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GLOBAL EXTERIOR CAUCHY PROBLEM FOR SPIN 3/2 ZERO REST-MASS FIELDS IN THE SCHWARZSCHILD SPACE-TIME

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1 Introduction

Spin 3/2 massless fields are of particular interest in two important domains of mathematical physics. Firstly in supersymmetry, a theory of gravitation with spin 3/2 source which studies intensively Einstein's vacuum equations coupled to the Rarita-Schwinger system (a particular form of spin 3/2 field equations), secondly in twistor theory where twistors, in flat space-time, can be interpreted as charges for such fields. This is due to the crucial role they seem to play in general relativity. Indeed, if we consider the Dirac equation for the first potential of a spin 3/2 zero rest-mass field

$$\nabla^{AA'}\sigma^{C}_{A'B'} = 0, \quad \sigma^{C}_{A'B'} = \sigma^{C}_{(A'B')}, \tag{1}$$

the vanishing of the Ricci curvature is the consistency condition for such an equation in curved space-time (see for example [2]). Such a close connection with Einstein's vacuum equations is quite remarquable. However, spin 3/2 fields have not to this day been studied from the point of view of hyperbolic partial differential equations. It is the purpose of this paper to set up a technical basis which will allow further analytic investigations in the future. We choose a particular Ricci-flat space-time : the Schwarzschild black-hole, on which we solve the global Cauchy problem for equation (1) for solutions with minimum regularity. This study is a first step towards the understanding of more difficult questions like the development of a time-dependent scattering theory for spin 3/2 fields on black-hole space-times. Besides, as mentionned above, it is related to an important issue in twistor theory : the interpretation of twistors as spin 3/2 charges. It is known (see [10]) that using a topological construction in flat space-time, one can give an alternative definition of a twistor as a charge for a spin 3/2 field. It is also known that such a construction in general Ricci-flat space-times has to be impossible. But it is not clear where the obstruction should arise in the construction itself. From [10], we know that this obstruction can be of two possible natures : either topological, no more will be said about this aspect here, or analytic, in which case it would be some pathological behavior of the propagator of the first potential modulo gauge. Solving the Cauchy problem for equation (1) in Schwarzschild's space-time will give us the beginning of an answer to this conundrum.

The article is divided as follows : in part 2, we give a general description of spin 3/2 field equations in flat and Ricci-flat space-times. In the third part, we describe in the Schwarzschild case the procedure given by the Newman-Penrose formalism for translating spinor equations into a form more suitable to a PDE-type analysis; this description is done in the simple case of the Dirac equation. Then, we apply this procedure to spin 3/2 equations. We obtain a system of coordinate-dependent partial differential equations for which we solve the global L^2-Cauchy problem in part 4. In this last part, the method is similar to what can be found in [1] and [9].

2 Spin 3/2 zero rest-mass fields in flat and Ricci-flat space-times

Wave equations with arbitrary spin were first introduced by P.A.M. Dirac [4] in 1936, using the fundamental relativistic relation between energy and momentum for a free particle with rest-mass m :

$$p_t^2 - p_x^2 - p_y^2 - p_z^2 - m^2 = 0$$

where p_t denotes the energy of the particle and p_x, p_y, p_z the three components of its momentum. His main purpose, as in the case of the electron wave-equation, was to obtain first order systems which make the application of Lorentz transformations straightforward. For a single particle of mass m and spin s, he defined a finite sequence of spinors describing the field and its successive potentials. He also proposed a method for taking into account the electromagnetic field. Following his work, M. Fierz and W. Pauli [5] discovered that his method led to inconsistent equations as soon as the spin was greater than 1. Keeping Dirac's equations in the force-free case, they introduced in the Lagrangian auxiliary tensors (for integral spin) or spinors (for half-integral spin) of lower rank and derived their equations from a variation principle without having to introduce extra conditions. W. Rarita and J. Schwinger [14] then found an alternative formalism for the description of particles with half-integral spin and gave a detailed description of the spin 3/2 case. They introduced a new Lagrangian, without auxiliary quantities, which anyway enabled them to take the electromagnetic field into account. A particularly interesting feature of their work is the fact that they did not describe spin 3/2 particles as fields but as the first potential of the field modulo gauge, which is exactly what we have to do in curved space-times.

In 1965, R. Penrose [11] described Dirac's equations for massless particles in the framework of 2-spinor formalism where they have a particularly synthetic expression. In the case of spin 3/2 zero rest-mass particles in Minkowski space-time, Dirac's equations can be expressed as follows : the field is described by a valence-3 spinor ψ satisfying

$$\psi_{A'B'C'} = \psi_{(A'B'C')}, \qquad \nabla^{AA'}\psi_{A'B'C'} = 0. \tag{2}$$

Locally at least, there exists a potential $\sigma_{A'B'}^{C}$ such that

$$\sigma_{A'B'}^{C} = \sigma_{(A'B')}^{C}, \qquad \nabla^{AA'}\sigma_{A'B'}^{C} = 0 \tag{3}$$

and

$$\psi_{A'B'C'} = \nabla_{CC'}\sigma_{A'B'}^{C}. \tag{4}$$

The gauge freedom for the choice of $\sigma_{A'B'}^{C}$ is given by

$$\sigma_{A'B'}^{C} \longmapsto \sigma_{A'B'}^{C} + \nabla_{B'}^{C}\pi_{A'}, \tag{5}$$

$\pi_{A'}$ being a solution of Weyl's neutrino equation (helicity +1/2)

$$\nabla^{AA'}\pi_{A'} = 0. \tag{6}$$

Note that $\nabla_{B'}^{C}\pi_{A'}$ is symmetrical in A', B' if and only if $\pi_{A'}$ satisfies the Weyl equation. Indeed,

$$\nabla_{B'}^{C}\pi_{A'} = \nabla_{(B'}^{C}\pi_{A')} + \nabla_{[B'}^{C}\pi_{A']} = \nabla_{(B'}^{C}\pi_{A')} + \nabla^{CC'}\pi_{C'}\varepsilon_{A'B'}.$$

We can also find, locally, a second potential $\rho_{A'}^{BC}$ satisfying

$$\rho_{A'}^{BC} = \rho_{A'}^{(BC)}, \qquad \nabla^{AA'}\rho_{A'}^{BC} = 0 \tag{7}$$

and

$$\sigma_{A'B'}^{C} = \nabla_{BB'}\rho_{A'}^{BC} \tag{8}$$

with the gauge freedom

$$\rho_{A'}^{BC} \longmapsto \rho_{A'}^{BC} + \nabla_{A'}^{B}\omega^{C} - i\varepsilon^{BC}\pi_{A'}, \quad \nabla_{BA'}\omega^{B} = -2i\pi_{A'}. \tag{9}$$

The coefficients in the previous line are chosen so that, when one considers for the gauge quantities ω^A and $\pi_{A'}$ the gauge transformations which leave the potentials unchanged, the new kind of gauge quantities exhibited will satisfy the twistor equation. Eventually, there is a third potential χ^{ABC}, which is a Hertz-type potential; it doesn't satisfy a first order wave equation but is a solution of the usual second order wave equation. Moreover, it is symmetric, thus

$$\chi^{ABC} = \chi^{(ABC)}, \quad \Box\chi^{ABC} = 0 \tag{10}$$

where

$$\Box = \nabla_{AA'}\nabla^{AA'} \tag{11}$$

and also

$$\rho_{A'}^{BC} = \nabla_{AA'}\chi^{ABC}. \tag{12}$$

Note that the field and the first two potentials also satisfy

$$\Box\psi_{A'B'C'} = 0, \quad \Box\sigma_{A'B'}^{C} = 0, \quad \Box\rho_{A'}^{BC} = 0. \tag{13}$$

The Rarita-Schwinger description of spin 3/2 massless fields takes only into account the first potential $\sigma_{A'B'}^{C}$ and makes no assumption about its symmetry. In this paper, we consider only the force-free case and we adopt the Dirac form.

The generalization of these equations to curved space-times requires caution. For zero rest-mass fields with spin $s > 1$, one needs to take account of Buchdahl's consistency conditions (see [2], [12]) which are algebraic conditions relating the solutions of the field equations to the conformal curvature (Weyl spinor) of the manifold. The case $s = 3/2$ has this remarkable feature that the vanishing of the Ricci curvature is the condition for a potential $\sigma_{A'B'}^{C}$ to be consistent in a curved space-time and we see that the zero rest-mass equations for this value of s are tied up with Einstein's vacuum equations. However, the first order wave-equations satisfied in the Minkowski case by $\psi_{A'B'C'}$ and $\rho_{A'}^{BC}$ are now inconsistent. Even if we tried to define the field from its first potential by

$$\psi_{A'B'C'} = \nabla_{CC'}\sigma_{A'B'}^{C},$$

we would find that the quantity thus obtained is not invariant under a gauge transformation of σ and therefore the definition is not satisfactory. It turns out the only description of the field we have access to is an indirect one, as the first potential σ modulo its gauge freedom, which coincides with the field derived from σ in flat space-time but replaces it on a curved Ricci-flat background. This is a Rarita-Schwinger type description of the field. For more details, see for example [10].

In summing up, the system we mean to study is

$$\nabla^{AA'}\sigma_{A'B'}^{C} = 0, \quad \sigma_{A'B'}^{C} = \sigma_{(A'B')}^{C} \tag{14}$$

together with the gauge freedom

$$\sigma_{A'B'}^{C} \longmapsto \sigma_{A'B'}^{C} + \nabla_{B'}^{C}\pi_{A'}, \quad \nabla^{AA'}\pi_{A'} = 0. \tag{15}$$

We also have the second potential $\rho_{A'}^{BC}$ satisfying

$$\rho_{A'}^{BC} = \rho_{A'}^{(BC)}, \quad \sigma_{A'B'}^{C} = \nabla_{BB'}\rho_{A'}^{BC} \tag{16}$$

with its gauge freedom

$$\rho_{A'}^{BC} \longmapsto \rho_{A'}^{BC} + \nabla_{A'}^{B}\omega^{C} - i\varepsilon^{BC}\pi_{A'}, \quad \nabla_{BA'}\omega^{B} = -2i\pi_{A'}. \tag{17}$$

$\rho_{A'}^{BC}$ no longer satisfies a first order wave equation and therefore we also lose the Hertz-type potential. Note that system (14) has 8 equations for only 6 independent unknowns, but one can reexpress two equations as constraints which are conserved by the evolution.

Remark 2.1 *It is not our purpose here to study the concept of first potential modulo gauge which is really an element of a sheaf cohomology class. We are only interested in studying the propagation of this somewhat strange quantity. The gauge being a simple Weyl neutrino field, it is known to have a "well-behaved" propagator on Ricci-flat space-times. Therefore, it is the potential itself, its gauge freedom left aside, on which we will focus our attention.*

Notations : We use essentially the same notations as in [12], [13] : abstract spinor indices are denoted by light-face sloping capital latin letters, numerical spinor indices are denoted by bold-face upright capital latin letters and take their values in $\{0,1\}$; abstract tensor labels are represented by little light-face sloping latin letters and correspond to a pair of spinor indices, one unprimed the other primed, clumped together. We use bold-face upright little latin letters for numerical tensor labels, which are not composite indices and take their values in $\{0,1,2,3\}$. Numerical indices referring to a null tetrad take their values in $\{1,2,3,4\}$ and are labelled by little latin letters enclosed in parentheses. Brackets on each side of a group of indices denote symmetrization and square brackets correspond to skew-symmetrization.

Let (M,g) be a Riemannian manifold, $\mathcal{C}_0^\infty(M)$ denotes the set of $\mathcal{C}^\infty$ functions with compact support in M, $H^k(M,g)$, $k\in\mathbb{N}$ is the Sobolev space, completion of $\mathcal{C}_0^\infty(M)$ for the norm

$$\|f\|^2_{H^k(M)} = \sum_{j=0}^{k} \int_M \left\langle \nabla^j f, \nabla^j f \right\rangle d\mu,$$

where ∇^j, $d\mu$ and $<,>$ are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric g. We write $L^2(M,g) = H^0(M,g)$.

The 2-dimensional euclidian sphere S^2_ω, $\omega=(\theta,\varphi)$, is endowed with its usual metric

$$d\omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \ , \ 0 \leq \theta \leq \pi \ , \ 0 \leq \varphi < 2\pi.$$

3 Application of the Newman-Penrose formalism to the translation of spin 1/2 and 3/2 equations in the Schwarzschild case

In the manifolds of general relativity, the translation of spinor field equations into a coordinate dependent form relies essentially on the choice of a null tetrad. Associated with a spin-frame, such a tetrad can be used to define the Infeld-Van der Waerden symbols; then, one can express the covariant derivative of spinor fields $\nabla_{AA'}$ in terms of partial derivatives in the coordinate basis.

