1}(\Sigma; \mathbb{C}^2); H^{p-l-1}(\Sigma; \mathbb{C}^4)\right)\right). \tag{29}$$

10

The proof of proposition 5.1 is given in appendix 1. We now give the proofs of the two theorems.

Proof of theorem 1: In order to solve the Cauchy problem for equation (25), we consider it as a perturbation of the simpler equation

$$\partial_t \phi = iA(t)\phi. \tag{30}$$

This is a first order linear symmetric hyperbolic system on $\mathbb{R}_t \times \Sigma$ and the well-posedness of the L^2-Cauchy problem for this class of equations is well-known. We appeal to standard results for the solution to this equation in trivial topology and then generalize to nontrivial topology. Then we solve the global Cauchy problem for (25) in $\mathcal{C}(\mathbb{R}_t; \mathcal{H})$ by a simple fixed-point argument, interpreting the potentials $Q_1(t)$ and $Q_2(t)$ as locally integrable functions in time with values in the Banach space of bounded linear operators on $\mathcal{H}$. The details of this proof are as follows.

The case of trivial topology.
When the topology is trivial, the whole space-time can be parametrized by $\mathbb{R}_t \times \mathbb{R}_x^3$ and the term $iA(t)\phi$ is of the form

$$iA(t)\phi = \sum_{i=1}^{3} a^i(t,x) \frac{\partial \phi}{\partial x^i}, \quad (t,x) \in \mathbb{R}^4.$$

The a^i's are hermitian 4×4 matrices whose coefficients are those of the space-like Infeld-Van der Waerden symbols multiplied by the lapse function N.

With our assumptions on the metric, we have immediately that

$$a^i \in \mathcal{C}^0\left(\mathbb{R}_t; \mathcal{C}_b^2(\mathbb{R}^3)\right) \hookrightarrow \mathcal{C}^0\left(\mathbb{R}_t; \mathcal{C}_b^1(\mathbb{R}^3)\right). \tag{31}$$

The well-posedness of the Cauchy problem in L^2 for this type of symmetric hyperbolic system is given, for example, in section 6.4 of Jerome, 1983, [5], culminating in theorem 6.4.5. This gives existence of a unique family of operators $\{\mathcal{U}(t,s)\}$ defined on $\mathbb{R}_{t,s}^2$ and satisfying the following properties (using the isomorphisms $\mathcal{H} \simeq L^2(\mathbb{R}^3; \mathbb{C}^4)$ and $H^1(\Sigma; \mathbb{C}^4) \simeq H^1(\mathbb{R}^3; \mathbb{C}^4)$):

(a) $\mathcal{U}$ is strongly continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}(\mathcal{H})$ with $\mathcal{U}(t,t) = \mathbb{1}$.

(b) $\mathcal{U}(t,s)\mathcal{U}(s,r) = \mathcal{U}(t,r)$.

(c) $\mathcal{U}(t,s): H^1(\Sigma; \mathbb{C}^4) \hookrightarrow H^1(\Sigma; \mathbb{C}^4)$ and $\mathcal{U}$ is strongly continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}(H^1(\Sigma; \mathbb{C}^4))$.

(d) $\frac{\partial}{\partial t}\mathcal{U}(t,s) = iA(t)\mathcal{U}(t,s)$, $\frac{\partial}{\partial s}\mathcal{U}(t,s) = -i\mathcal{U}(t,s)A(s)$ which both exist in the strong sense in $\mathcal{L}(H^1(\Sigma; \mathbb{C}^4), \mathcal{H})$ and are strongly continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}(H^1(\Sigma; \mathbb{C}^4), \mathcal{H})$.

Thus, the global Cauchy problem is solved for (30) in the case of trivial topology.

The case of nontrivial topology.
We cover Σ by balls and apply the result for trivial topology in the domain of dependence of each ball. The finite propagation speed for equation (30) yields, for a short time, a solution global on Σ from the solutions in the domains of dependence. This will be enough to prove global existence and uniqueness of solutions of (30).

The propagation speed of equation (30) is estimated at each point $(t,x) \in \mathbb{R}^4$ by

$$3\mathrm{Sup}\left\{\|a^i(t,x)\|; \ 1 \leq i \leq 3, \ x \in \mathbb{R}^3\right\} \tag{32}$$

which, in turn, is controlled by a positive continuous function of the variable t only: $C(t)$. It is thus bounded uniformly in space and locally uniformly in time. The bound (32) is given in theorem 3.1 in Racke, 1992, [12], for $\mathcal{C}^1$ solutions, however, we are by no means certain of the existence of solutions with such regularity here. The theorem is in effect still true for H^1-valued solutions whose existence is guaranteed by property (c) above (the proof remains essentially the same modulo a couple of technical modifications). The result

11

is readily extended to minimum regularity solutions using the density of H^1 in L^2 and the continuity of the solutions with respect to their initial data. In non trivial topology, this estimate on the propagation speed can be obtained locally in the same manner using local coordinate charts; the uniform control only depends on uniform estimates on the metric and its derivatives which are true by assumption. Therefore, we still have existence of a continuous function $C(t)$ such that for each $(t,x) \in \mathbb{R} \times \Sigma$, the propagation speed at (t,x) is estimated by $C(t)$.

When Σ is not topologically trivial, we use this uniformly finite propagation speed to localize the problem into open sets of trivial topology. Consider a compact time interval $[-T,T]$, $T > 0$, and $t_0 \in [-T,T[$. Let $\{\Omega_i\}_{1 \le i \le n}$ be a finite covering of Σ by open sets of trivial topology and evolve each set Ω_i into its domain of dependence from time $t = t_0$, i.e. we evolve the boundary $\partial\Omega_i$ of Ω_i along the flow of the vector field

$$v(t,x)\nu^a + \frac{1}{\sqrt{2}}T^a$$

where ν^a is the interior normal to $\partial\Omega_i$ and $v(t,x)$ the propagation speed at the point $(t,x) \in \mathbb{R} \times \Sigma$ in the direction ν^a. (This is equivalent to evolving Ω_i into its domain of dependence for the metric g, i.e. along null-geodesics orthogonal to $\partial\Omega_i$, since the characteristics for equation (30) are the null geodesics.) For $t \ge t_0$ we obtain open sets $\Omega_i(t)$, $1 \le i \le n$, $\Omega_i(t_0) = \Omega_i$. Because of the uniformly finite propagation speed on $[-T,T] \times \Sigma$, we can choose the covering $\{\Omega_i\}$ such that for some $\varepsilon > 0$ and for $t \in [t_0, t_0 + \varepsilon]$ the family $\{\Omega_i(t)\}_{1 \le i \le n}$ is still a covering of Σ by topologically trivial open sets. Moreover, the length ε of the time interval can be kept constant for all $t_0 \in [-T,T[$.