On the manifold $\mathbb{R}_t\times]0,+\infty[_r\times S^2_\omega$, we introduce the coordinate basis $g_{\mathbf{a}}{}^a$. The relation to the more usual geometrical notations is given by

$$g_0{}^a \equiv \frac{\partial}{\partial t} \quad , \quad g_1{}^a \equiv \frac{\partial}{\partial r} \quad , \quad g_2{}^a \equiv \frac{\partial}{\partial \theta} \quad , \quad g_3{}^a \equiv \frac{\partial}{\partial \varphi}. \tag{18}$$

The associated covariant dual basis is denoted by $g_a{}^{\mathbf{a}}$. The Schwarzschild metric on our manifold has the form

$$g_{ab}dx^a dx^b = F dt^2 - F^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{19}$$

where $F(r) = 1-2M/r$, M being the mass of the black-hole and $r_0 = 2M$ the radius of the black-hole. We can, without loss of generality, assume $2M=1$ and $F = 1-1/r$. This can be obtained simply by multiplying the metric by the conformal weight $1/(2M)^2$ and replacing t and r by $t/(2M)$,

$r/(2M)$. The tetrad defined by

$$l^a = \frac{1}{\sqrt{2}}\left(F^{-1/2}{g_0}^a + F^{1/2}{g_1}^a\right), \tag{20}$$

$$n^a = \frac{1}{\sqrt{2}}\left(F^{-1/2}{g_0}^a - F^{1/2}{g_1}^a\right), \tag{21}$$

$$m^a = \frac{1}{r\sqrt{2}}\left({g_2}^a + \frac{i}{\sin\theta}{g_3}^a\right), \tag{22}$$

$$\overline{m}^a = \frac{1}{r\sqrt{2}}\left({g_2}^a - \frac{i}{\sin\theta}{g_3}^a\right), \tag{23}$$

is a null tetrad satisfying the orthonormality conditions

$$l^a n_a = 1 \quad , \quad m^a \overline{m}_a = -1. \tag{24}$$

It is chosen so that the "extent" of all the vectors is the same, where we define the extent of a vector as

$$\|V\| = |g_{ab}| V^a V^b. \tag{25}$$

The associated null covectors are

$$l_a = g_{ab} l^b = \frac{1}{\sqrt{2}}\left(F^{1/2}{g_a}^0 - F^{-1/2}{g_a}^1\right), \tag{26}$$

$$n_a = g_{ab} n^b = \frac{1}{\sqrt{2}}\left(F^{1/2}{g_a}^0 + F^{-1/2}{g_a}^1\right), \tag{27}$$

$$m_a = g_{ab} m^b = \frac{-r}{\sqrt{2}}\left({g_a}^2 + i\sin\theta {g_a}^3\right), \tag{28}$$

$$\overline{m}_a = g_{ab} \overline{m}^b = \frac{-r}{\sqrt{2}}\left({g_a}^2 - i\sin\theta {g_a}^3\right). \tag{29}$$

Using this null tetrad, we convert the Dirac-Weyl electron wave equation into a system of partial differential equations and check that we find the standard form of the Dirac equation on the Schwarzschild metric. To this purpose, we begin by calculating the Infeld-Van der Waerden symbols and the spin coefficients.

Considering the null tetrad $l^a, n^a, m^a, \overline{m}^a$ as being associated with a spin-frame $\varepsilon_0^A = o^A$, $\varepsilon_1^A = \iota^A$, i.e.

$$l^a = o^A o^{A'} \quad , \quad n^a = \iota^A \iota^{A'} \quad , \quad m^a = o^A \iota^{A'} \quad , \quad \overline{m}^a = \iota^A o^{A'}, \tag{30}$$

the Infeld-Van der Waerden symbols are defined by

$$g_{\mathbf{AA'}}{}^{\mathbf{a}} = \begin{pmatrix} l^{\mathbf{a}} & m^{\mathbf{a}} \\ \overline{m}^{\mathbf{a}} & n^{\mathbf{a}} \end{pmatrix} \quad , \quad g_{\mathbf{a}}{}^{\mathbf{AA'}} = \begin{pmatrix} n_{\mathbf{a}} & -\overline{m}_{\mathbf{a}} \\ -m_{\mathbf{a}} & l_{\mathbf{a}} \end{pmatrix}. \tag{31}$$

Replacing the values of the components of the null vectors l^a, m^a, $\overline{m}^a$, n^a in the previous expressions, we get

$$g_{\mathbf{AA'}}{}^0 = \frac{F^{-1/2}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad g_{\mathbf{AA'}}{}^1 = \frac{F^{1/2}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{32}$$

$$g_{\mathbf{AA'}}{}^2 = \frac{1}{r\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad g_{\mathbf{AA'}}{}^3 = \frac{1}{r\sqrt{2}\sin\theta}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{33}$$

$$g_0{}^{\mathbf{AA'}} = \frac{F^{1/2}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad g_1{}^{\mathbf{AA'}} = \frac{F^{-1/2}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{34}$$

$$g_2{}^{\mathbf{AA'}} = \frac{r}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad g_3{}^{\mathbf{AA'}} = \frac{r\sin\theta}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{35}$$

Then, we calculate the spin-coefficients of the Newman-Penrose formalism :

$$\begin{aligned}
&\kappa = \gamma_{311}, \quad \rho = \gamma_{314}, \quad \varepsilon = \tfrac{1}{2}\left(\gamma_{211} + \gamma_{341}\right),\\
&\sigma = \gamma_{313}, \quad \mu = \gamma_{243}, \quad \gamma = \tfrac{1}{2}\left(\gamma_{212} + \gamma_{342}\right),\\
&\lambda = \gamma_{244}, \quad \tau = \gamma_{312}, \quad \alpha = \tfrac{1}{2}\left(\gamma_{214} + \gamma_{344}\right),\\
&\nu = \gamma_{242}, \quad \pi = \gamma_{241}, \quad \beta = \tfrac{1}{2}\left(\gamma_{213} + \gamma_{343}\right).
\end{aligned} \tag{36}$$

The Ricci rotation-coefficients $\gamma_{(i)(j)(k)}$ are defined by

$$\gamma_{(i)(j)(k)} = \frac{1}{2}\left(\lambda_{(i)(j)(k)} + \lambda_{(k)(i)(j)} - \lambda_{(j)(k)(i)}\right) \tag{37}$$

where

$$\lambda_{(i)(j)(k)} = e_{(j)\mathbf{a},\mathbf{b}}\left({e_{(i)}}^{\mathbf{a}}{e_{(k)}}^{\mathbf{b}} - {e_{(k)}}^{\mathbf{a}}{e_{(i)}}^{\mathbf{b}}\right), \tag{38}$$

${e_{(i)}}^{\mathbf{a}}$ denoting the components of the elements of the null tetrad :

$${e_1}^{\mathbf{a}} = l^{\mathbf{a}} \quad , \quad {e_2}^{\mathbf{a}} = n^{\mathbf{a}} \quad , \quad {e_3}^{\mathbf{a}} = m^{\mathbf{a}} \quad , \quad {e_4}^{\mathbf{a}} = \overline{m}^{\mathbf{a}}. \tag{39}$$

The notation $_{,\mathbf{a}}$ corresponds to the derivation with respect to the **a**-th variable. Due to the anti-symmetry of $\lambda_{(i)(j)(k)}$ in i, k, we have 24 coefficients to evaluate instead of 64. Among these, the non-zero coefficients are

$$\lambda_{211} = \lambda_{221} = \frac{F'F^{-1/2}}{2\sqrt{2}} \ , \ \lambda_{432} = -\lambda_{243} = -\lambda_{431} = \lambda_{143} = \frac{F^{1/2}}{r\sqrt{2}} \ , \tag{40}$$

$$\lambda_{443} = -\lambda_{433} = \frac{\cot\theta}{r\sqrt{2}}. \tag{41}$$

Whence the values of the spin-coefficients :

$$\kappa = \sigma = \lambda = \tau = \nu = \pi = 0, \tag{42}$$

$$\rho = \mu = -\frac{F^{1/2}}{r\sqrt{2}} \quad , \quad \varepsilon = \gamma = \frac{F'F^{-1/2}}{4\sqrt{2}} \quad , \quad \beta = -\alpha = \frac{\cot\theta}{2r\sqrt{2}}. \tag{43}$$

We now use the previous calculations to rewrite in terms of partial derivatives the Dirac-Weyl equation, for an electron of mass m :

$$\left\{\begin{array}{l} \nabla^{A}_{A'}\phi_A = \mu\chi_{A'}, \quad \mu = \left(\hbar\sqrt{2}\right)^{-1} m, \\ \nabla^{A'}_{A}\chi_{A'} = \mu\phi_A. \end{array}\right. \tag{44}$$

For the first equation, we have

$$\nabla^{A}_{A'}\phi_A = \varepsilon^{AB}\nabla_{BA'}\phi_A = \mu\chi_{A'}.$$

The components of $\varepsilon^{AB}\nabla_{BA'}\phi_A$ in the spin-frame o^A, ι^A are

$$\Phi_{\mathbf{A}'} = {\varepsilon_{\mathbf{A}'}}^{A'}\varepsilon^{AB}\nabla_{BA'}\phi_A = \varepsilon^{\mathbf{AB}}\Psi_{\mathbf{ABA}'},$$

$\Psi_{\mathbf{ABA}'}$ denoting the components of $\nabla_{BA'}\phi_A$ in the spin-frame, i.e.

$$\Psi_{\mathbf{ABA}'} = {\varepsilon_{\mathbf{A}'}}^{A'}{\varepsilon_{\mathbf{A}}}^{A}{\varepsilon_{\mathbf{B}}}^{B}\nabla_{BA'}\phi_A \ .$$

Using coordinate partial derivatives and the spin-coefficients, these components can be expressed in the form (see [12])

$$\Psi_{\mathbf{ABA'}} = g_{\mathbf{BA'}}{}^{\mathbf{b}}\phi_{\mathbf{A},\mathbf{b}} - \phi_{\mathbf{A}_0}\gamma_{\mathbf{BA'A}}{}^{\mathbf{A}_0} \tag{45}$$

where the $\gamma_{\mathbf{BA'A}}{}^{\mathbf{A}_0}$ are the spin-coefficients arranged in the following manner

$\gamma_{\mathbf{AB'C}}{}^{\mathbf{D}} =$ (46)

D / **C** / **AB′**	D = 0, C = 0	D = 1, C = 0	D = 0, C = 1	D = 1, C = 1
$0\ 0'$	ε	$-\kappa$	π	$-\varepsilon$
$1\ 0'$	α	$-\rho$	λ	$-\alpha$
$0\ 1'$	β	$-\sigma$	μ	$-\beta$
$1\ 1'$	γ	$-\tau$	ν	$-\gamma$

The two components $\Phi_{0'}$ and $\Phi_{1'}$ have the form

$$\Phi_{0'} = \Psi_{010'} - \Psi_{100'} \quad , \quad \Phi_{1'} = \Psi_{011'} - \Psi_{101'}$$

and we only have to evaluate the terms $\Psi_{010'}$, $\Psi_{100'}$, $\Psi_{011'}$ and $\Psi_{101'}$.

$$\begin{aligned}
\Psi_{010'} &= g_{10'}{}^{2}\phi_{0,\theta} + g_{10'}{}^{3}\phi_{0,\varphi} - \phi_0\gamma_{10'0}{}^{0} - \phi_1\gamma_{10'0}{}^{1} \\
&= \frac{1}{\sqrt{2}}\left\{\left(\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\phi_0 + \frac{\cot\theta}{2r}\phi_0 - \frac{F^{1/2}}{r}\phi_1\right\}, \\
\Psi_{100'} &= g_{00'}{}^{0}\phi_{1,t} + g_{00'}{}^{1}\phi_{1,r} - \phi_0\gamma_{00'1}{}^{0} - \phi_1\gamma_{00'1}{}^{1} \\
&= \frac{1}{\sqrt{2}}\left\{\left(F^{-1/2}\frac{\partial}{\partial t} + F^{1/2}\frac{\partial}{\partial r}\right)\phi_1 + \frac{F'F^{-1/2}}{4}\phi_1\right\}, \\
\Psi_{011'} &= g_{11'}{}^{0}\phi_{0,t} + g_{11'}{}^{1}\phi_{0,r} - \phi_0\gamma_{11'0}{}^{0} - \phi_1\gamma_{11'0}{}^{1} \\
&= \frac{1}{\sqrt{2}}\left\{\left(F^{-1/2}\frac{\partial}{\partial t} - F^{1/2}\frac{\partial}{\partial r}\right)\phi_0 - \frac{F'F^{-1/2}}{4}\phi_0\right\}, \\
\Psi_{101'} &= g_{01'}{}^{2}\phi_{1,\theta} + g_{01'}{}^{3}\phi_{1,\varphi} - \phi_0\gamma_{01'1}{}^{0} - \phi_1\gamma_{01'1}{}^{1} \\
&= \frac{1}{\sqrt{2}}\left\{\left(\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\phi_1 + \frac{\cot\theta}{2r}\phi_1 + \frac{F^{1/2}}{r}\phi_0\right\}.
\end{aligned}$$

Thus, we can now write equation

$$\nabla^{A}_{A'}\phi_A = \mu\chi_{A'}$$

in terms of partial derivatives, which gives us two partial differential equations

$$-F^{-1/2}\frac{\partial}{\partial t}\phi_1 - F^{1/2}\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{F'}{4F}\right)\phi_1 + \frac{1}{r}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)\phi_0 - \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\phi_0 = \mu\sqrt{2}\chi_{0'} \ , \tag{47}$$

$$F^{-1/2}\frac{\partial}{\partial t}\phi_0 - F^{1/2}\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{F'}{4F}\right)\phi_0 - \frac{1}{r}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)\phi_1 - \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\phi_1 = \mu\sqrt{2}\chi_{1'} \ . \tag{48}$$

As for the second equation

$$\nabla^{A'}_{A}\chi_{A'} = \varepsilon^{A'B'}\nabla_{AB'}\chi_{A'} = \mu\phi_A,$$

we put

$$\Phi_{\mathbf{A}} = \varepsilon_{\mathbf{A}}{}^{A}\nabla_{A}^{A'}\chi_{A'} = \varepsilon_{\mathbf{A}}{}^{A}\varepsilon_{\mathbf{A'}}{}^{A'}\varepsilon_{\mathbf{B'}}{}^{B'}\varepsilon^{\mathbf{A'B'}}\nabla_{AB'}\chi_{A'} = \varepsilon^{\mathbf{A'B'}}\Psi_{\mathbf{AA'B'}}\ .$$