Considering some initial data $\phi_0 \in \mathcal{H}$ at time t_0, in each domain of dependence, (using the result for trivial topology) we have existence of a unique solution ϕ_i, continuous in time with values in L^2 such that

$$\phi_i|_{t=t_0} = \phi_0|_{\Omega_i}.$$

The uniqueness of the solution in each domain of dependence guarantees that for $t \in [t_0, t_0 + \varepsilon]$

$$\phi_i(t)|_{\Omega_i(t) \cap \Omega_j(t)} = \phi_j(t)|_{\Omega_i(t) \cap \Omega_j(t)}$$

whenever the intersection $\Omega_i(t) \cap \Omega_j(t)$ is non-empty. Hence, it makes sense to define

$$\phi(t,x) = \phi_i(t,x), \quad \text{for } x \in \Omega_i(t). \tag{33}$$

Each function ϕ_i is continuous in time with values in L^2 in the domain of dependence of Ω_i. Furthermore, for each $t \in [t_0, t_0 + \varepsilon]$, a function f defined on Σ_t is in $L^2(\Sigma)$ if and only if $f|_{\Omega_i(t)} \in L^2(\Omega_i(t))$, $1 \le i \le n$ and we have

$$\|f\|^2_{L^2(\Sigma)} \le \sum_{i=1}^{n} \|f\|^2_{L^2(\Omega_i(t))} \le n\|f\|^2_{L^2(\Sigma)}.$$

Therefore

$$\phi \in \mathcal{C}([t_0, t_0 + \varepsilon]; \mathcal{H}).$$

Further, ϕ restricted to the domain of dependence of each Ω_i is a solution of (30) in the sense of distributions, so that ϕ is a solution of (30) in the sense of distributions on $[t_0, t_0 + \varepsilon] \times \Sigma$. (Note that the initial value condition is satisfied by definition $\phi(t_0)|_{\Omega_i} = \phi_i(t_0)|_{\Omega_i} = \phi_0|_{\Omega_i}$ so that $\phi|_{t=t_0} = \phi_0$.)

To sum up, given some initial data $\phi_0 \in \mathcal{H}$ at $t = s$, consider $T > 0$ with $|T| > |s|$, then propagate the solution forward from $t = s$, step by step on time intervals of length ε, up to $t = T$. Reversing time in equation (30) allows us to propagate backwards in exactly the same manner down to $t = -T$. This guarantees the existence and uniqueness of solutions of (30) in $\mathcal{C}([-T,T]; \mathcal{H})$ for each $T > 0$, i.e. the global Cauchy problem for (30) is solved independently of the topology.

Furthermore, on each domain of dependence and small time interval there is a unique local propagator $\mathcal{U}_i(t, s)$ satisfying properties (a), (b), (c), and (d). The local uniqueness allows us to patch the $\mathcal{U}_i$'s together and to construct a global propagator which we denote $\mathcal{U}(t, s)$, $t, s \in \mathbb{R}$, satisfying properties (a), (b), (c), and (d).

Extension to perturbed Hamiltonian

We give a general result of well-posedness of the Cauchy problem for a class of perturbations of (30) and prove that Q_1 and Q_2 fit into this class.

Proposition 5.2 *Given $Q \in L^1_{loc}(\mathbb{R}_t; \mathcal{L}(\mathcal{H}))$, the Cauchy problem for the perturbed equation*

$$\partial_t \phi = iA(t)\phi + Q(t)\phi \tag{34}$$

is well-posed in $\mathcal{H}$ in the sense that, $\forall s \in \mathbb{R}$ and $\forall \phi_0 \in \mathcal{H}_s$, equation (34) has a unique solution $\phi_s(t)$ such that

$$\phi_s(t) \in \mathcal{C}(\mathbb{R}_t; \mathcal{H}), \ \phi_s|_{t=s} = \phi_0. \tag{35}$$

The propagator for equation (34) $\mathcal{W}(t, u) : \phi_s(u) \longmapsto \phi_s(t)$ is a continuous semi-group on $\mathcal{H}$ satisfying properties (i) and (ii) of theorem 1.

The proof of proposition 5.2 is a standard fixed point argument. It is included for completeness as a second appendix.

In order to solve the minimum regularity Cauchy problem for (25), we now need to check that

$$Q_1, Q_2 \in L^1_{loc}(\mathbb{R}_t; \mathcal{L}(\mathcal{H})).$$

The local term Q_1 simply involves multiplying the components of ϕ by spin-coefficients, components of the extrinsic curvature and $N^{AA'}$, all of them multiplied by the lapse function N. With our hypotheses on the metric, all these quantities are in $\mathcal{C}^0(\mathbb{R}_t, \mathcal{C}^1_b(\Sigma))$. Hence, we have immediately

$$Q_1 \in \mathcal{C}^0(\mathbb{R}_t; \mathcal{L}(\mathcal{H})) \hookrightarrow L^1_{loc}(\mathbb{R}_t; \mathcal{L}(\mathcal{H})). \tag{36}$$

The non-local potential Q_2 results from the application of a non-local operator to a spinor ψ_D

$$-2\mathrm{D}_{(AB}(\mathrm{D}^{-1})^D_{C)}\psi_D$$

where ψ_D is obtained by contracting ϕ with quantities in $\mathcal{C}^0(\mathbb{R}_t, \mathcal{C}^1_b(\Sigma))$. Consequently, the operator

$$\phi \longmapsto \psi_D = N K_{DEFG}\phi^{EFG} + N^b \phi_{bD}$$

is in $\mathcal{C}(\mathbb{R}_t; \mathcal{L}(\mathcal{H}; L^2(\Sigma; \mathbb{C}^2)))$. Moreover, using proposition 5.1, we see that the non local operator belongs to the space

$$\mathcal{C}(\mathbb{R}_t; \mathcal{L}(L^2(\Sigma; \mathbb{C}^2); \mathcal{H})).$$

This implies that the non-local potential Q_2 has the following regularity

$$Q_2 \in \mathcal{C}^0(\mathbb{R}_t; \mathcal{L}(\mathcal{H})) \hookrightarrow L^1_{loc}(\mathbb{R}_t; \mathcal{L}(\mathcal{H})), \tag{37}$$

and this concludes the proof of theorem 1. $\square$

Proof of theorem 2:

We follow the same steps as for theorem 1. We take the metric to be of class (k, δ), $k \geq 4$, $\delta > -1/2$. We first solve the Cauchy problem in H^m, $1 \leq m \leq k - 3$, for the free equation (30) on a topologically trivial space-time. In this case, using global coordinates (t, x^i) on $\mathbb{R} \times \mathbb{R}^3$, equation (30) has the form

$$\frac{\partial \phi}{\partial t} = \sum_{i=1}^{3} a^i(t, x) \frac{\partial \phi}{\partial x^i}$$

13

where the 4×4 matrices a^i are hermitian and with entries in $\mathcal{C}^l\left(\mathbb{R}_t; \mathcal{C}_b^{k-l-2}(\mathbb{R}_x^3)\right)$ for $0 \leq l \leq k-2$. The well-posedness of the Cauchy problem in H^m, $1 \leq m \leq k-3$, for such symmetric hyperbolic systems can be found in, for example, Racke, 1992, [12], theorem 3.3 (for m large enough); or the proof of theorem 6.4.5 in Jerome, 1984, [5] can be adapted using the identifying operator $S^m = (Id - \Delta)^{m/2}$ from $H^m(\mathbb{R}^3)$ onto $L^2(\mathbb{R}^3)$ instead of S from H^1 onto L^2. Hence, we see that the propagator $\{\mathcal{U}(t,s)\}$ for equation (30) satisfies the stronger version of properties (a), (b), (c), (d):

(a) $\mathcal{U}$ is strongly continuous on $\mathbb{R}^2$ to $\mathcal{L}(\mathcal{H})$ with $\mathcal{U}(t,t) = Id$.

(b) $\mathcal{U}(t,s)\mathcal{U}(s,r) = \mathcal{U}(t,r)$.

(c') $\mathcal{U}(t,s) : H^m(\Sigma; \mathbb{C}^4) \hookrightarrow H^m(\Sigma; \mathbb{C}^4)$ for $1 \leq m \leq k-3$, and $\mathcal{U}$ is strongly continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}(H^m(\Sigma; \mathbb{C}^4))$.