As before, $\Psi_{\mathbf{AA'B'}}$ can be expressed in the form (see [12])

$$\Psi_{\mathbf{AA'B'}} = g_{\mathbf{AB'}}{}^{\mathbf{a}}\chi_{\mathbf{A'},\mathbf{a}} - \chi_{\mathbf{A}'_0}\overline{\gamma}_{\mathbf{AB'A'}}{}^{\mathbf{A}'_0} \tag{49}$$

and the components of the spinor $\overline{\gamma}_{AB'C'}{}^{D'}$, complex-conjugate of $\gamma_{BA'C}{}^{D}$, are given by

$$\overline{\gamma}_{\mathbf{AB'C'}}{}^{\mathbf{D'}} = \overline{\gamma_{\mathbf{BA'C}}{}^{\mathbf{D}}}. \tag{50}$$

After the same type of calculations as before, we obtain the following partial differential form of our second equation :

$$-F^{-1/2}\frac{\partial}{\partial t}\chi_{1'} - F^{1/2}\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{F'}{4F}\right)\chi_{1'} + \frac{1}{r}\left(\frac{\partial}{\partial \theta} + \frac{1}{2}\cot\theta\right)\chi_{0'} + $$

$$+\frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\chi_{0'} = \mu\sqrt{2}\phi_0\ , \tag{51}$$

$$F^{-1/2}\frac{\partial}{\partial t}\chi_{0'} - F^{1/2}\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{F'}{4F}\right)\chi_{0'} - \frac{1}{r}\left(\frac{\partial}{\partial \theta} + \frac{1}{2}\cot\theta\right)\chi_{1'}$$

$$+\frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\chi_{1'} = \mu\sqrt{2}\phi_1\ . \tag{52}$$

If we introduce the basis of Dirac matrices

$$\gamma^0 = i\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^\alpha = i\begin{pmatrix} 0 & \sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix} \quad \alpha = 1, 2, 3, \tag{53}$$

σ^α, $\alpha = 0, 1, 2, 3$, denoting the Pauli spin-matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{54}$$

putting

$$u_1 = \chi_{0'}, \quad u_2 = \chi_{1'}, \quad , u_3 = \phi_1, \quad u_4 = -\phi_0, \tag{55}$$

we see that the four-spinor

$$\Psi = {}^t(u_1, u_2, u_3, u_4) \tag{56}$$

satisfies the usual massive Dirac equation (see [9])

$$\left\{iF^{\frac{-1}{2}}\gamma^0\frac{\partial}{\partial t} + iF^{\frac{1}{2}}\gamma^1\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{F'}{4F}\right) + \frac{i}{r}\gamma^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)\right.$$

$$\left. + \frac{i}{r\sin\theta}\gamma^3\frac{\partial}{\partial\varphi} - \mu\sqrt{2}\right\}\Psi = 0 \tag{57}$$

for a particle of mass $\mu\sqrt{2}$ and spin 1/2 on the Schwarzschild metric. The scaling of the mass is due to the choice of extent for the vectors of the null tetrad. Multiplying each null vector by $\hbar^{-1}$, we obtain the same equation with the right mass m.

We now apply the method described above to equation (14) and we evaluate its components in terms of partial derivatives in a coordinate basis. Putting

$$\phi_{B'}^{AC} = \nabla^{AA'}\sigma_{A'B'}^{C} = \varepsilon^{AD}\varepsilon^{A'D'}\nabla_{DD'}\sigma_{A'B'}^{C} = \varepsilon^{AD}\varepsilon^{A'D'}\Psi_{DD'A'B'}{}^{C}, \tag{58}$$

we have in a spin-frame o^A, ι^A

$$\phi_{\mathbf{B'}}^{\mathbf{AC}} = \varepsilon^{\mathbf{AD}}\varepsilon^{\mathbf{A'D'}}\Psi_{\mathbf{DD'A'B'}}^{\ \ \ \ \ \ \mathbf{C}} \tag{59}$$

and $\Psi_{\mathbf{DD'A'B'}}^{\ \ \ \ \ \ \mathbf{C}}$ can be expressed in the form

$$\Psi_{\mathbf{DD'A'B'}}^{\ \ \ \ \ \ \mathbf{C}} = g_{\mathbf{DD'}}{}^{\mathbf{d}}\ \sigma_{\mathbf{A'B'},\mathbf{d}}^{\mathbf{C}} + \sigma^{\mathbf{C}_0}{}_{\mathbf{A'B'}}\ \gamma_{\mathbf{DD'C}_0}{}^{\mathbf{C}}$$
$$-\sigma_{\mathbf{A}_0'\mathbf{B'}}^{\mathbf{C}}\ \overline{\gamma}_{\mathbf{DD'A'}}{}^{\mathbf{A}_0'} - \sigma_{\mathbf{A'B}_0'}^{\mathbf{C}}\ \overline{\gamma}_{\mathbf{DD'B'}}{}^{\mathbf{B}_0'}. \tag{60}$$

The 8 components of $\phi_{B'}^{AC}$ are

$$\phi_{0'}^{00} = \varepsilon^{0\mathbf{D}}\varepsilon^{\mathbf{A'D'}}\Psi_{\mathbf{DD'A'}0'}^{\ \ \ \ \ \ 0} = \Psi_{11'0'0'}^{\ \ \ \ \ \ 0} - \Psi_{10'1'0'}^{\ \ \ \ \ \ 0}\ \ ,\ \ \phi_{0'}^{01} = \Psi_{11'0'0'}^{\ \ \ \ \ \ 1} - \Psi_{10'1'0'}^{\ \ \ \ \ \ 1}$$

$$\phi_{1'}^{00} = \Psi_{11'0'1'}^{\ \ \ \ \ \ 0} - \Psi_{10'1'1'}^{\ \ \ \ \ \ 0}\ \ ,\ \ \phi_{1'}^{01} = \Psi_{11'0'1'}^{\ \ \ \ \ \ 1} - \Psi_{10'1'1'}^{\ \ \ \ \ \ 1}$$

$$\phi_{0'}^{10} = \varepsilon^{1\mathbf{D}}\varepsilon^{\mathbf{A'D'}}\Psi_{\mathbf{DD'A'}0'}^{\ \ \ \ \ \ 0} = \Psi_{00'1'0'}^{\ \ \ \ \ \ 0} - \Psi_{01'0'0'}^{\ \ \ \ \ \ 0}\ \ ,\ \ \phi_{0'}^{11} = \Psi_{00'1'0'}^{\ \ \ \ \ \ 1} - \Psi_{01'0'0'}^{\ \ \ \ \ \ 1}$$

$$\phi_{1'}^{10} = \Psi_{00'1'1'}^{\ \ \ \ \ \ 0} - \Psi_{01'0'1'}^{\ \ \ \ \ \ 0}\ \ ,\ \ \phi_{1'}^{11} = \Psi_{00'1'1'}^{\ \ \ \ \ \ 1} - \Psi_{01'0'1'}^{\ \ \ \ \ \ 1}.$$

Evaluating the 16 required components of $\Psi_{DD'A'B'}^{\ \ \ \ \ \ C}$ using the Infeld-Van der Waerden symbols (32), (33), (34), (35) and the spin-coefficients (42), (43), (46), we eventually find the components of $\sqrt{2}\phi_{B'}^{AC}$. The quantities $\phi_{1'}^{00}$ and $\phi_{0'}^{10}$ (resp. $\phi_{1'}^{01}$ and $\phi_{0'}^{11}$) both involve the time derivative of the same component of σ, namely $\sigma_{0'1'}^{0}$ (resp. $\sigma_{0'1'}^{1}$). We keep

$$\phi_{1'}^{01} = 0\ \ ,\ \ \phi_{0'}^{10} = 0$$

as evolution equations and replace

$$\phi_{1'}^{00} = 0\ \ ,\ \ \phi_{0'}^{11} = 0$$

by

$$\phi_{0'}^{10} - \phi_{1'}^{00} = 0\ \ ,\ \ \phi_{0'}^{11} - \phi_{1'}^{01} = 0. \tag{61}$$

The two equations (61) do not involve time derivatives and are merely constraints on the solutions of the six evolution equations. Hence, we can write (14) under the following form : putting

$$U = {}^t\left(\sigma_{0'0'}^0\ ,\ \sigma_{0'1'}^0\ ,\ \sigma_{1'1'}^0\ ,\ \sigma_{0'0'}^1\ ,\ \sigma_{0'1'}^1\ ,\ \sigma_{1'1'}^1\right), \tag{62}$$

we have

$$\frac{\partial U}{\partial t} = \begin{pmatrix} h\sigma_{0'0'}^0 + \frac{F^{1/2}}{r}\overline{L_1}\ \sigma_{0'1'}^0 \\ -\left(h+\frac{F}{r}\right)\sigma_{0'1'}^0 - \frac{F}{r}\sigma_{0'0'}^1 + \frac{F^{1/2}}{r}L_3\ \sigma_{0'0'}^0 \\ -\left(h+\frac{F'}{2}\right)\sigma_{1'1'}^0 - \frac{F}{r}\sigma_{0'1'}^1 + \frac{F^{1/2}}{r}L_2\ \sigma_{0'1'}^0 \\ \left(h+\frac{F'}{2}\right)\sigma_{0'0'}^1 + \frac{F}{r}\sigma_{0'1'}^0 + \frac{F^{1/2}}{r}\overline{L_2}\ \sigma_{0'1'}^1 \\ \left(h+\frac{F}{r}\right)\sigma_{0'1'}^1 + \frac{F}{r}\sigma_{1'1'}^0 + \frac{F^{1/2}}{r}\overline{L_3}\ \sigma_{1'1'}^1 \\ -h\sigma_{1'1'}^1 + \frac{F^{1/2}}{r}L_1\ \sigma_{0'1'}^1 \end{pmatrix} = HU \tag{63}$$

where

$$h = F\left(\frac{\partial}{\partial r} + \frac{F'}{4F} + \frac{1}{r}\right), \tag{64}$$

$$L_k = \frac{\partial}{\partial\theta} + \left(k - \frac{3}{2}\right)\cot\theta + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} \quad , \quad k = 1, 2, 3, \tag{65}$$

$$\overline{L_k} = \frac{\partial}{\partial\theta} + \left(k - \frac{3}{2}\right)\cot\theta - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} \quad , \quad k = 1, 2, 3, \tag{66}$$

together with the two constraints

$$2h\sigma^0_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\sigma^0_{0'1'} + \frac{F}{r}\sigma^1_{0'0'} - \frac{F^{1/2}}{r}\left(L_3\ \sigma^0_{0'0'} - \overline{L_2}\ \sigma^0_{1'1'}\right) = 0, \tag{67}$$

$$2h\sigma^1_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\sigma^1_{0'1'} + \frac{F}{r}\sigma^0_{1'1'} - \frac{F^{1/2}}{r}\left(L_2\ \sigma^1_{0'0'} - \overline{L_3}\ \sigma^1_{1'1'}\right) = 0. \tag{68}$$

We can also check by translating the components of the gauge quantity $\nabla^C_{A'}\pi_{B'}$ into a coordinate dependent form that its symmetry in A', B' is equivalent to

$$F^{\frac{-1}{2}}\frac{\partial}{\partial t}\pi_{0'} - F^{\frac{1}{2}}\left(\frac{\partial}{\partial r} + \frac{F'}{4F} + \frac{1}{r}\right)\pi_{0'} - \frac{1}{r}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)\pi_{1'} + \frac{i}{r\sin\theta}\pi_{1'} = 0,$$

$$F^{\frac{-1}{2}}\frac{\partial}{\partial t}\pi_{1'} + F^{\frac{1}{2}}\left(\frac{\partial}{\partial r} + \frac{F'}{4F} + \frac{1}{r}\right)\pi_{1'} - \frac{1}{r}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right)\pi_{0'} - \frac{i}{r\sin\theta}\pi_{0'} = 0,$$

which is the partial differential form of (see equations (51) and (52)) :

$$\nabla^{AA'}\pi_{A'} = 0.$$

4 Global Cauchy problem in the successive domains of H

We introduce the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H} = \left\{L^2\left(]1, +\infty[_r \times S^2_\omega;\ F^{-1}dr^2 + r^2d\omega^2\right)\right\}^6 \tag{69}$$

and the successive domains of H in $\mathcal{H}$, $D(H^0)$ being identified with $\mathcal{H}$,

$$D(H^k) = \left\{U \in \mathcal{H};\quad H^jU \in \mathcal{H},\ 1 \le j \le k\right\},\quad k \in \mathbb{N}^*. \tag{70}$$