(d') $\frac{\partial}{\partial t}\mathcal{U}(t,s) = iA(t)\mathcal{U}(t,s)$, $\frac{\partial}{\partial s}\mathcal{U}(t,s) = -i\mathcal{U}(t,s)A(s)$ which both exist in the strong sense in $\mathcal{L}(H^m(\Sigma; \mathbb{C}^4); H^{m-1}(\Sigma; \mathbb{C}^4))$, $1 \leq m \leq k-3$, and are continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}(H^m(\Sigma; \mathbb{C}^4); H^{m-1}(\Sigma; \mathbb{C}^4))$.

Just as in the proof of theorem 1, we can show that $\mathcal{U}(t,s)$ satisfies (a), (b), (c'), (d') when Σ is topologically non trivial using the finite propagation speed for equation (30) and localizing the study in domains of dependence of topologically trivial open sets.

We now give, as before for minimum regularity solutions, an existence and uniqueness theorem for solutions in $H^m(\Sigma)$, $1 \leq m \leq k-3$, to a wide class of perturbations of (30).

Proposition 5.3 *Given an operator*

$$Q \in L_{loc}^1\left(\mathbb{R}_t; \mathcal{L}(H^m(\Sigma; \mathbb{C}^4))\right), \ for \ m \ such \ that \ 1 \leq m \leq k-3, \tag{38}$$

the Cauchy problem for the perturbed equation

$$\partial_t \phi = iA(t)\phi + Q(t)\phi \tag{39}$$

is well-posed in $H^m(\Sigma; \mathbb{C}^4)$. More precisely, for $s \in \mathbb{R}$ and $\phi_0 \in H^m(\Sigma; \mathbb{C}^4)$, equation (39) has a unique solution ϕ such that

$$\phi \in \mathcal{C}(\mathbb{R}_t; H^m(\Sigma; \mathbb{C}^4)), \ \phi|_{t=s} = \phi_0. \tag{40}$$

The propagator for equation (39) $\mathcal{W}(t,u) : \ \phi_s(u) \longmapsto \phi_s(t)$ is a continuous semi-group on $\mathcal{H}$ satisfying properties (i), (ii) and (iii) of theorems 1 and 2.
Further, if Q is assumed to be continuous in time, i.e.

$$Q \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{L}\left(H^m(\Sigma; \mathbb{C}^4)\right)\right)$$

for m such that $1 \leq m \leq k-3$, then $\mathcal{W}$ satisfies a property analogous to (iv) in theorem 2:
$\frac{\partial}{\partial t}\mathcal{W}(t,s) = (iA(t) + Q(t))\mathcal{W}(t,s)$, $\frac{\partial}{\partial s}\mathcal{W}(t,s) = -\mathcal{W}(t,s)(iA(s) + Q(s))$. *Both derivatives exist in the strong sense in $\mathcal{L}\left(H^m(\Sigma; \mathbb{C}^4), H^{m-1}(\Sigma; \mathbb{C}^4)\right)$ and are continuous on $\mathbb{R}_{t,s}^2$ to $\mathcal{L}\left(H^m(\Sigma; \mathbb{C}^4), H^{m-1}(\Sigma; \mathbb{C}^4)\right)$.*

The proof of proposition 5.3 can be found in appendix 2 after the proof of proposition 5.2.

Thus it remains to show that, with greater regularity we have assumed of the metric, Q_1 and Q_2 are suitably bounded operators. Since the metric is of class (k, δ), $\delta > -1/2$, the potential Q_1 is a multiplication operator by quantities in $\mathcal{C}^l\left(\mathbb{R}_t; \mathcal{C}_b^{k-l-3}(\Sigma)\right)$, $0 \leq l \leq k-3$ so that

$$Q_1 \in \mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(H^{k-l-3}(\Sigma; \mathbb{C}^4)\right)\right), \ 0 \leq l \leq k-3.$$

The multiplication part of the non-local potential Q_2 is

$$\phi \longmapsto \psi_D = NK_{DEFG}\phi^{EFG} + N^b\phi_{bD}$$

14

and is in $\mathcal{C}^l\left(\mathbb{R}_t;\mathcal{L}\left(H^{k-l-3}(\Sigma;\mathbb{C}^4);H^{k-l-3}(\Sigma;\mathbb{C}^2)\right)\right)$, $0\le l\le k-3$, for the same reason as for Q_1. Using proposition 5.1, we see that the non-local operator in Q_2

$$\psi_D\longmapsto -2\mathrm{D}_{(AB}(\mathrm{D}^{-1})_{C)}\psi_D$$

belongs to

$$\mathcal{C}^l\left(\mathbb{R}_t;\mathcal{L}\left(H^{k-l-3}(\Sigma;\mathbb{C}^2);H^{k-l-3}(\Sigma;\mathbb{C}^4)\right)\right),\ 0\le l\le k-3.$$

Consequently, the potentials Q_1 and Q_2 both have the same regularity

$$Q_1,Q_2\in\mathcal{C}^l\left(\mathbb{R}_t;\mathcal{L}\left(H^{k-l-3}(\Sigma;\mathbb{C}^4)\right)\right),\ 0\le l\le k-3. \tag{41}$$

In particular, for $1\le m\le k-3$, the potentials Q_1 and Q_2 belong to $\mathcal{C}(\mathbb{R}_t;\mathcal{L}(H^m(\Sigma;\mathbb{C}^4)))$. This guarantees the well-posedness of the Cauchy problem in $H^m(\Sigma;\mathbb{C}^4)$ for equation (25) thanks to proposition 5.3, and the fact that the propagator $\mathcal{V}$ satisfies properties (iii) and (iv).

Now, for $0\le l\le k-4$, the first order operator $A(t)$ satisfies

$$A(t)\in\mathcal{C}^l\left(\mathbb{R}_t;\mathcal{L}\left(H^{k-l-3}(\Sigma;\mathbb{C}^4);H^{k-l-4}(\Sigma;\mathbb{C}^4)\right)\right). \tag{42}$$

Using this and the regularity of Q_1 and Q_2, we read in an obvious manner directly from equation (25) that the solution $\phi_s\in\mathcal{C}(\mathbb{R}_t;H^m(\Sigma))$, $1\le m\le k-3$, associated with some initial time $s\in\mathbb{R}$ and some initial data $\phi_0\in H^m(\Sigma)$, has in fact the additional regularities

$$\phi_s\in\mathcal{C}^l\left(\mathbb{R}_t;H^{m-l}(\Sigma;\mathbb{C}^4)\right),\ 0\le l\le m. \tag{43}$$

This concludes the proof of theorem 2. $\square$

6 The constraint equations

In this section we construct the projection operator onto the subspace of solutions to the constraint equation (6). We denote by $\mathbb{D}$ the constraint operator on a space-like hypersurface Σ_t, $t\in\mathbb{R}$

$$\mathbb{D}:\phi=\phi_{AA'B}\longmapsto\mathbb{D}\phi=\mathrm{D}^{AB}\phi_{AA'B} \tag{44}$$

considered as an unbounded operator from $\mathcal{H}$ to $L^2\left(\Sigma;\mathbb{C}^2\right)$.

Theorem 3 *If the metric g is of class $(4,\delta)$, $\delta>-1/2$, and satisfies (H), the operator*

$$P(t)=1-\mathbb{D}^*\left(\mathbb{D}\mathbb{D}^*\right)^{-1}\mathbb{D} \tag{45}$$

is well-defined and bounded on $\mathcal{H}$. It is the orthogonal projector onto the closed subspace of $\mathcal{H}$:

$$\mathcal{K}_t=\{\phi\in\mathcal{H};\mathbb{D}\phi=0\} \tag{46}$$

and consequently $\|P(t)\|=1$ provided $\mathcal{K}_t\ne\{0\}$.

Proof of theorem 3: The main problem in the definition of the projector P is to show that $(\mathbb{D}\mathbb{D}^*)^{-1}$ exists. We do this by showing that $\mathbb{D}\mathbb{D}^*$ is injective on $\mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)$. For $\alpha\in\mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)$,

$$(\alpha,\mathbb{D}\mathbb{D}^*\alpha)_{L^2(\Sigma)}=\|\mathbb{D}^*\alpha\|_2^2, \tag{47}$$

so it is sufficient to prove the injectivity of $\mathbb{D}^*$ on $\mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)$. If $\alpha\in\mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)$ is such that $\mathbb{D}^*\alpha=0$, α must be a 3-surface twistor because, reintroducing indices (flipping the primed index with $T^{AA'}$) we have

$$\mathbb{D}^*\alpha=\mathrm{D}_{(AB}\alpha_{C)}. \tag{48}$$

The compact support of α then entails $\alpha_A=0$ (in Tod (1983) it is shown that a solution to the 3-surface twistor equation, together with its first derivative, satisfy a transport equation

15

so that, if it vanishes together with its first derivative at a point, it vanishes everywhere). Hence, $\mathbb{D}\mathbb{D}^*$ is a bijection from $\mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)$ onto the subspace of $L^2\left(\Sigma;\mathbb{C}^2\right)$:

$$\text{Ran}(\mathbb{D}\mathbb{D}^*)_{\mathcal{C}_0^\infty} = \left\{\mathbb{D}\mathbb{D}^*\alpha;\ \alpha \in \mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)\right\} \subset \mathcal{C}_0^0\left(\Sigma;\mathbb{C}^2\right). \tag{49}$$

Note that $\text{Ran}(\mathbb{D}\mathbb{D}^*)_{\mathcal{C}_0^\infty}$ is only a set of continuous functions because of the weak regularity of the metric. We now introduce the subspace $\mathcal{F}$ of $\mathcal{H}$ defined by

$$\mathcal{F} = \mathcal{K} \oplus \text{Ran}(\mathbb{D}^*)_{\mathcal{C}_0^\infty} \tag{50}$$

where

$$\text{Ran}(\mathbb{D}^*)_{\mathcal{C}_0^\infty} = \left\{\mathbb{D}^*\alpha;\ \alpha \in \mathcal{C}_0^\infty\left(\Sigma;\mathbb{C}^2\right)\right\} \subset \mathcal{C}_0^1\left(\Sigma;\mathbb{C}^4\right). \tag{51}$$

We have obviously

$$\mathbb{D}\mathcal{F} = \text{Ran}(\mathbb{D}\mathbb{D}^*)_{\mathcal{C}_0^\infty}$$

since $\mathbb{D}\mathcal{K} = \{0\}$ by definition, which allows us to define P on $\mathcal{F}$. Moreover, $\text{Ran}(\mathbb{D}^*)_{\mathcal{C}_0^\infty}$ is dense in $\mathcal{K}^\perp$ since if $\phi \in \mathcal{H}$ is orthogonal to $\text{Ran}(\mathbb{D}^*)_{\mathcal{C}_0^\infty}$, we must have $\phi \in \mathcal{K}$. Hence $\mathcal{F}$ is dense in $\mathcal{H}$ and is stable under P. We have

$$P : \mathcal{F} \longrightarrow \mathcal{K} , \quad P|_\mathcal{K} = Id_\mathcal{K} , \quad P|_{\text{Ran}(\mathbb{D}^*)_{\mathcal{C}_0^\infty}} = 0. \tag{52}$$

Thus we can define P^2 on $\mathcal{F}$ and we see immediately that

$$P^2 = P \quad \text{on } \mathcal{F}. \tag{53}$$

Properties (52) and (53) together with the self-adjointness of P on $\mathcal{F}$ and the density of $\mathcal{F}$ in $\mathcal{H}$ show that P can be extended in a unique way as a bounded self-adjoint operator from $\mathcal{H}$ to itself. This operator, which we still denote by P, is the orthogonal projector onto $\mathcal{K}$ and its norm is 1 provided $\mathcal{K} \neq \{0\}$. The orthogonal projector onto $\mathcal{K}^\perp$ is $1 - P$ and its norm is 1 since

$$\mathcal{K}^\perp = \overline{\text{Ran}(\mathbb{D}^*)_{\mathcal{C}_0^\infty}} \neq \{0\}. \quad \square \tag{54}$$

Remark 6.1 *We have seen in section 3.2.3 that the quantity $C_{A'}$, which is proportional to the constraints, satisfies the differential equation*

$$\nabla_T\left(NC^{A'}\right) + \sqrt{2}KNC^{A'} = 0.$$

Thus if $C^{A'}$ is zero initially, it remains zero and, conversely, if $C^{A'}$ is non zero initially, it remains non zero. A consequence of this result is that if the initial data ϕ_0 at time s belongs to the constrained subspace $\mathcal{K}_s$, then at each time t, the solution $\phi(t)$ belongs to $\mathcal{K}_t$.

7 Conclusion and perspectives

7.1 Analytic aspects

Concerning the purely analytic aspect of this work, one would like in the short term to improve the results in two ways:

- *The maximal regularity of the solution in terms of the regularity of the metric.* As things stand, for a metric of class (k,δ), $k \geq 4$, $\delta > -1/2$, i.e. belonging to $\mathcal{C}^l\left(\mathbb{R}_t; H_\delta^{k-l}(\Sigma)\right)$, $0 \leq l \leq k$, the highest regularity we obtain for the solution is $\mathcal{C}^l\left(\mathbb{R}_t; H^{k-l-3}(\Sigma)\right)$, $0 \leq l \leq k-3$. This loss of three degrees of smoothness has two causes. The nature of the equation entails the loss of one degree because the potentials are derivatives of the metric. The other two degrees are lost through the use of Sobolev's embedding theorems

$$H_\delta^k(\Sigma) \hookrightarrow \mathcal{C}_{\delta'}^l(\Sigma) , \quad l\langle k - \frac{3}{2}, \quad \delta'\langle\delta + \frac{3}{2}$$

16

for k and l integers. If, instead of considering the coefficients of the potentials as elements of Hölder spaces (through Sobolev's embedding results), we keep the full information that these coefficients belong to weighted Sobolev spaces, we should be able to prove the existence of solutions in spaces of higher smoothness using product theorems between Sobolev spaces. In spite of the technical difficulty, a strong incentive for doing this is the fact that the spin connection is a pair of spin 3/2 fields with only one degree of smoothness less than the metric. One should therefore expect to be able to define solutions with a loss of one and not three degrees of smoothness.

- *Fall-off at infinity of spin 3/2 fields.* We specified hypotheses on the metric in terms of weighted Sobolev spaces because they allowed us to control the fall-off of the metric at infinity as well as its regularity. One naturally expects to relate the behavior of the metric at infinity to that of the spin connection. This would become crucial if we wish to extract some information on the metric from the connection; there is no way one could hope to control the fall-off of the metric at infinity if nothing is known on the fall-off of the connection. For these reasons, it seems necessary to prove that the propagator for the spin 3/2 equation acts stably and continuously on weighted Sobolev spaces. The major difficulty would be to prove the existence in these spaces of solutions to the principal part of the equation which we denoted

$$\partial_t \phi = A(t)\phi.$$

To our knowledge, at the time of writing there are no general results addressing the well-posedness of the Cauchy problem in weighted Sobolev spaces for symmetric hyperbolic equations (although see page 65 of Rendall 1997). As for the potentials and the non-local term, they would in fact be more naturally dealt with in the framework of weighted Sobolev spaces (a large part of the complication of the proof of proposition 5.1, and the strange space $\mathbb{H}^k$ which we need to introduce, comes from the fact that we need to control standard Sobolev norms and not weighted Sobolev norms).