We also consider the spaces $\mathcal{H}_c$ and $D(H^k)_c$, $k \in \mathbb{N}^*$, of the elements of $\mathcal{H}$ and $D(H^k)$, $k \in \mathbb{N}^*$, which satisfy the constraint equations (67), (68); i.e. if we write (67) in the following way

$$AU = 0,\quad A = \left(-\frac{F^{1/2}}{r}L_3\ ,\ 2h + \frac{2F}{r} - \frac{F'}{2}\ ,\ \frac{F^{1/2}}{r}\overline{L_2}\ ,\ \frac{F}{r}\ ,\ 0\ ,\ 0\right), \tag{71}$$

and in the same manner (68) becomes

$$BU = 0,\quad B = \left(0\ ,\ 0\ ,\ \frac{F}{r}\ ,\ -\frac{F^{1/2}}{r}L_2\ ,\ 2h + \frac{2F}{r} - \frac{F'}{2}\ ,\ \frac{F^{1/2}}{r}\overline{L_3}\right), \tag{72}$$

then, we have simply

$$\mathcal{H}_c = KerA\bigcap KerB \tag{73}$$

where $KerA$ is the kernel of A in $\mathcal{H}$, and for $k \in \mathbb{N}^*$,

$$D(H^k)_c = (KerA)_{D(H^k)}\bigcap(KerB)_{D(H^k)} = KerA\bigcap KerB\bigcap D(H^k) \tag{74}$$

where $(KerA)_{D(H^k)}$ is the kernel of A in $D(H^k)$. The spaces $\mathcal{H}_c$ and $D(H^k)_c$ are the functional spaces in which the solutions of (63), (67), (68) will live. Before we prove the existence and uniqueness of such solutions, which will be done in theorem 4.1, it is interesting to note that spin 3/2 fields have a natural conserved quantity given by integration of the Rarita-Schwinger 3-form on a space-like hypersurface and that this conserved quantity is not positive definite. Therefore, there is no hope of using it to define a norm on the space of solutions and the evolution will not be unitary. This absence of natural self-adjointness framework will be the first problem to deal with if we wish to develop a time-dependent scattering theory for spin 3/2 fields in the Schwarzschild space-time. The conserved quantity is described in the following proposition

Proposition 4.1 *The Rarita-Schwinger 3-form*

$$\beta = i\sigma_{aC'}\ \overline{\sigma}_{bC}\ dx^a \wedge dx^b \wedge dx^c \tag{75}$$

is divergence-free, $\sigma^A_{B'C'}$ denoting the spinor, symmetric in B', C', whose components satisfy (63), (67) and (68). In other words, if we consider the sesqui-linear form (obtained by integration of the 3-form on a space-like hypersurface) defined for $\xi, \eta \in \mathcal{H}$ by

$$\begin{aligned} <\xi,\eta>_\beta \ &= \left(\xi^0_{0'0'},\eta^0_{0'0'}\right)_{L^2} + \left(\xi^1_{1'1'},\eta^1_{1'1'}\right)_{L^2} + \left(\xi^1_{0'1'},\eta^1_{0'1'}\right)_{L^2} + \left(\xi^0_{0'1'},\eta^0_{0'1'}\right)_{L^2} \\ &+ \left(\xi^0_{1'1'},\eta^1_{0'1'}\right)_{L^2} + \left(\xi^1_{0'1'},\eta^0_{1'1'}\right)_{L^2} + \left(\xi^1_{0'0'},\eta^0_{0'1'}\right)_{L^2} + \left(\xi^0_{0'1'},\eta^1_{0'0'}\right)_{L^2} \end{aligned} \tag{76}$$

where $(\ ,\ )_{L^2}$ denotes the standard scalar product on $L^2\left(]1,+\infty[_r \times S^2_\omega, F^{-1}dr^2\ \ +r^2d\omega^2\right)$, then for any $U, V \in D(H)_c$

$$< HU, V >_\beta = - < U, HV >_\beta \tag{77}$$

and if $U \in \mathcal{C}\left(\mathbb{R}_t;\mathcal{H}_c\right)$ is a solution of (63), the quantity $< U,U >_\beta$ is conserved throughout time.

The proof of proposition 4.1 will use several notations which we shall introduce later for the proof of our main theorem. Therefore we shall postpone it until the end of the paragraph, after the proof of theorem 4.1 has been completed. Let us now state our main existence and uniqueness result

Theorem 4.1 *For any initial data $U_0 \in \mathcal{H}_c$ (resp. $U_0 \in D(H^k)_c$, $k \in \mathbb{N}^*$), equation (63) admits a unique solution U such that*

$$U \in \mathcal{C}\left(\mathbb{R}_t;\mathcal{H}_c\right) \qquad \left(\textit{resp. } U \in \mathcal{C}\left(\mathbb{R}_t; D(H^k)_c\right)\right) \tag{78}$$

and

$$U|_{t=0} = U_0. \tag{79}$$

Note that if $U_0 \in D(H^k)_c$, $k \in \mathbb{N}^*$, the solution U has the following additional regularities which are immediate consequences of (63)

$$U \in \bigcap_{j=0}^{k} \mathcal{C}^j\left(\mathbb{R}_t; D(H^{k-j})_c\right). \tag{80}$$

Proof of theorem 4.1 : The essential idea is to use the symmetry of the space-time in order to separate the variables and reduce the problem to solving an evolution equation in one space dimension. This is done by decomposing equations (63), (67) and (68) into spin-weighted spherical harmonics. We prove global existence and uniqueness results on each sub-space of given angular dependence. Then, theorem 4.1 follows from an energy estimate.

If we introduce a new variable r_*, called the Regge-Wheeler or tortoise coordinate, defined by

$$r_* = r + \ln(r-1) \tag{81}$$

and satisfying

$$\frac{dr_*}{dr} = F^{-1} = \left(1 - \frac{1}{r}\right)^{-1}, \tag{82}$$

if we define the isometry

$$\begin{array}{rcl} j : L^2\left(]1, +\infty[_r \times S^2_\omega, F^{-1}dr^2 + r^2 d\omega^2\right) & \longrightarrow & L^2\left(\mathbb{R}_{r_*} \times S^2_\omega; dr_*^2 + d\omega^2\right) \\ f & \longmapsto & rF^{1/4} f, \end{array} \tag{83}$$

then we have

$$jh = \partial_{r_*} j. \tag{84}$$

Thus, we can use this isometry to simplify the expression of system (63). Multiplying each equation by $rF^{1/4}$ and putting

$$V = {}^t\left(\zeta^0_{0'0'}, \zeta^0_{0'1'}, \zeta^0_{1'1'}, \zeta^1_{0'0'}, \zeta^1_{0'1'}, \zeta^1_{1'1'}\right) = rF^{1/4} U \tag{85}$$

we get the following system which is equivalent to (63)

$$\frac{\partial V}{\partial t} = \begin{pmatrix} \partial_{r_*}\zeta^0_{0'0'} + \frac{F^{1/2}}{r}\overline{L_1}\ \zeta^0_{0'1'} \\ -\partial_{r_*}\zeta^0_{0'1'} - \frac{F}{r}\zeta^0_{0'1'} - \frac{F}{r}\zeta^1_{0'0'} + \frac{F^{1/2}}{r} L_3\ \zeta^0_{0'0'} \\ -\partial_{r_*}\zeta^0_{1'1'} - \frac{F'}{2}\zeta^0_{1'1'} - \frac{F}{r}\zeta^1_{0'1'} + \frac{F^{1/2}}{r} L_2\ \zeta^0_{0'1'} \\ \partial_{r_*}\zeta^1_{0'0'} + \frac{F'}{2}\zeta^1_{0'0'} + \frac{F}{r}\zeta^0_{0'1'} + \frac{F^{1/2}}{r}\overline{L_2}\ \zeta^1_{0'1'} \\ \partial_{r_*}\zeta^1_{0'1'} + \frac{F}{r}\zeta^1_{0'1'} + \frac{F}{r}\zeta^0_{1'1'} + \frac{F^{1/2}}{r}\overline{L_3}\ \zeta^1_{1'1'} \\ -\partial_{r_*}\zeta^1_{1'1'} + \frac{F^{1/2}}{r} L_1\ \zeta^1_{0'1'} \end{pmatrix} = \tilde{H} V \tag{86}$$

and the constraints (67), (68) become

$$2\partial_{r_*}\zeta^0_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\zeta^0_{0'1'} + \frac{F}{r}\zeta^1_{0'0'} - \frac{F^{1/2}}{r}\left(L_3\ \zeta^0_{0'0'} - \overline{L_2}\ \zeta^0_{1'1'}\right) = 0, \tag{87}$$

$$2\partial_{r_*}\zeta^1_{0'1'} + \left(\frac{2F}{r} - \frac{F'}{2}\right)\zeta^1_{0'1'} + \frac{F}{r}\zeta^0_{1'1'} - \frac{F^{1/2}}{r}\left(L_2\ \zeta^1_{0'0'} - \overline{L_3}\ \zeta^1_{1'1'}\right) = 0, \tag{88}$$

which we write

$$\tilde{A}V = 0, \quad \tilde{A} = \left(-\frac{F^{1/2}}{r}L_3\ ,\ 2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\ ,\ \frac{F^{1/2}}{r}\overline{L_2}\ ,\ \frac{F}{r}\ ,\ 0\ ,\ 0\right), \tag{89}$$

$$\tilde{B}V = 0, \quad \tilde{B} = \left(0\ ,\ 0\ ,\ \frac{F}{r}\ ,\ -\frac{F^{1/2}}{r}L_2\ ,\ 2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\ ,\ \frac{F^{1/2}}{r}\overline{L_3}\right). \tag{90}$$

The operator $\tilde{H}$ on

$$\tilde{\mathcal{H}} = \left\{L^2\left(\mathbb{R}_{r_*} \times S^2_\omega; dr_*^2 + d\omega^2\right)\right\}^6 \tag{91}$$

is isometric to H on $\mathcal{H}$. Hence, it suffices to prove the theorem for the system (86), (87), (88) associated with $\tilde{H}$. We define the successive domains of $\tilde{H}$ in $\tilde{\mathcal{H}}$

$$D(\tilde{H}^k) = \left\{V \in \tilde{\mathcal{H}}; H^j V \in \tilde{\mathcal{H}}, 1 \leq j \leq k\right\},\ k \in \mathbb{N}^*, \tag{92}$$

with the norm

$$\|V\|^2_{D(\tilde{H}^k)} = \sum_{l=0}^{k} \|H^l V\|^2_{\tilde{H}} \ , \ k \in \mathbb{N}^* \tag{93}$$

and in the same manner as previously

$$\tilde{\mathcal{H}}_c = Ker\tilde{A} \bigcap Ker\tilde{B}, \tag{94}$$

$$D(\tilde{H}^k)_c = Ker\tilde{A} \bigcap Ker\tilde{B} \bigcap D(\tilde{H}^k). \tag{95}$$

We will also need the Sobolev spaces

$$\mathbb{H}^k = \left\{ H^k \left(\mathbb{R}_{r_*}; dr_*^2 \right) \right\}^6 , \quad k \in \mathbb{N}. \tag{96}$$

Let us now separate the variables in equations (86), (87) and (88) by means of special functions called spin-weighted spherical harmonics. Ordinary spherical harmonics, or spherical functions, arise when considering the action of the three-dimensional rotation group O_3 on scalar functions defined on the unit 2-sphere. For each $l \in \mathbb{N}$, there is a system of $2l+1$ functions

$$\left\{ Y^l_n(\theta, \varphi) \in L^2(S^2) \right\}_{-l \leq n \leq l} , \tag{97}$$

the spherical functions of order l, which is invariant under the group O_3. The definition of these functions, which merely expresses the invariance of the family under the infinitesimal generators of the representation of O_3 acting on $L^2(S^2)$, is the following

$$i \frac{\partial}{\partial \varphi} Y^l_n = n Y^l_n, \tag{98}$$

$$\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - \frac{n^2}{\sin^2\theta} + l(l+1) \right\} Y^l_n = 0, \tag{99}$$

$$e^{i\varphi} \left(\frac{\partial}{\partial\theta} + n \cot\theta \right) Y^l_n = \left| \begin{array}{l} -i\sqrt{(l-n+1)(l+n)} \ Y^l_{n-1}, \ n > -l, \\ 0, \ n = -l, \end{array} \right. \tag{100}$$

$$e^{-i\varphi} \left(\frac{\partial}{\partial\theta} - n \cot\theta \right) Y^l_n = \left| \begin{array}{l} -i\sqrt{(l-n)(l+n+1)} \ Y^l_{n+1}, \ n < l, \\ 0, \ n = l. \end{array} \right. \tag{101}$$

On the subspace of $L^2(S^2)$ generated by the spherical harmonics of order l, the representation of O_3 becomes irreducible. When l varies in $\mathbb{N}$, we obtain a family of irreducible representations of O_3 of all integral orders. This family is in fact the decomposition into irreducible parts of the representation of O_3 acting on $L^2(S^2)$. It follows that when normalized, the spherical harmonics

$$\left\{ Y^l_n \right\}_{l \in \mathbb{N}, -l \leq n \leq l} \tag{102}$$

are a Hilbert basis of $L^2(S^2)$.

If we study the action of O_3 on more complex quantities like vector, tensor or spinor-valued functions on S^2, other representations will appear, characterized by a non-zero spin-weight. The spin-weight describes the nature of the influence of a rotation around the direction of the North pole on the value at the North pole of quantities on which the representation acts. More precisely, a quantity with spin-weight m will, under such a rotation of angle φ, be multiplied by $e^{im\varphi}$. Let us give a concrete example with a vector-valued function on S^2. The value at the North-pole is simply a vector in $\mathbb{R}^3$ whose origin is at this point. Let us denote a_r, a_θ and a_φ its components in spherical coordinates and put

$$a_+ = a_\theta + i a_\varphi, \quad a_- = a_\theta - i a_\varphi. \tag{103}$$

If we rotate this vector around the direction of the North pole by an angle φ_1, the components are transformed in the following manner

$$\tilde{a}_r = a_r, \quad \tilde{a}_\theta = a_\theta \cos\varphi_1 - a_\varphi \sin\varphi_1, \quad \tilde{a}_\varphi = a_\theta \sin\varphi_1 + a_\varphi \cos\varphi_1$$

and therefore, a_+ and a_- become

$$\tilde{a}_+ = \tilde{a}_\theta + i\tilde{a}_\varphi = e^{i\varphi_1} a_+,$$

$$\tilde{a}_- = \tilde{a}_\theta - i\tilde{a}_\varphi = e^{-i\varphi_1} a_-.$$

We see that a_r, a_+ and a_- are transformed independently of each other. What we have done here is to decompose the representation of O_3 acting on vector-valued functions on the sphere into a sum of three different representations. The representation acting on a_r is identical to the one acting on scalar functions (the radial part of the vector always remains orthogonal to the sphere when a rotation acts, only the value of the component is modified), the corresponding spin-weight is 0. The representation acting on a_+ has spin-weight $+1$ since under a rotation of angle φ_1 around the North pole, a_+ is multiplied by $e^{+i\varphi_1}$ and the spin-weight associated to a_- is -1. For a tensor field of order $k \in \mathbb{N}$ on the sphere, we can in a similar way decompose the tensor into parts of spin-weights $-k$, $-k+1$,..., 0,..., $k-1$, k. Finally, the representation acting on 2-spinor fields of rank $k \in \mathbb{N}$ on S^2 can be decomposed as the sum of simple or multiple representations with spin-weights $-\frac{k}{2}$, $-\frac{k}{2}+1$,...,$\frac{k}{2}-1$, $\frac{k}{2}$. Note that when k is odd, the spin-weights involved are half-integers and not integers. In the case of spinor fields of even rank and tensor fields, the representations obtained are single-valued, i.e. representations of O_3 as such. However, for spinor fields of odd rank, we obtain the double-valued representations which are representations of the universal (two-fold) covering of O_3: SU_2.

For each spin-weight m, $2m \in \mathbb{Z}$, for each l such that $l - |m| \in \mathbb{N}$, we have an irreducible representation of order l acting on weighted scalar fields on the 2-sphere with spin-weight m. The invariant family of functions associated with it are called the spin-weighted spherical harmonics of order l and spin-weight m. They are defined in the same manner as ordinary spherical harmonics by

$$W^l_{mn}(\theta,\varphi) = e^{-in\varphi} u^l_{mn}(\theta) \in L^2(S^2), \tag{104}$$

$$\frac{d^2 u^l_{mn}}{d\theta^2} + \cot\theta \frac{du^l_{mn}}{d\theta} + \left[l(l+1) - \frac{n^2 - 2mn\cos\theta + m^2}{\sin^2\theta} \right] u^l_{mn} = 0, \tag{105}$$

$$\frac{du^l_{mn}}{d\theta} - \frac{m - n\cos\theta}{\sin\theta} u^l_{mn} = \left| \begin{array}{l} -i\left[(l+n)(l-n+1)\right]^{1/2} u^l_{m,n-1}, \ n > -l, \\ 0, \ n = -l, \end{array} \right. \tag{106}$$

$$\frac{du^l_{mn}}{d\theta} + \frac{m - n\cos\theta}{\sin\theta} u^l_{mn} = \left| \begin{array}{l} -i\left[(l+n+1)(l-n)\right]^{1/2} u^l_{m,n+1}, \ n < l, \\ 0, \ n = l \end{array} \right. \tag{107}$$

and we normalize them by

$$\int_0^\pi \left| u^l_{mn}(\theta) \right|^2 \sin\theta d\theta = \frac{1}{2\pi}. \tag{108}$$

We also have the following recurrence relations which will be useful to us

$$\frac{du^l_{mn}}{d\theta} - \frac{n - m\cos\theta}{\sin\theta} u^l_{mn} = -i\left[(l+m)(l-m+1)\right]^{1/2} u^l_{m-1,n}, \tag{109}$$

$$\frac{du^l_{mn}}{d\theta} + \frac{n - m\cos\theta}{\sin\theta} u^l_{mn} = -i\left[(l+m+1)(l-m)\right]^{1/2} u^l_{m+1,n}. \tag{110}$$

For a given spin-weight m, the family of irreducible representations obtained for $l \geq |m|$, $l - |m| \in \mathbb{N}$, is the decomposition into irreducible parts of the representation acting on weighted scalar functions on S^2 with spin-weight m. It follows that for any spin-weight m, $2m \in \mathbb{Z}$, if we define the set of indices

$$\mathcal{I}_m = \{(l,n);\; l-|m| \in \mathbb{N},\; l-|n| \in \mathbb{N}\}\,, \tag{111}$$

the family

$$\left\{W^l_{mn}(\theta,\varphi)\right\}_{(l,n)\in\mathcal{I}_m} \tag{112}$$

is a Hilbert basis of $L^2\left(S^2; d\omega^2\right)$. For $m=0$, we recover ordinary spherical harmonics, i.e.

$$W^l_{0n} \equiv Y^l_n. \tag{113}$$

For a more detailed account on spin-weighted spherical harmonics, their construction and properties, see for example [6].

Square integrable weighted scalar functions on $I\times S^2$, $I\subset\mathbb{R}$, with spin-weight m can be expanded uniquely in a series of the form

$$\sum_{l=|m|}^{+\infty}\sum_{n=-l}^{l} a^l_n(r) W^l_{mn}(\theta,\varphi), \quad a^l_n \in L^2(I) \tag{114}$$

and this expansion will be invariant under rotations. Thus we have a way of separating the variables for equations which are invariant under rotations, like zero restmass field equations in spherically symmetric backgrounds. The advantage of using the Newman-Penrose formalism for translating 2-spinor equations is that we obtain a form in which the spinors are already decomposed into components of different spin-weights. Therefore, it allows us to separate the variables without further manipulations on the equation. In our case, putting

$$\mathcal{W}^l_n = {}^t\left(W^l_{-\frac{3}{2},n}, W^l_{-\frac{1}{2},n}, W^l_{\frac{1}{2},n}, W^l_{-\frac{1}{2},n}, W^l_{\frac{1}{2},n}, W^l_{\frac{3}{2},n}\right),\ (l,n)\in\mathcal{I}_{1/2}, \tag{115}$$

with the convention

$$W^{\frac{1}{2}}_{\pm\frac{3}{2},\frac{1}{2}} \equiv W^{\frac{1}{2}}_{\pm\frac{3}{2},-\frac{1}{2}} \equiv 0, \tag{116}$$

we have

$$\tilde{\mathcal{H}} = \bigoplus_{(l,n)\in\mathcal{I}_{1/2}} \mathbb{H}^0 \otimes \mathcal{W}^l_n \tag{117}$$

and for $k\in\mathbb{N}^*$

$$D(\tilde{H}^k) = \bigoplus_{(l,n)\in\mathcal{I}_{1/2}} \mathbb{H}^k \otimes \mathcal{W}^l_n; \tag{118}$$

the spaces $\mathbb{H}^k$, $k\in\mathbb{N}$ being defined in (96). We separate the variables in equations (86), (87) and (88) using the basis $\mathcal{W}^l_n$ of $\left[L^2(S^2)\right]^6$. This comes down to saying that the component $\sigma^0_{0'0'}$ of $\sigma^A_{B'C'}$ has spin-weight $-3/2$, that $\sigma^0_{0'1'}$ and $\sigma^1_{0'0'}$ have spin-weight $-1/2$, $\sigma^0_{1'1'}$ and $\sigma^1_{0'1'}$ have spin-weight $1/2$ and $\sigma^1_{1'1'}$ has spin-weight $3/2$.

Lemma 4.1 *For $(l,n)\in\mathcal{I}_{1/2}$, if*

$$V = \mathbf{V}\otimes\mathcal{W}^l_n,\ \mathbf{V} = {}^t\left(f^0_{0'0'}, f^0_{0'1'}, f^0_{1'1'}, f^1_{0'0'}, f^1_{0'1'}, f^1_{1'1'}\right) \in \mathbb{H}^k,\ k\in\mathbb{N}^*,$$

then

$$\tilde{H}V \in \mathbb{H}^{k-1}\otimes\mathcal{W}^l_n \tag{119}$$

and we can write $\tilde{H}V = \left(\tilde{H}_l \mathbf{V}\right) \otimes \mathcal{W}^l_n$, *with*

$$\tilde{H}_l \mathbf{V} = \begin{pmatrix} \partial_{r_*} f^0_{0'0'} - i\alpha_1 \frac{F^{1/2}}{r} f^0_{0'1'} \\ -\partial_{r_*} f^0_{0'1'} - \frac{F}{r} f^0_{0'1'} - \frac{F}{r} f^1_{0'0'} - i\alpha_1 \frac{F^{1/2}}{r} f^0_{0'0'} \\ -\partial_{r_*} f^0_{1'1'} - \frac{F'}{2} f^0_{1'1'} - \frac{F}{r} f^1_{0'1'} - i\alpha_2 \frac{F^{1/2}}{r} f^0_{0'1'} \\ \partial_{r_*} f^1_{0'0'} + \frac{F'}{2} f^1_{0'0'} + \frac{F}{r} f^0_{0'1'} - i\alpha_2 \frac{F^{1/2}}{r} f^1_{0'1'} \\ \partial_{r_*} f^1_{0'1'} + \frac{F}{r} f^1_{0'1'} + \frac{F}{r} f^0_{1'1'} - i\alpha_1 \frac{F^{1/2}}{r} f^1_{1'1'} \\ -\partial_{r_*} f^1_{1'1'} - i\alpha_1 \frac{F^{1/2}}{r} f^1_{0'1'} \end{pmatrix} \tag{120}$$

where

$$\alpha_1 = \left[\left(l - \frac{1}{2}\right)\left(l + \frac{3}{2}\right)\right]^{1/2} \quad , \quad \alpha_2 = l + \frac{1}{2}. \tag{121}$$

Moreover, V satisfies he constraints (i.e. $V \in D(H^k)_c$) if and only if $\mathbf{V}$ *satisfies the following radial constraint equations*

$$\tilde{A}_l \mathbf{V} = \left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right) f^0_{0'1'} + \frac{F}{r} f^1_{0'0'} + \frac{i\alpha_1 F^{\frac{1}{2}}}{r} f^0_{0'0'} - \frac{i\alpha_2 F^{\frac{1}{2}}}{r} f^0_{1'1'} = 0, \tag{122}$$

$$\tilde{B}_l \mathbf{V} = \left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right) f^1_{0'1'} + \frac{F}{r} f^0_{1'1'} - \frac{i\alpha_1 F^{\frac{1}{2}}}{r} f^1_{1'1'} + \frac{i\alpha_2 F^{\frac{1}{2}}}{r} f^1_{0'0'} = 0. \tag{123}$$

Proof of lemma 4.1: We just need to calculate $\tilde{H}V$ and the constraints using the recurrence relations (109) and (110). The first component of $\tilde{H}V$ is

$$\left(\tilde{H}V\right)^0_{0'0'} = \partial_{r_*}\left(f^0_{0'0'}(r_*) W^l_{-\frac{3}{2}n}(\theta,\varphi)\right) +$$
$$\frac{F^{1/2}}{r}\left(\partial_\theta - \frac{1}{2}\cot\theta - \frac{i}{\sin\theta}\partial_\varphi\right)\left(f^0_{0'1'} W^l_{-\frac{1}{2}n}(\theta,\varphi)\right).$$

From (109) and the definition of W^l_{mn} we see that

$$\left(\partial_\theta - \frac{1}{2}\cot\theta - \frac{i}{\sin\theta}\partial_\varphi\right) W^l_{-\frac{1}{2}n}(\theta,\varphi) = \left(\partial_\theta - \frac{n - \frac{1}{2}\cot\theta}{\sin\theta}\right) u^l_{-\frac{1}{2}n}(\theta) e^{-in\varphi}$$

$$= -i\left[\left(l - \frac{1}{2}\right)\left(l + \frac{3}{2}\right)\right]^{1/2} u^l_{-\frac{3}{2}n}(\theta) e^{-in\varphi} = -i\alpha_1 W^l_{-\frac{3}{2}n}(\theta,\varphi)$$

whence the first component of $\tilde{H}V$ can be written

$$\left(\tilde{H}V\right)^0_{0'0'} = \left(\partial_{r_*} f^0_{0'0'} - i\alpha_1 \frac{F^{1/2}}{r} f^0_{0'1'}\right) W^l_{-\frac{3}{2}n}.$$

The same explicit calculation for the five other components entails

$$\tilde{H}V = \begin{pmatrix} \left(\partial_{r_*} f^0_{0'0'} - i\alpha_1 \frac{F^{1/2}}{r} f^0_{0'1'}\right) W^l_{-\frac{3}{2}n} \\ \left(-\partial_{r_*} f^0_{0'1'} - \frac{F}{r} f^0_{0'1'} - \frac{F}{r} f^1_{0'0'} - i\alpha_1 \frac{F^{1/2}}{r} f^0_{0'0'}\right) W^l_{-\frac{1}{2}n} \\ \left(-\partial_{r_*} f^0_{1'1'} - \frac{F'}{2} f^0_{1'1'} - \frac{F}{r} f^1_{0'1'} - i\alpha_2 \frac{F^{1/2}}{r} f^0_{0'1'}\right) W^l_{\frac{1}{2}n} \\ \left(\partial_{r_*} f^1_{0'0'} + \frac{F'}{2} f^1_{0'0'} + \frac{F}{r} f^0_{0'1'} - i\alpha_2 \frac{F^{1/2}}{r} f^1_{0'1'}\right) W^l_{-\frac{1}{2}n} \\ \left(\partial_{r_*} f^1_{0'1'} + \frac{F}{r} f^1_{0'1'} + \frac{F}{r} f^0_{1'1'} - i\alpha_1 \frac{F^{1/2}}{r} f^1_{1'1'}\right) W^l_{\frac{1}{2}n} \\ \left(-\partial_{r_*} f^1_{1'1'} - i\alpha_1 \frac{F^{1/2}}{r} f^1_{0'1'}\right) W^l_{\frac{3}{2}n} \end{pmatrix} = \left(\tilde{\mathbb{H}}_l \mathbf{V}\right) \otimes \mathcal{W}^l_n .$$