7.2 Applications to the vacuum equations

There are a number of potential applications to the analysis of the vacuum equations. We discuss (1) the direct analysis of the vacuum equations in the sense of proving long time existence and uniqueness, (2) the analysis via an inverse scattering approach using the helicity 3/2 equations as a type of Lax pair for the vacuum equations. A further application that we do not discuss here is to the definition in general relativity of twistors as charges of helicity 3/2 fields. It may well be possible to relate this to the second topic above.

7.2.1 The direct analysis

In this application we are interested in the analysis of the initial value problem for the Einstein vacuum equations. We will assume that we are using a maximal slicing condition in what follows as this simplifies some of the considerations.

This application arises from the fact that the spin connection can be naturally identified with a pair of helicity 3/2 fields. One defines the spin connection by choosing a spinor dyad $\varepsilon_{\underline{B}}^A = (o^A, \iota^A)$, where the concrete index $\underline{B} = 0, 1$. (The dyad is not assumed to be normalized in the following.) We then define

$$\gamma_{A\underline{B}} = (\gamma_{A0}, \gamma_{A1}) = (\mathrm{d}o_A, \mathrm{d}\iota_A).$$

The spin connection can be expressed in this spin frame as the 2×2 matrix of 1-forms $\gamma_{\underline{B}}^A$. However, we will prefer to consider it as the pair of helicity 3/2 fields, $\gamma_{A\underline{B}}$ where $\underline{B} = 0, 1$. Superficially these helicity 3/2 fields are pure gauge. However, pure gauge fields have the form $\mathrm{d}\nu_A$ where $\nu_A \to 0$ at ∞ whereas (o^A, ι^A) must be asymptotically constant to be a

spinor dyad. Furthermore, we have that the fields $\gamma_{A\underline{B}}$ have nonzero norm with respect to the natural helicity $3/2$ inner product. We have

$$\|\gamma_{A0}\|^2 = i \int_{\Sigma_t} \gamma_{A0} \wedge \bar{\gamma}_{A'0'} \wedge \mathrm{d}x^{AA'} = \lim_{R\to\infty} \int_{S_R} io_A \mathrm{d}\bar{o}_{A'} \wedge \mathrm{d}x^{AA'} \,.$$

The expression in the middle is the Sparling three-form, the Hamiltonian density for general relativity (Mason & Frauendiener 1991) and the right hand expression is the Witten-Nester expression for the total energy in vacuum of the gravitational field (Witten 1981, Nester 1983, Mason & Frauendiener 1991). Thus the norms of these fields are null components of the ADM energy momentum 4 vector and vanish iff the space-time is vacuum (assuming that it is also asymptotically flat at space-like infinity).

The gauge freedom absorbs the freedom of choice of a spin frame, and the gauge choice of the preceding sections leads to the imposition of the Witten equations on the spinor dyad. There are precisely two independent asymptotically constant solutions of the Witten equation (Parker & Taubes 1982). As the trace of the extrinsic curvature vanishes, these can be taken to be $SU(2)$ conjugates, $\iota^A = T^{AA'}\bar{o}_{A'}$, as the Witten equation is then self-adjoint. We will henceforth assume that we have chosen a maximal slicing and that (o^A, ι^A) have been chosen to be $SU(2)$ conjugates. [There is a possible problem here in that the dyad could fail to be linearly independent at some points in the interior. This can only happen if the function $f = \iota^A o_A = T^{AA'} o_A \bar{o}_{A'}$ vanishes at some point or points and hence if both components of o^A vanish in some spin-frame. This requires the vanishing of two complex functions and so generically this will not happen on the 3-manifold Σ. It may be possible to show that f does not vanish in general (at least outside an event horizon).]

Thus, the results given in the present work for Rarita-Schwinger fields also apply directly to the constraint and evolution equations for the spin connection that follow from the vacuum equations. One can think of these results as giving a priori estimates for the evolution of the connection. It is hoped that these will be important in establishing long time and/or large data existence theorems for the vacuum equations. The particular advantage one has over other approaches is that one is using the physical energy of the system and that one is working with a less complicated equation that is of lower order in the derivatives of the metric.

7.2.2 Inverse scattering

It has been proposed by a number of authors that the Rarita-Schwinger equations can be thought of as a linear system or Lax Pair for the vacuum equations as the vacuum equations are the consistency conditions for these equations. The reduction of the equations given above can be used to obtain a formulation that is reminiscent of a Lax pair but with some important differences. The form of the evolution equations given in equation (9) is not formally skew Hermitian when the constraints are not imposed. In order to have unitary evolution even in the case where the constraints are not satisfied one can write the evolution equation above in Hamiltonian form as $(\partial_t - H)\phi = 0$ and then replace that evolution by $(\partial_t - H')\phi = 0$ where $H' = \frac{1}{2}(H - H^\dagger)$ and $H^\dagger$ is the Hermitian conjugate of H. This will agree with the evolution above on solutions to the constraints (in fact one can check that this merely, in effect, changes H by subtracting off the hermitian conjugate of the non-local term which vanishes on the solutions to the constraint equations). The evolution preserves the constraints, and hence preserves the subspace of solutions to the constraints. Thus, if P is the projector onto the constraints, we have

$$[\partial_t - H', P] = 0 \,, \tag{55}$$

and this consistency condition is equivalent to the Einstein vacuum equations. This should be compared with the Lax pair formulation of the Korteweg de Vries equation, $[\partial_t - A, L] = 0$ where

$$L = \partial_x^2 + 2u(x,t) \,, \quad \text{and} \quad A = \partial_x^3 + 3u\partial_x + v \,.$$

18

The main difference between the two situations is that L is an elliptic operator for each t, whereas P is a projector.

The main feature of an integrable system is that there should exist 'sufficiently many' constants of motion in involution (an infinite number in the case of partial differential equations). This arises from the operator L in the Lax pair as its spectrum is, formally, constant as the equation $[\partial_t - A, L] = 0$ implies that L evolves by conjugation (more precisely, the constants arise as coefficients in the asymptotic expansion in the variable λ about $\lambda = \infty$ of trace$(L - \lambda)^{-1}$). Here the spectrum of P is $\{0, 1\}$ and so there are no interesting constants of the motion. This is as it should be as general relativity is not an integrable system and one does not expect constants of the motion over and above the ADM quantities.

The other important use of the Lax pair is in the scattering transform and its inverse. This transform, roughly speaking, is the map from initial data $u(x, t)$ at some fixed t to the asymptotic parts of the integral kernel for $(L - \lambda)^{-1}$ as $x \to \pm\infty$. Such a transform is conceivable also in our case. It would be a map from initial data sets for general relativity to the asymptotics at spatial infinity for the kernel of the operator P. This will be studied in due course.