We can apply the same method to the constraint equations. From (87) we can write

$$\tilde{A}V = \left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right) f^0_{0'1'} W^l_{-\frac{1}{2}n} + \frac{F}{r} f^1_{0'0'} W^l_{-\frac{1}{2}n} - \frac{F^{1/2}}{r}\left(\partial_\theta + \frac{3}{2}\cot\theta + \frac{i}{\sin\theta}\partial_\varphi\right) f^0_{0'0'} W^l_{-\frac{3}{2}n} + \frac{F^{1/2}}{r}\left(\partial_\theta + \frac{1}{2}\cot\theta - \frac{i}{\sin\theta}\partial_\varphi\right) f^0_{1'1'} W^l_{\frac{1}{2}n}.$$

By (110) and (109) respectively we see that

$$-\left(\partial_\theta + \frac{3}{2}\cot\theta + \frac{i}{\sin\theta}\partial_\varphi\right) W^l_{-\frac{3}{2}n} = i\alpha_1 W^l_{-\frac{1}{2}n}$$

and

$$\left(\partial_\theta + \frac{1}{2}\cot\theta - \frac{i}{\sin\theta}\partial_\varphi\right) W^l_{\frac{1}{2}n} = -i\alpha_2 W^l_{-\frac{1}{2}n}$$

whence

$$\tilde{A}V = \left[\left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right) f^0_{0'1'} + \frac{F}{r} f^1_{0'0'} + i\alpha_1 \frac{F^{1/2}}{r} f^0_{0'0'} - i\alpha_2 \frac{F^{1/2}}{r} f^0_{1'1'}\right] W^l_{-\frac{1}{2}n}.$$

And in the same manner we prove

$$\tilde{B}V = \left[\left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right) f^1_{0'1'} + \frac{F}{r} f^0_{1'1'} - \frac{i\alpha_1 F^{\frac{1}{2}}}{r} f^1_{1'1'} + \frac{i\alpha_2 F^{\frac{1}{2}}}{r} f^1_{0'0'}\right] W^l_{\frac{1}{2}n}$$

which proves lemma 4.1. □

It follows from the previous lemma that $\tilde{\mathcal{H}}_c$ and $D(\tilde{H}^k)_c$, $k \in \mathbb{N}^*$, can themselves be decomposed into spin-weighted spherical harmonics in the following way :

$$\tilde{\mathcal{H}}_c = \bigoplus_{(l,n)\in\mathcal{I}_{1/2}} \mathbb{H}^0_{cl} \otimes \mathcal{W}^l_n, \quad D(\tilde{H}^k)_c = \bigoplus_{(l,n)\in\mathcal{I}_{1/2}} \mathbb{H}^k_{cl} \otimes \mathcal{W}^l_n \tag{124}$$

where, for $k \in \mathbb{N}$, $l \geq 1/2$,

$$\mathbb{H}^k_{cl} = \left\{\mathbf{V} \in \mathbb{H}^k; (121),\ (122),\ (123)\ hold\right\}. \tag{125}$$

For $(l,n) \in \mathcal{I}_{1/2}$, $\tilde{H}$ can be written on each $\mathbb{H}^k$, $k \in \mathbb{N}$, as

$$\tilde{H}_l = E\partial_{r_*} + P_l(r) \tag{126}$$

where

$$E = diag(1,-1,-1,1,1,-1) \tag{127}$$

and the potential

$$P_l(r) = \begin{pmatrix} 0 & -i\alpha_1 \frac{F^{1/2}}{r} & 0 & 0 & 0 & 0 \\ -i\alpha_1 \frac{F^{1/2}}{r} & -\frac{F}{r} & 0 & -\frac{F}{r} & 0 & 0 \\ 0 & -i\alpha_2 \frac{F^{1/2}}{r} & -\frac{F'}{2} & 0 & -\frac{F}{r} & 0 \\ 0 & \frac{F}{r} & 0 & \frac{F'}{2} & -i\alpha_2 \frac{F^{1/2}}{r} & 0 \\ 0 & 0 & \frac{F}{r} & 0 & \frac{F}{r} & -i\alpha_1 \frac{F^{1/2}}{r} \\ 0 & 0 & 0 & 0 & -i\alpha_1 \frac{F^{1/2}}{r} & 0 \end{pmatrix} \tag{128}$$

is clearly a bounded operator on $\mathbb{H}^k$. Moreover, $E\partial_{r_*}$ is skew-adjoint on $\mathbb{H}^k$ with dense domain $\mathbb{H}^{k+1}$. We can now state and prove the following existence and uniqueness result

Lemma 4.2 *Let $V_0 \in \mathbb{H}^k \otimes \mathcal{W}_n^l$, $k \in \mathbb{N}$, $(l,n) \in \mathcal{I}_{1/2}$, equation (86) has a unique solution V such that*

$$V \in \mathcal{C}\left(\mathbb{R}_t; \mathbb{H}^k \otimes \mathcal{W}_n^l\right), \quad V\mid_{t=0} = V_0 \tag{129}$$

and we have an exponential control on the norm of the solution :

$$\exists C_{kl} > 0; \quad \forall t \in \mathbb{R} \quad \|V(t)\|_{\mathbb{H}^k} \le e^{C_{kl}|t|} \|V_0\|_{\mathbb{H}^k}. \tag{130}$$

We also have

$$V \in \bigcap_{j=0}^{k} \mathcal{C}^j \left(\mathbb{R}_t; \mathbb{H}^{k-j} \otimes \mathcal{W}_n^l\right). \tag{131}$$

Moreover, if $\{V_0^m\}_{m\in\mathbb{N}}$ is a sequence in $\mathbb{H}^k \otimes \mathcal{W}_n^l$ such that

$$V_0^m \longrightarrow V_0 \ \ in\ \mathbb{H}^k \otimes \mathcal{W}_n^l, \quad m \to +\infty,$$

for each m, we have a solution of (86)

$$V^m \in \mathcal{C}\left(\mathbb{R}_t; \mathbb{H}^k \otimes \mathcal{W}_n^l\right) \tag{132}$$

associated with V_0^m and the sequence V^m satisfies

$$V^m \longrightarrow V \ in\ \mathcal{C}\left(\mathbb{R}_t; \mathbb{H}^k \otimes \mathcal{W}_n^l\right), \quad m \to +\infty, \tag{133}$$

i.e. the convergence is uniform on each compact of $\mathbb{R}_t$. The solutions are continuous with respect to their initial data. Note that the propagation speed is lower than or equal to 1.

Proof of lemma 4.2: We use essentially a fixed point (or Picart) method. Firstly, we express the evolution system (86) with the initial data condition

$$V|_{t=0} = V_0 \in \mathbb{H}^k \otimes \mathcal{W}_n^l$$

as an integral equation

$$V(t) = \mathcal{S}V(t) \tag{134}$$

where

$$\mathcal{S}V(t) = e^{tE\frac{\partial}{\partial r_*}}V_0 + \int_0^t e^{(t-s)E\frac{\partial}{\partial r_*}}P_l(r)V(s)ds. \quad (135)$$

The solution of (134) on $\mathcal{C}(0,T;\mathbb{H}^k \otimes \mathcal{W}^l_n)$, $T \in \mathbb{R}$, is equivalent to the solution of (86) in the same space with the initial data condition. For $T \in \mathbb{R}$, the space $\mathcal{C}(0,T;\mathbb{H}^k \otimes \mathcal{W}^l_n)$ is stable under the functional $\mathcal{S}$ and for T small enough, $\mathcal{S}$ is a strict contraction on the closed ball

$$\left\{ V \in \mathcal{C}(0,T;\mathbb{H}^k \otimes \mathcal{W}^l_n); \forall t \in [0,T],\ \|V(t)\|_{\mathbf{H}^k} \leq 2\|V_0\|_{\mathbf{H}^k} \right\}. \quad (136)$$

By a standard convexity argument, this gives the existence and uniqueness of local solutions of (86). The boundedness of P_l entails by Gronwall's lemma that a solution of (134) in $\mathcal{C}(0,T;\mathbb{H}^k \otimes \mathcal{W}^l_n)$ must satisfy the estimate

$$\|V(t)\|_{\mathbf{H}^k} \leq e^{C_{kl}|t|}\|V_0\|_{\mathbf{H}^k} \ \ \textit{where}\ C_{kl} = \|P_l\|_{\mathcal{L}(\mathbf{H}^k)}, \quad (137)$$

$\mathcal{L}(\mathbb{H}^k)$ being the space of continuous linear mappings from $\mathbb{H}^k$ to itself. This a priori estimate guarantees global existence for local solutions. Together with the linearity of the equation, it also entails uniqueness of solutions as well as the continuity with respect to initial data. □

The next step is to see that the constraints are conserved by the evolution. We start by proving that the hamiltonian operator on each angular dependence conserves the constraints

Lemma 4.3 *Let $V = \mathbf{V} \otimes \mathcal{W}^l_n \in \mathbb{H}^0 \otimes \mathcal{W}^l_n$, $(l,n) \in \mathcal{I}_{1/2}$. In the sense of distributions we have*

$$\tilde{A}_l\tilde{H}_l\mathbf{V} = -\left(\partial_{r_*} + \frac{F}{r}\right)\tilde{A}_l\mathbf{V}, \quad \tilde{B}_l\tilde{H}_l\mathbf{V} = +\left(\partial_{r_*} + \frac{F}{r}\right)\tilde{B}_l\mathbf{V}. \quad (138)$$

Proof of lemma 4.3 It is just a long and explicit calculation without any difficulty. We denote

$$\mathbf{V} = {}^t\left(f^0_{0'0'}, f^0_{0'1'}, f^0_{1'1'}, f^1_{0'0'}, f^1_{0'1'}, f^1_{1'1'}\right).$$

The quantity

$$\tilde{A}_l\tilde{H}_l\mathbf{V} = \left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right)\left(\tilde{H}_l\mathbf{V}\right)^0_{0'1'} + \frac{F}{r}\left(\tilde{H}_l\mathbf{V}\right)^1_{0'0'}$$
$$+\frac{i\alpha_1F^{1/2}}{r}\left(\tilde{H}_l\mathbf{V}\right)^0_{0'0'} - \frac{i\alpha_2F^{1/2}}{r}\left(\tilde{H}_l\mathbf{V}\right)^0_{1'1'}, \quad (139)$$

where

$$\left(\tilde{H}_l\mathbf{V}\right)^0_{0'0'} = \partial_{r_*}f^0_{0'0'} - i\alpha_1\frac{F^{1/2}}{r}f^0_{0'1'}$$

is the first component of $\tilde{H}_l\mathbf{V}$, $\left(\tilde{H}_l\mathbf{V}\right)^0_{0'1'}$ the second, etc..., is defined in the sense of distributions. We develop (139) completely using the expression of the components of $\tilde{H}\mathbf{V}$. We do the same with

$$-\left(\partial_{r_*} + \frac{F}{r}\right)\tilde{A}_l\mathbf{V} = -\left(\partial_{r_*} + \frac{F}{r}\right)\left[\left(2\partial_{r_*} + \frac{2F}{r} - \frac{F'}{2}\right)f^0_{0'1'}\right.$$
$$\left.+\frac{F}{r}f^1_{0'0'} + \frac{i\alpha_1F^{1/2}}{r}f^0_{0'0'} - \frac{i\alpha_2F^{1/2}}{r}f^0_{1'1'}\right] \quad (140)$$

and we substract the latter from the former. Remembering that $\partial_{r_*} = F\partial_r$ as well as the values of α_1 and α_2, we will find that all the terms cancel one another. The same sort of painstaking exercise gives

$$\tilde{B}_l\tilde{H}_l\mathbf{V} = \left(\partial_{r_*} + \frac{F}{r}\right)\tilde{B}_l\mathbf{V} \quad (141)$$

and lemma 4.3 is thus proved. □

As an immediate consequence of (138), for $\mathbf{V} \in \mathbb{H}^0$

$$\tilde{A}_l \mathbf{V} = 0 \ \Rightarrow \ \tilde{A}_l \tilde{H}_l \mathbf{V} = 0, \ \ \tilde{B}_l \mathbf{V} = 0 \ \Rightarrow \ \tilde{B}_l \tilde{H}_l \mathbf{V} = 0.$$

The upshot of this result is that the constraints are indeed conserved by the evolution. The precise statement is given in the following lemma

Lemma 4.4 *For any initial data $V_0 \in \mathbb{H}^k \otimes \mathcal{W}^l_n$, $k \in \mathbb{N}$, $(l,n) \in \mathcal{I}_{1/2}$, satisfying the constraints*

$$\tilde{A} V_0 = 0, \ \tilde{B} V_0 = 0,$$

the solution V of (86) in $\mathcal{C}\left(\mathbb{R}_t ; \mathbb{H}^k \otimes \mathcal{W}^l_n\right)$ such that $V|_{t=0} = V_0$ satisfies the constraints, i.e.

$$V \in \bigcap_{j=0}^{k} \mathcal{C}^j \left(\mathbb{R}_t ; \mathbb{H}^{k-j}_{cl} \otimes \mathcal{W}^l_n\right).$$

The case $k \geq 1$ is a straightforward consequense of the case $k = 0$. However, the proof of lemma 4.4 in the case $k = 0$ is slightly technical and will be detailed after the proof of the main theorem.