Appendix 1: proof of proposition 5.1

We denote by $\mathcal{D}$ the Witten operator (following the notation in [9])

$$\psi_A \longmapsto (\mathcal{D}\psi)_B = \mathrm{D}_B^A \psi_A \tag{56}$$

and by $\not{D}$ the symmetrized space-like derivative

$$\psi_C \longmapsto (\not{D}\psi)_{ABC} = \mathrm{D}_{(AB}\psi_{C)}. \tag{57}$$

$\mathcal{D}$ and $\not{D}$ are respectively a contracted and a symmetrized form of the space-like part D of the covariant derivative on (M, g). Remembering that $\mathrm{D} = \overline{\mathrm{D}} + \kappa$ where κ is a combination of the extrinsic curvature, we see that $\mathcal{D}$ and $\not{D}$ are both first order operators such that the coefficients of their first order part belong to $\mathcal{C}^l\left(\mathbb{R}_t; \mathcal{C}_b^{k-l-2}(\Sigma)\right)$, $0 \leq l \leq k - 2$. From the definition of $\mathbb{H}^p$, it is obvious that $\tilde{\mathrm{D}}$ is a bounded operator from $\mathbb{H}^p(\Sigma)$ to $H^{p-1}(\Sigma)$. Consequently, the first order parts of $\not{D}$ and $\mathcal{D}$ belong to

$$\mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(\mathbb{H}^{k-l-2}(\Sigma), H^{k-l-3}(\Sigma)\right)\right), \ 0 \leq l \leq k - 3.$$

Concerning the zero order parts of these two operators, their coefficients are either spin coefficients or come from the extrinsic curvature. Using remark 4.1, we can interpret these as elements of $\mathcal{C}^l(\mathbb{R}_t; \mathcal{C}_{\alpha+1}^{k-l-3}(\Sigma))$, $0 \leq l \leq k - 3$, $\alpha < \delta + 3/2$, $\alpha \leq 1$. In particular, since $\delta + 3/2 \geq 0$, we see that the coefficients of the zero order parts of $\mathcal{D}$ and $\not{D}$ belong to the function spaces

$$\mathcal{C}^l\left(\mathbb{R}_t; \mathcal{C}_1^{k-l-3}(\Sigma)\right) \hookrightarrow \mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(\mathbb{H}^{k-l-3}(\Sigma), H^{k-l-3}(\Sigma)\right)\right)$$

$$\hookrightarrow \mathcal{C}^l\left(\mathbb{R}_t; \mathcal{L}\left(\mathbb{H}^{k-l-2}(\Sigma), H^{k-l-3}(\Sigma)\right)\right), \ 0 \leq l \leq k - 3,$$

(the weight 1 acts continuously from L_{-1}^2 to L^2 and thus changes elements of $\mathbb{H}^p$ into elements of H^p). Hence the regularity part of properties 1) and 2) is proved and we now need to establish the invertibility of $\mathcal{D}$ on H^{p-1} with values in $\mathbb{H}^p$ at each time.

We start by showing that $\mathcal{D}$ is an isomorphism from $\mathbb{H}^1 = H_{-1}^1$ onto L^2 at each time. This property is established in [9] for smooth asymptotically flat metrics. In order to generalize this result, we consider the metric g as a limit, in the class (k, δ), of a continuous family of smooth metrics g_λ, $\lambda \in [0, 1[$, of class (∞, δ) and that satisfy hypothesis (H). More precisely, this means that $g_\lambda - g$ is continuous in λ on $[0, 1[$ with values on $\mathcal{C}^l\left(\mathbb{R}_t; H_\delta^{k-l}(\Sigma)\right)$, $0 \leq l \leq k$, and as $\lambda \to 1$, we have

$$g_\lambda - g \longrightarrow 0 \quad \text{in} \quad \mathcal{C}^l\left(\mathbb{R}_t; H_\delta^{k-l}(\Sigma)\right), \ 0 \leq l \leq k.$$

In this manner, we interpret the operator $\mathcal{D}(t)$ at each time t as the limit of the continuous family of Witten operators $\mathcal{D}_\lambda(t)$ associated with the smooth metrics g_λ, i.e.

$$\forall t \in \mathbb{R}, \quad \lim_{\lambda \to 1} \mathcal{D}(t) - \mathcal{D}_\lambda(t) = 0 \ \text{ in } \ \mathcal{L}\left(H^1_{-1}(\Sigma), L^2(\Sigma)\right).$$

Each operator $\mathcal{D}_\lambda(t)$, $\lambda \in [0,1[$, $t \in \mathbb{R}$, is an isomorphism from $H^1_{-1}(\Sigma)$ onto $L^2(\Sigma)$. Using theorem 6.4 in [1], we see that we only need to prove the injectivity of $\mathcal{D}(t)$ in order to show that it is an isomorphism. Indeed, the operator $\mathcal{D}(t)$ satisfies the hypotheses of theorem 6.4 in [1]: it is an elliptic operator of the form

$$\mathcal{D}(t) = A_1(t)\tilde{\mathrm{D}} + A_0(t)$$

with $A_1(t) - B \in H^k_\gamma(\Sigma)$, $-3/2 < \gamma < min(-1/2, \delta)$ (taking account of the m/r term in the metric) where B is a $\mathcal{C}^\infty$ tensor field on Σ which is constant outside a compact set (we can take for B the matrix associated with the first order part of the Witten operator for the smooth background metric $\tilde{h}$) and $A_0(t) \in H^{k-1}_{\gamma+1}(\Sigma)$.

In order to prove the injectivity of $\mathcal{D}(t)$, we first note that the identity

$$\|\mathrm{D}\psi\|_2 = \|\mathcal{D}\psi\|_2 \tag{58}$$

which (see equation 5.1 in [9]) is true at each time t for the operators D_λ and $\mathcal{D}_\lambda$ associated with g_λ is still valid by continuity for the operators D and $\mathcal{D}$ (at each time t) associated with g. Hence, we only need to prove the injectivity of D at each time on $H^1_{-1}(\Sigma)$.

Consider $\psi \in H^1_{-1}(\Sigma; \mathbb{C}^2)$ such that $\mathrm{D}\psi = 0$. Using remark 4.2 and $\mathrm{D}^2\psi = \mathrm{D}(\mathrm{D}\psi) = 0$, we have $\psi \in H^2_{-1}(\Sigma; \mathbb{C}^2)$ whence ψ is continuous on Σ. We have in fact more: knowing that ψ is continuous and $\mathrm{D}\psi = 0$, we deduce that $\overline{\mathrm{D}}\psi$ is continuous and since the metric is $\mathcal{C}^2$ and the connection $\mathcal{C}^1$, this implies that ψ is of class $\mathcal{C}^1$. This is enough for us to apply the proof of lemma 4.3 in [9], replacing the explicit fall-off assumptions on ψ by the condition $\psi \in L^2_{-1}(\Sigma)$.