In order to establish the global existence in $\mathcal{C}\left(\mathbb{R}_t ; \tilde{\mathcal{H}}_c\right)$ and $\mathcal{C}\left(\mathbb{R}_t ; D(\tilde{H}^k)_c\right)$, $k \geq 1$, of the solutions of (86), we use an energy estimate. Let us consider the set

$$\mathcal{K} = \left\{ V = \sum_{finite} V^{ln} \otimes \mathcal{W}^l_n ; \ V^{ln} \in \mathcal{C}^\infty_0(\mathbb{R}_{r_*}), \ \tilde{A}_l V^{ln} = 0, \ \tilde{B}_l V^{ln} = 0 \right\}. \tag{142}$$

$\mathcal{K}$ is obviously dense in $\tilde{\mathcal{H}}_c$ and $D(\tilde{H}^k)_c$, $k \geq 1$ and we can define on $\mathcal{K}$ a propagator $\mathcal{V}$ for (86) by linearity. Indeed, if

$$V_0 = \sum_{finite} V^{ln}_0 \otimes \mathcal{W}^l_n \in \mathcal{K} \tag{143}$$

we denote by V^{ln} the solution of (86) associated with the initial data $V^{ln}_0 \otimes \mathcal{W}^l_n$ and the action on V_0 of the propagator $\mathcal{V}$ at each time $t \in \mathbb{R}$ will be given by

$$\mathcal{V}(t) V_0 = \sum V^{ln}(t). \tag{144}$$

By construction, $\mathcal{V}$ satisfies the following properties :

$$\mathcal{V}(t+s) = \mathcal{V}(t)\mathcal{V}(s), \ \forall t, s \in \mathbb{R}, \tag{145}$$

$$\mathcal{V}(0) = \mathbb{1}_{\mathcal{K}} \tag{146}$$

and for any $V_0 \in \mathcal{K}$ we have

$$\mathcal{V}(t) V_0 \in \mathcal{C}\left(\mathbb{R}_t ; \tilde{\mathcal{H}}_c\right) \bigcap \left[\bigcap_{k=1}^{+\infty} \bigcap_{j=0}^{k} \mathcal{C}^j \left(\mathbb{R}_t ; D(\tilde{H}^{k-j})_c\right) \right]. \tag{147}$$

The energy estimate will enable us to extend $\mathcal{V}$ to the whole of $\tilde{\mathcal{H}}_c$. Let $V_0 \in \mathcal{K}$ and

$$V = \ {}^t \left(\zeta^0_{0'0'}, \zeta^0_{0'1'}, \zeta^0_{1'1'}, \zeta^1_{0'0'}, \zeta^1_{0'1'}, \zeta^1_{1'1'}\right) = \mathcal{V}(t) V_0.$$

We have

$$\frac{\partial}{\partial t} |V|^2 = iHV\overline{V} + V\overline{iHV} = 2Re(\overline{V} iHV).$$

If we integrate this identity on $[0,t]\times \mathbb{R}_{r_*}\times S^2_\omega$, we obtain

$$\|V(t)\|^2_{\tilde{\mathcal{H}}}-\|V(0)\|^2_{\tilde{\mathcal{H}}}=2\int_0^t\int_{\mathbf{R}\times S^2} Re\left(\overline{V}(s)iHV(s)\right)dr_*d\omega ds. \tag{148}$$

From the second and fifth lines of (86), using (120) and the constraints, we find

$$\partial_t\zeta^0_{0'1'}=\partial_{r_*}\zeta^0_{0'1'}+\left(\frac{F}{r}-\frac{F'}{2}\right)\zeta^0_{0'1'}+\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{1}{2}\cot\theta-\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^0_{1'1'},$$

$$\partial_t\zeta^1_{0'1'}=-\partial_{r_*}\zeta^1_{0'1'}-\left(\frac{F}{r}-\frac{F'}{2}\right)\zeta^1_{0'1'}+\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{1}{2}\cot\theta+\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^1_{0'0'}.$$

Multiplying the first equality by $2\zeta^0_{0'1'}$ and the second by $2\zeta^1_{0'1'}$, we integrate them on $[0,t]\times\mathbb{R}_{r_*}\times S^2_\omega$. Adding the results to (148) and developing the right handside of (148), we obtain

$$\begin{aligned}&\|V(t)\|^2_{\tilde{\mathcal{H}}}+\|\zeta^0_{0'1'}(t)\|^2_{L^2}+\|\zeta^1_{0'1'}(t)\|^2_{L^2}\\&-\left(\|V(0)\|^2_{\tilde{\mathcal{H}}}+\|\zeta^0_{0'1'}(0)\|^2_{L^2}+\|\zeta^1_{0'1'}(0)\|^2_{L^2}\right)=I(t)\end{aligned} \tag{149}$$

where $\|.\|_{L^2}$ denotes the norm in $L^2(\mathbb{R}\times S^2;dr_*^2+d\omega^2)$ and

$$\begin{aligned}I(t)=2\int_0^t\int_{\mathbf{R}\times S^2}Re\Big\{&\overline{\zeta^0_{0'0'}}\partial_{r_*}\zeta^0_{0'0'}+\overline{\zeta^0_{0'0'}}\frac{F^{1/2}}{r}\left(\partial_\theta-\frac{1}{2}\cot\theta-\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^0_{0'1'}\\
&-\overline{\zeta^0_{0'1'}}\partial_{r_*}\zeta^0_{0'1'}-\overline{\zeta^0_{0'1'}}\frac{F}{r}\left(\zeta^0_{0'1'}+\zeta^1_{0'0'}\right)+\overline{\zeta^0_{0'1'}}\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{3}{2}\cot\theta+\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^0_{0'0'}\\
&-\overline{\zeta^0_{1'1'}}\partial_{r_*}\zeta^0_{1'1'}-\overline{\zeta^0_{1'1'}}\frac{F'}{2}\zeta^0_{1'1'}-\overline{\zeta^0_{1'1'}}\frac{F}{r}\zeta^1_{0'1'}+\overline{\zeta^0_{1'1'}}\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{1}{2}\cot\theta+\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^0_{0'1'}\\
&+\overline{\zeta^0_{0'1'}}\partial_{r_*}\zeta^0_{0'1'}+\overline{\zeta^0_{0'1'}}\left(\frac{F}{r}-\frac{F'}{2}\right)\zeta^0_{0'1'}+\overline{\zeta^0_{0'1'}}\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{1}{2}\cot\theta-\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^0_{1'1'}\\
&-\overline{\zeta^1_{0'1'}}\partial_{r_*}\zeta^1_{0'1'}-\overline{\zeta^1_{0'1'}}\left(\frac{F}{r}-\frac{F'}{2}\right)\zeta^1_{0'1'}+\overline{\zeta^1_{0'1'}}\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{1}{2}\cot\theta+\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^1_{0'0'}\\
&+\overline{\zeta^1_{0'0'}}\partial_{r_*}\zeta^1_{0'0'}+\overline{\zeta^1_{0'0'}}\frac{F'}{2}\zeta^1_{0'0'}+\overline{\zeta^1_{0'0'}}\frac{F}{r}\zeta^0_{0'1'}+\overline{\zeta^1_{0'0'}}\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{1}{2}\cot\theta-\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^1_{0'1'}\\
&+\overline{\zeta^1_{0'1'}}\partial_{r_*}\zeta^1_{0'1'}+\overline{\zeta^1_{0'1'}}\frac{F}{r}\left(\zeta^1_{0'1'}+\zeta^0_{1'1'}\right)+\overline{\zeta^1_{0'1'}}\frac{F^{1/2}}{r}\left(\partial_\theta+\frac{3}{2}\cot\theta-\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^1_{1'1'}\\
&-\overline{\zeta^1_{1'1'}}\partial_{r_*}\zeta^1_{1'1'}+\overline{\zeta^1_{1'1'}}\frac{F^{1/2}}{r}\left(\partial_\theta-\frac{1}{2}\cot\theta+\frac{i}{\sin\theta}\partial_\varphi\right)\zeta^1_{0'1'}\Big\}dr_*d\omega ds.\end{aligned} \tag{150}$$

It is easy to see, using integrations by parts in θ and φ, that the terms in $I(t)$ involving the angular operators L_k or $\overline{L_k}$ all cancel one another. The same is true for the terms involving a partial derivative with respect to r_*. The remaining terms are of the form

$$2Re\int_0^t\int_{\mathbf{R}\times S^2}\overline{g}\frac{F}{r}fdr_*d\omega ds\quad and\quad 2Re\int_0^t\int_{\mathbf{R}\times S^2}\overline{g}\frac{F'}{2}fdr_*d\omega ds$$

and can be estimated by

$$2\int_0^t\|V(s)\|^2_{\tilde{\mathcal{H}}}ds.$$

Hence, we have the following estimate for $t \in \mathbb{R}$

$$\|V(t)\|^2_{\tilde{\mathcal{H}}} \leq 2\|V_0\|^2_{\tilde{\mathcal{H}}} + C \int_0^t \|V(s)\|^2_{\tilde{\mathcal{H}}} ds \ , \quad C > 0$$

and using Gronwall's lemma, we obtain

Lemma 4.5 *There exists a constant $C > 0$ such that for $V_0 \in \mathcal{K}$ and $t \in \mathbb{R}$*

$$\|\mathcal{V}(t)V_0\|^2_{\tilde{\mathcal{H}}} \leq 2e^{C|t|}\|V_0\|^2_{\tilde{\mathcal{H}}} \tag{151}$$

where $\mathcal{V}$ is the propagator in $\mathcal{K}$ for equation (86).

Moreover, if we consider

$$\tilde{H}V_0 = \sum_{finite} \tilde{H}_l V_0^{ln} \otimes \mathcal{W}_n^l,$$

it is an element of $\mathcal{K}$ and $\tilde{H}V^{ln}(t) \otimes \mathcal{W}_n^l$ is the solution of (86) in the space $\mathcal{C}\left(\mathbb{R}_t; \mathbb{H}^k \otimes \mathcal{W}_n^l\right)$, $k \in \mathbb{N}$, associated with the initial data $V_0^{ln} \otimes \mathcal{W}_n^l$, i.e. for all $t \in \mathbb{R}$ we have

$$\mathcal{V}(t)\left(\tilde{H}V_0\right) = \sum_{finite} \tilde{H}_l V^{ln}(t) \otimes \mathcal{W}_n^l = \tilde{H}\mathcal{V}(t)V_0 = \tilde{H}V(t)$$

whence

$$\|\tilde{H}V(t)\|^2_{\tilde{\mathcal{H}}} \leq 2\|\tilde{H}V_0\|^2_{\tilde{\mathcal{H}}} e^{C|t|}.$$

Repeating the process an arbitrary number of times, we can write for any $k \in \mathbb{N}^*$

$$\|V(t)\|^2_{D(\tilde{H}^k)} \leq 2\|V_0\|^2_{D(\tilde{H}^k)} e^{C|t|}. \tag{152}$$

Lemma 4.5 and estimate (152) show that for any $t \in \mathbb{R}$, $\mathcal{V}(t)$ is a bounded operator on $\mathcal{K}$ for the norms in $\tilde{\mathcal{H}}$ and $D(\tilde{H}^k)$, $k \geq 1$. This allows us to extend $\mathcal{V}(t)$, $t \in \mathbb{R}$, as an element of the following space

$$\mathcal{V}(t) \in \mathcal{L}\left(\tilde{\mathcal{H}}_c\right) \bigcap \left[\bigcap_{k=1}^{+\infty} \mathcal{L}\left(D(\tilde{H}^k)_c\right)\right], \tag{153}$$

$\mathcal{L}\left(\tilde{\mathcal{H}}_c\right)$ denoting the space of bounded linear mappings from $\tilde{\mathcal{H}}_c$ to itself. We have the following control on the norm of $\mathcal{V}(t)$ in all these spaces

$$\|\mathcal{V}(t)\| \leq \sqrt{2}e^{C'|t|}, \ C' = \frac{C}{2} > 0, \ t \in \mathbb{R}. \tag{154}$$

By continuity, $\mathcal{V}$ satisfies

$$\mathcal{V}(t+s) = \mathcal{V}(t)\mathcal{V}(s), \ in \ \mathcal{L}\left(\tilde{\mathcal{H}}_c\right), \ t, s \in \mathbb{R}, \tag{155}$$

$$\mathcal{V}(0) = \mathbb{I}_{\mathcal{L}(\tilde{\mathcal{H}}_c)} \tag{156}$$

and for $V_0 \in \mathcal{L}\left(\tilde{\mathcal{H}}_c\right)$ (resp. $D(\tilde{H}^k)_c$, $k \geq 1$)

$$\mathcal{V}(t)V_0 \in \mathcal{C}\left(\mathbb{R}_t; \mathcal{L}\left(\tilde{\mathcal{H}}_c\right)\right) \quad \left(resp. \ \mathcal{C}\left(\mathbb{R}_t; D(\tilde{H}^k)_c\right)\right) \tag{157}$$

as a locally uniform limit of continuous functions. If we consider $V_0 \in \tilde{\mathcal{H}}_c$, by continuity in the sense of distributions, $\mathcal{V}(t)V_0$ is a solutions of (86) in $\mathcal{C}\left(\mathbb{R}_t; \tilde{\mathcal{H}}_c\right)$ such that $(\mathcal{V}(t)V_0)|_{t=0} = V_0$ and this solution is unique as a consequence of the uniqueness of solutions on each angular dependence. Thus, theorem 4.1 is proved. We shall denote $\mathcal{U}$ the propagator for equation (63). $\mathcal{U}$ can be defined explicitely from $\mathcal{V}$ by

$$\mathcal{U}(t)U_0 = \left(rF^{1/4}\right)^{-1} \mathcal{V}(t) \left[rF^{1/4}U_0\right] \tag{158}$$

and $\mathcal{U}$ on $\mathcal{H}_c$ is isometric to $\mathcal{V}$ on $\tilde{H}$ in the same way that H is isometric to $\tilde{H}$. □