Considering the point O with respect to which the quantity $r(x)$ is calculated, we define for $\rho > 0$ the 2-sphere

$$S^2_\rho = \{x \in \Sigma;\ r(x) = \rho\}.$$

We can assume here that $r(x)$ is calculated using the metric $h(t)$ and not $\tilde{h}$. Supposing there is a point $x \in S^2_\rho$ such that $\psi(x) \neq 0$, then by continuity, there is a whole open set $\mathcal{O}$ in S^2_ρ such that

$$\exists \alpha > 0;\ \ \forall x \in \mathcal{O},\ \ |\psi(x)| \geq \alpha.$$

Differentiating $|\psi|^2$ as was done in [9], we see that, wherever ψ is non zero, any partial derivative of $\ln|\psi|$ in a local basis is bounded in norm by the norm of the extrinsic curvature at this point. From the assumption that the metric g is of class (k, δ) and using remark 4.1, we infer the existence for each $\varepsilon \in]0, \delta + 3/2[$ of a constant $C_\varepsilon > 0$ such that, at each point x of Σ

$$|K_{ab}| \leq \frac{C_\varepsilon}{(1 + r^2)^{\frac{1+\varepsilon}{2}}} \ .$$

Consequently, at each point where ψ is non zero, for $\varepsilon \in]0, \delta + 3/2[$, there exists $K_\varepsilon > 0$ such that any partial derivative of $\ln|\psi|$ satisfies

$$\partial \ln|\psi| \geq \frac{-K_\varepsilon}{(1 + r^2)^{\frac{1+\varepsilon}{2}}} \geq \frac{-K_\varepsilon}{r^{1+\varepsilon}}. \tag{59}$$

We now place ourselves on a geodesic emerging from O and going through a point $x_0 \in \mathcal{O}$. We express inequality (59) for the derivative with respect to r, and integrate it on the geodesic, out from x_0 to a point x $(r(x) > \rho)$ such that ψ remains non zero on the part of the geodesic between x_0 and x. We obtain

$$|\psi(x)| \geq |\psi(x_0)| exp\left[\varepsilon K_\varepsilon \left(\frac{1}{|x|^\varepsilon} - \frac{1}{|x_0|^\varepsilon}\right)\right]. \tag{60}$$

By continuity, it follows that $\psi(x)$ cannot vanish (and therefore (60) is valid) on the geodesic from x_0 out to infinity. Thus, propagating $\mathcal{O}$ outwards along the geodesic flow emanating from point O and using inequality (60), we contradict the fact that $\psi \in L^2_{-1}$, since in this sector of Σ, we have

$$\frac{1}{1+r^2}|\psi(x)|^2 \geq |\psi(x_0)|^2 exp\left[-\frac{2\varepsilon K_\varepsilon}{|x_0|^\varepsilon}\right]\frac{1}{1+r^2}$$

and $(1+r^2)^{-1}$ is not integrable at infinity on Σ. All this proves that ψ has to be identically zero on S^2_ρ for any $\rho > 0$, i.e. $\psi \equiv 0$. D is therefore injective at each time on $H^1_{-1}(\Sigma)$, from which we infer that $\mathcal{D}$ is an isomorphism from $H^1_{-1}(\Sigma)$ onto $L^2(\Sigma)$ at each time.

We now wish to prove that $\mathcal{D}(t)$ is an isomorphism from $\mathbb{H}^p(\Sigma;\mathbb{C}^2)$ onto $H^{p-1}(\Sigma;\mathbb{C}^2)$, $1 \leq p \leq k-2$. Using the fact that this property is true for $p=1$, it suffices to see that the norms $\|\mathcal{D}(t)\psi\|_{H^{p-1}}$ and $\|\psi\|_{\mathbb{H}^p}$ are equivalent. We prove this by induction. We have this property for $p=1$ and we assume it to hold for $p=1,...,m-1$, with $2 \leq m \leq k-2$. Taking advantage of remark 4.2, we use the definition of the Sobolev norms involving the derivative D instead of $\overline{D}$:

$$\|\psi\|^2_{\mathbb{H}^m} = \|\psi\|^2_{\mathbb{H}^{m-1}} + \|D^m\psi\|^2_{L^2}$$

and (at least locally), ∂^α denoting a partial derivative of order $|\alpha|$ with α a multi-index,

$$\|D^m\psi\|^2_{L^2} = \sum_{|\alpha|=m-1}\|\partial^\alpha D\psi\|^2_{L^2}.$$

We have

$$\partial^\alpha D\psi = D\left(\partial^\alpha\psi\right) + \text{lower order terms}.$$

The squared norms of the terms of lower order but still at least of order 1 can be estimated by

$$C\sum_{1\leq|\alpha|\leq m-1}\|\partial^\alpha\psi\|^2_{L^2}$$

and the term of order 0 which comes from the derivation of spin coefficients and the extrinsic curvature is estimated by (using the fall-off of these quantities at infinity)

$$C\|\psi\|^2_{L^2_{-1}}.$$

Hence, we have

$$\|\psi\|^2_{\mathbb{H}^m} \leq C\left(\|\psi\|^2_{L^2_{-1}} + \sum_{1\leq|\alpha|\leq m-1}\|\partial^\alpha\psi\|^2_{L^2}\right) + \|\psi\|^2_{\mathbb{H}^{m-1}} + \sum_{|\alpha|=m-1}\|D(\partial^\alpha\psi)\|^2_{L^2}.$$

Using (58) and the definition of $\mathbb{H}^{m-1}$, this inequality becomes

$$\|\psi\|^2_{\mathbb{H}^m} \leq C\|\psi\|^2_{\mathbb{H}^{m-1}} + \sum_{|\alpha|=m-1}\|\mathcal{D}(\partial^\alpha\psi)\|^2_{L^2}.$$

Again, we can write

$$\mathcal{D}\partial^\alpha\psi = \partial^\alpha\mathcal{D}\psi + \text{lower order terms}$$

where the norm of the lower order terms is controlled by $\|\psi\|_{\mathbb{H}^{m-1}}$. Finally, we obtain the inequality

$$\|\psi\|^2_{\mathbb{H}^m} \leq C\|\psi\|^2_{\mathbb{H}^{m-1}} + \sum_{|\alpha|=m-1}\|\partial^\alpha(\mathcal{D}\psi)\|^2_{L^2}.$$

and using the equivalence of $\|\psi\|_{\mathbb{H}^{m-1}}$ and $\|\mathcal{D}\psi\|_{H^{m-2}}$ gives

$$\|\psi\|^2_{\mathbb{H}^m} \leq C\|\mathcal{D}\psi\|^2_{H^{m-1}}.$$

The other inequality has already been obtained by the continuity of $\mathcal{D}$ from $\mathbb{H}^p$ to H^{p-1}, $1 \leq p \leq k-2$. This proves proposition 5.1. $\square$

Appendix 2: Proof of propositions 5.2 and 5.3

Proof of proposition 5.2:

In order to use a fixed point method, we express equation (34) with the specification of initial data

$$\phi|_{t=s} = \phi_0 \in \mathcal{H} \tag{61}$$

as the integral equation

$$\phi(t) = \mathcal{S}\phi(t), \quad \text{where} \quad \mathcal{S}\phi(t) = \mathcal{U}(t,s)\phi_0 + \int_s^t \mathcal{U}(t,\tau)Q(\tau)\phi(\tau)d\tau. \tag{62}$$

A function $\phi \in \mathcal{C}([s,T],\mathcal{H})$ satisfies (62) if and only if it is a solution of (34) in the sense of distributions and it satisfies the initial data condition (61). The space $\mathcal{C}([s,T];\mathcal{H})$ is stable under the functional $\mathcal{S}$ and for $|T-s|$ small enough, $\mathcal{S}$ is a strict contraction on the closed ball

$$\mathcal{B}_{s,T,\phi_0} = \{\phi \in \mathcal{C}([s,T];\mathcal{H}); \ \|\phi(t)\|_{\mathcal{H}} \leq 2\|\phi_0\|_{\mathcal{H}} \ \forall t \in [s,T]\}.$$

By a standard argument, this guarantees the existence of a unique fixed point of $\mathcal{S}$ in $\mathcal{C}([s,T];\mathcal{H})$ for $|T-s|$ small enough. Furthermore, using Gronwall's lemma, the uniform boundedness of $\mathcal{U}$ and Q on each compact time interval entails a uniform a priori bound on $\|\phi(t)\|_{\mathcal{H}}$ on each compact time interval. This is enough to prove the existence on the whole time axis of the solution ϕ_s of (62) with values in $\mathcal{H}$ and there is a continuous function $K : \mathbb{R}^2 \to]0,+\infty[$ such that

$$\|\phi_s(t)\|_{\mathcal{H}} \leq K(t,s)\|\phi_0\|_{\mathcal{H}}, \quad \forall t,s \in \mathbb{R}. \tag{63}$$

Denoting

$$\phi_s(t) = \mathcal{W}(t,s)\phi_0,$$

the family of operators $\{\mathcal{W}(t,s)\}$ satisfies

$$\mathcal{W}(t,s)\mathcal{W}(s,r) = \mathcal{W}(t,r)$$

by local uniqueness of the solution and for each $(t,s) \in \mathbb{R}^2$, $\mathcal{W}(t,s) \in \mathcal{L}(\mathcal{H})$ as an obvious consequence of (63) and of the linearity of the equation. The strong continuity of $\mathcal{W}$ on $\mathbb{R}^2$ to $\mathcal{L}(\mathcal{H})$ and the fact that $\mathcal{W}(t,t) = Id$ are entailed by the strong continuity of $\mathcal{U}$ and $\mathcal{U}(t,t) = Id$ on the one hand and on the other hand by the fact that the norm in $\mathcal{L}(\mathcal{H})$ of the integral term

$$\phi_0 \longmapsto \int_s^t \mathcal{U}(t,s)Q(\tau)\phi_s(\tau)d\tau$$

goes to zero when $|t-s| \to 0$, which is an immediate consequence of (63) and the locally uniform bounds on $\mathcal{U}$ and Q. $\square$

Proof of proposition 5.3:

It is mostly identical to the proof of proposition 5.2 after replacing the space $\mathcal{H}$ by $H^m(\Sigma;\mathbb{C}^4)$. Only the last property (analogous to (iv) in theorem 2) is new and needs to be checked.

Following (62), the propagator $\mathcal{W}$ can be expressed implicitly as:

$$\mathcal{W}(t,s) = \mathcal{U}(t,s) + \int_s^t \mathcal{U}(t,\tau)Q(\tau)\mathcal{W}(\tau,s)Id \, d\tau.$$

Consequently, the difference quotients for $h \in \mathbb{R}$

$$W_h^1 = \frac{\mathcal{W}(t+h,s) - \mathcal{W}(t,s)}{h} = \frac{\mathcal{W}(t+h,t) - Id}{h}\mathcal{W}(t,s)$$

and

$$W_h^2 = \frac{\mathcal{W}(t,s+h) - \mathcal{W}(t,s)}{h} = \mathcal{W}(t,s)\frac{\mathcal{W}(s,s+h) - Id}{h},$$

take the form:

$$W_h^1 = \left\{ \frac{\mathcal{U}(t+h,t) - Id}{h} + \frac{1}{h} \int_t^{t+h} \mathcal{U}(t+h,\tau)Q(\tau)\mathcal{W}(\tau,t)Id \ d\tau \right\} \mathcal{W}(t,s)$$

and

$$W_h^2 = \mathcal{W}(t,s) \left\{ \frac{\mathcal{U}(s,s+h) - Id}{h} + \frac{1}{h} \int_{s+h}^s \mathcal{U}(s,\tau)Q(\tau)\mathcal{W}(\tau,s+h)Id \ d\tau \right\}.$$

We know from property (d') of $\mathcal{U}$ that, as $h \to 0$,

$$\frac{\mathcal{U}(t+h,t) - Id}{h} \longrightarrow iA(t) \quad \text{and} \quad \frac{\mathcal{U}(s,s+h) - Id}{h} \longrightarrow -iA(s)$$

in $\mathcal{L}\left(H^m(\Sigma;\mathbb{C}^4), H^{m-1}(\Sigma;\mathbb{C}^4)\right)$. As an obvious consequence of the continuity of $\mathcal{U}$, Q and $\mathcal{W}$, we also have as $h \to 0$

$$\frac{1}{h} \int_t^{t+h} \mathcal{U}(t+h,\tau)Q(\tau)\mathcal{W}(\tau,t)Id \ d\tau \longrightarrow Q(t) \ \text{ in } \ \mathcal{L}\left(H^m(\Sigma;\mathbb{C}^4)\right)$$

and

$$\frac{1}{h} \int_{s+h}^s \mathcal{U}(s,\tau)Q(\tau)\mathcal{W}(\tau,s+h)Id \ d\tau \longrightarrow -Q(s) \ \text{ in } \ \mathcal{L}\left(H^m(\Sigma;\mathbb{C}^4)\right).$$

This together with

$$\mathcal{U}, \mathcal{W} \in \mathcal{C}\left(\mathbb{R}^2_{t,s}; \mathcal{L}\left(H^m(\Sigma;\mathbb{C}^4)\right)\right) \ , \ A \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{L}\left(H^m(\Sigma;\mathbb{C}^4), H^{m-1}(\Sigma;\mathbb{C}^4)\right)\right)$$

establishes the last property in proposition 5.3 and concludes the proof. $\square$

References

[1] Y. Choquet-Bruhat, D. Christodoulou, (1981) *Elliptic problems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity*, Acta Math., **146**, pp. 129-150.

[2] D. Christodoulou, S. Klainerman, (1993) *The global nonlinear stability of the Minkowski space*, Princeton Mathematical series 41, Princeton University Press.

[3] R.P. Geroch, (1970) *The domain of dependence*, J. Math. Phys., **11**, pp. 437-439.

[4] E. Hebey, (1996) *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Mathematics 1635, Springer Verlag.

[5] J.W. Jerome, (1983) *Approximation of nonlinear evolution systems*, Vol. 64 of Mathematics in Science and Engineering, Academic Press.

[6] L.J.Mason & J.Frauendiener, (1991) *The Sparling 3-form, Ashtekar variables and quasi-local mass*, in Twistors in mathematics and physics, eds R.Baston & T.Bailey, C.U.P, p.189-217. .

[7] J.M. Nester, (1983) *The gravitational Hamiltonian, Asymptotic Behaviour of Mass and Space-Time Geometry*, (Lecture Notes in Physics 202) ed. F J Flaherty (Berlin: Springer) pp 155–63.

[8] J.-P. Nicolas, (1997) *Global exterior Cauchy problem for spin 3/2 zero rest-mass fields in the Schwarzchild space-time*, Commun. in PDE, **22**(3&4), 465-502.

[9] T. Parker, C.H. Taubes, (1982) *On Witten's proof of the positive energy theorem*, Commun. Math. Phys., **84**, 223-238.

[10] R. Penrose, (1992) *Twistors as charges for helicity 3/2 fields*, in 'From superstrings to the real superworld, Proceedings of the international school of subnuclear physics, Erice', ed. A. Zichichi, World Scientific.

[11] R. Penrose, W. Rindler, (1984 & 1986) *Spinors and space-time*, Vol. I & II, Cambridge monographs on mathematical physics, Cambridge University Press.

[12] R. Racke, (1992) *Lectures on nonlinear evolution equations: Initial value problems*, Aspects of Mathematics Vol. 19, Vieweg.

[13] A. Rendall, (1997) *An introduction to the Einstein-Vlasov system*, in Mathematics of Gravitation, ed. P. Chrusciel, Banach Centre Publications, **41**, pp. 35-68.

[14] A. Sen, (1982) *Quantum Theory of Spin* 3/2 *Field in Einstein Spaces*, Internat. J. Theoretical Physics, **21**, 1, pp. 1-35.

[15] E. Witten, (1981) *A new proof of the positive energy theorem*, Commun. Math. Physics **80** 381-402.