Proof of lemma 4.2 : Let us consider for some $(l,n) \in \mathcal{I}_{1/2}$

$$V_0 \in \mathbb{H}^0_{cl} \otimes \mathcal{W}^l_n.$$

We know from (138) that $\tilde{A}_l\tilde{H}_lV_0 = 0$ and $\tilde{B}_l\tilde{H}_lV_0 = 0$. Denoting by $\mathcal{V}_l$ the propagator of the equation in $\mathbb{H}^0 \otimes \mathcal{W}^l_n$, we wish to prove that for any $t \in \mathbb{R}$

$$\tilde{A}_l\mathcal{V}_l(t)V_0 = 0 \quad and \quad \tilde{B}_l\mathcal{V}_l(t)V_0 = 0.$$

To this purpose, we express $\mathcal{V}_l(t)V_0$, $t \in \mathbb{R}^*$, as the limit

$$\mathcal{V}_l(t)V_0 = \lim_{k\to+\infty} \left(I - \frac{t}{k}\tilde{H}_l\right)^{-k} V_0 \quad in \quad \mathbb{H}^0 \otimes \mathcal{W}^l_n \tag{159}$$

We are justified in writing so since $\mathcal{V}_l$ is a strongly continuous one parameter *group* on $\mathbb{H}^0 \otimes \mathcal{W}^l_n$ (see [7]) and moreover the limit (159) is uniform with respect to t in each interval of the form $[\varepsilon, 1/\varepsilon[$. Indeed, we see immediately through an argument of convergence in the sense of distributions that the operator $\tilde{H}_l$ is closed and is therefore the infinitesimal generator of $\mathcal{V}_l$. The idea is now to prove that, under some suitable assumptions, the operator $\left(I - \frac{t}{n}\tilde{H}_l\right)^{-1}$ conserves the constraints. The estimate (130) tells us that for $|Re(\lambda)| > C_{0l}$, where $Re(\lambda)$ denotes the real part of λ, λ cannot be an eigenvalue of $\tilde{H}_l$ in $\mathbb{H}^0 \otimes \mathcal{W}^l_n$ and the resolvent

$$R(\lambda, \tilde{H}_l) = (\lambda - \tilde{H}_l)^{-1} \tag{160}$$

is a bounded operator on $\mathbb{H}^0 \otimes \mathcal{W}^l_n$. We wish to prove that for $\lambda \in \mathbb{R}$, $|\lambda| > K_l = Max\{1, C_{0l}\}$, we have the following implication

$$u \in \mathbb{H}^0_{cl} \otimes \mathcal{W}^l_n \quad \Rightarrow \quad (\lambda - \tilde{H}_l)^{-1}u \in \mathbb{H}^0_{cl} \otimes \mathcal{W}^l_n. \tag{161}$$

Let us consider $u \in \mathbb{H}^0_{cl} \otimes \mathcal{W}^l_n$. We put

$$w = (\lambda - \tilde{H}_l)^{-1}u. \tag{162}$$

With this definition, we have

$$w \in \mathbb{H}^0 \otimes \mathcal{W}^l_n, \quad u = (\lambda - \tilde{H}_l)w \in \mathbb{H}^0_{cl} \otimes \mathcal{W}^l_n.$$

A straightforward consequence is that both w and $\tilde{H}_lw$ belong to $\mathbb{H}^0 \otimes \mathcal{W}^l_n$ and therefore, w considered as only a function of r_* belongs to $(H^1(\mathbb{R}_{r_*}))^6$. In other words, we have

$$w \in \mathbb{H}^1 \otimes \mathcal{W}^l_n.$$

If now we put

$$v = \tilde{A}_lw \tag{163}$$

then obviously v belongs to $L^2(\mathbb{R}_{r_*})$ and using (138), the fact that $(\lambda - \tilde{H}_l)w$ satisfies the constraints implies

$$\lambda v + \left(\partial_{r_*} + \frac{F}{r}\right) v = 0, \tag{164}$$

i.e. v is an eigenfunction of $\partial_{r_*} + \frac{F}{r}$ in $L^2(\mathbb{R}_{r_*})$ associated with the eigenvalue $-\lambda$. It is very easy to see that if μ is a real eigenvalue for $\partial_{r_*} + \frac{F}{r}$ associated with an eigenfunction $f \in L^2(\mathbb{R}_{r_*})$ (the angular variables have no importance whatsoever here), then μ has to satisfy

$$|\mu| \leq \left\| \frac{F}{r} \right\|_{L^\infty} \leq 1.$$

Since by assumption $|\lambda| > 1$, v must be identically zero, which means

$$\tilde{A}_l w = 0$$

and in the same manner we obtain

$$\tilde{B}_l w = 0$$

which proves the implication (161). Using this result, we have immediately that for $0 < |t| < 1/K_l$ and for $n \in \mathbb{N}^*$ the operator

$$\left(I - \frac{t}{n} \tilde{H}_l \right)^{-1}$$

conserves the constraints and so does

$$\left(I - \frac{t}{n} \tilde{H}_l \right)^{-n}.$$

Since both operators $\tilde{A}_l$ and $\tilde{B}_l$ on $\mathbb{H}^0 \otimes \mathcal{W}_n^l$ are closed, the limit (159) implies that $\mathcal{V}_l(t)$ conserves the constraints for $0 < |t| < 1/K_l$ and consequently for all $t \neq 0$ using the group property. □

Proof of proposition 4.1: In all its generality, the Sparling 3-form (see [13]) plays a very important role in General Relativity. It is defined by

$$\beta = i \overline{W}_{aC'} W_{bC} dx^a \wedge dx^b \wedge dx^c \tag{165}$$

where

$$W_{aB} = \nabla_a \lambda_B, \tag{166}$$

λ_B being a solution of the Sen-Witten equation on a foliation of space-like hypersurfaces. The property

$$d\beta = 0 \tag{167}$$

is a necessary and sufficient condition for Einstein's vacuum equations to hold, together with the connection ∇_a being tortion-free. The 3-form that we are considering here

$$\beta = i \sigma_{aC'} \overline{\sigma}_{bC} dx^a \wedge dx^b \wedge dx^c$$

is a special case where $\overline{W}_{aB'}$ has been replaced with a spin 3/2 potential. It is convenient to express β as

$$\beta = i e^{abcd} \sigma_{bC'} \overline{\sigma}_{dC} X_a \ , \quad X_a = \frac{1}{6} e_{abcd} dx^b \wedge dx^c \wedge dx^d, \tag{168}$$

e_{abcd} being the Levi-Civita tensor, i.e.

$$e_{abcd} = e_{[abcd]} \ , \quad e_{0123} = 1. \tag{169}$$

e_{abcd} can be expressed in terms of ε spinors as

$$e_{abcd} = i \varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'D'} \varepsilon_{B'C'} - i \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'C'} \varepsilon_{B'D'} \tag{170}$$

and the equivalence between the 2 expressions of β is straightforward. Using (170), we have

$$\beta = - \left(\sigma_{BB'}{}^{B'} \overline{\sigma}^{A'BA} - \sigma_{BB'}{}^{A'} \overline{\sigma}^{B'AB} \right) X_a. \tag{171}$$

$\sigma^{C}_{A'B'}$ being symmetric in A', B', $\sigma_{BB'}{}^{B'}=0$ and

$$\beta=\sigma_{BB'}{}^{A'}\overline{\sigma}^{B'AB}X_a. \tag{172}$$

If we want to integrate β on the space-like hypersurface

$$S=]1,+\infty[_r\times S^2_\omega, \tag{173}$$

we choose

$$X_a=t_a\mathcal{S} \tag{174}$$

where t_a is the unit normal to S, i.e.

$$t^a=F^{-1/2}g_0{}^a,\quad t_a=F^{1/2}g_a{}^0 \tag{175}$$

and $\mathcal{S}$ is the 3-volume element on S, given by the square root of the determinant of the induced metric on S

$$\mathcal{S}=F^{-1/2}r^2\sin\theta drd\theta d\varphi. \tag{176}$$

Hence, we find that

$$\int_S\beta=\int_S\sigma_{BB'}{}^{A'}\overline{\sigma}^{B'AB}g_a{}^0r^2\sin\theta drd\theta d\varphi \tag{177}$$

and we just need to evaluate $\sigma_{BB'}{}^{A'}\overline{\sigma}^{B'AB}g_a{}^0$. Using concrete indices, we have

$$\sigma_{BB'}{}^{A'}\overline{\sigma}^{B'AB}g_a{}^0=\sigma_{BB'}{}^{A'}\overline{\sigma}^{B'AB}g_{AA'}{}^0=\sigma_{\mathbf{BB'}}{}^{\mathbf{A'}}\overline{\sigma}^{\mathbf{B'AB}}g_{\mathbf{AA'}}{}^0$$

and the components $g_{\mathbf{AA'}}{}^0$ of $g_a{}^0$ in the spin-frame are the components of the first Infeld-Van der Waerden symbol. After calculation and putting

$$\zeta^A_{B'C'}=rF^{1/4}\sigma^A_{B'C'}, \tag{178}$$

we find that the integral of β over S can be expressed as

$$\int_S\beta=\frac{-1}{\sqrt{2}}\left\{\left(\zeta^0_{0'0'},\zeta^0_{0'0'}\right)+\left(\zeta^1_{1'1'},\zeta^1_{1'1'}\right)+\left(\zeta^1_{0'1'},\zeta^1_{0'1'}\right)+\left(\zeta^0_{0'1'},\zeta^0_{0'1'}\right)\right.$$
$$\left.+2Re\left(\zeta^0_{1'1'},\zeta^1_{0'1'}\right)+2Re\left(\zeta^0_{0'1'},\zeta^1_{0'0'}\right)\right\} \tag{179}$$

where $(.,.)$ denotes the scalar product on $L^2\left(\mathbb{R}_{r_*}\times S^2_\omega;dr_*^2+d\omega^2\right)$. From the previous expression, we define the sesqui-linear form for $\xi,\eta\in\tilde{\mathcal{H}}$

$$\ll\xi,\eta\gg_\beta=\left(\xi^0_{0'0'},\eta^0_{0'0'}\right)+\left(\xi^1_{1'1'},\eta^1_{1'1'}\right)+\left(\xi^1_{0'1'},\eta^1_{0'1'}\right)+\left(\xi^0_{0'1'},\eta^0_{0'1'}\right)$$
$$+\left(\xi^0_{1'1'},\eta^1_{0'1'}\right)+\left(\xi^1_{0'1'},\eta^0_{1'1'}\right)+\left(\xi^1_{0'0'},\eta^0_{0'1'}\right)+\left(\xi^0_{0'1'},\eta^1_{0'0'}\right). \tag{180}$$

We want to see that if V is a solution of (86) in $\mathcal{C}\left(\mathbb{R}_t;\tilde{\mathcal{H}}_c\right)$, then $\ll V,V\gg_\beta$ is conserved throughout time. Let us first prove that for $\xi,\eta\in D(\tilde{H})_c$

$$\ll\tilde{H}\xi,\eta\gg_\beta=-\ll\xi,\tilde{H}\eta\gg_\beta. \tag{181}$$

Considering the particular case when

$$\xi,\eta\in\mathcal{C}^\infty_0\left(\mathbb{R}_{r_*}\right)\otimes\mathcal{W}^l_n,\quad(l,n)\in\mathcal{I}_{1/2},$$

and ξ, η satisfy the constraints, it is a tedious but straightforward calculation to check that (181) holds. Consequently, (181) has to be satisfied by any two ξ, η in $\mathcal{K}$. Indeed, if I_1 and I_2 are two finite subsets of $\mathcal{I}_{1/2}$ and

$$\xi = \sum_{(l,n)\in I_1} \xi^{ln} \otimes \mathcal{W}^l_n \in \mathcal{K}, \quad \eta = \sum_{(l,n)\in I_2} \eta^{ln} \otimes \mathcal{W}^l_n \in \mathcal{K},$$

then

$$\ll \tilde{H}\xi, \eta \gg_\beta = \ll \sum_{(l,n)\in I_1} \tilde{H}_l \xi^{ln} \otimes \mathcal{W}^l_n, \sum_{(l,n)\in I_2} \eta^{ln} \otimes \mathcal{W}^l_n \gg_\beta .$$

By orthogonality of spin-weighted spherical harmonics, we can take the sum out of the scalar product and we obtain

$$\ll \tilde{H}\xi, \eta \gg_\beta = \sum_{(l,n)\in I_1\cup I_2} \ll \tilde{H}(\xi^{ln} \otimes \mathcal{W}^l_n), \eta^{ln} \otimes \mathcal{W}^l_n \gg_\beta$$

with the convention

$$\xi^{ln} = 0 \; if \; (l,n) \in I_2 - I_1, \quad \eta^{ln} = 0 \; if \; (l,n) \in I_1 - I_2.$$

We can now use (181) on each angular dependence

$$\ll \tilde{H}\xi, \eta \gg_\beta = - \sum_{(l,n)\in I_1\cup I_2} \ll \xi^{ln} \otimes \mathcal{W}^l_n, \tilde{H}(\eta^{ln} \otimes \mathcal{W}^l_n) \gg_\beta$$

$$= - \ll \sum_{(l,n)\in I_1} \xi^{ln} \otimes \mathcal{W}^l_n, \tilde{H} \sum_{(l,n)\in I_2} \eta^{ln} \otimes \mathcal{W}^l_n \gg_\beta = - \ll \xi, \tilde{H}\eta \gg_\beta .$$

Hence, (181) holds on $\mathcal{K}$ and by density of $\mathcal{K}$ in $D(\tilde{H})_c$, (181) is satisfied by any two $\xi, \eta \in D(\tilde{H})_c$. Note that this is equivalent to (77). As a simple consequence, we can see that for $V_0 \in D(\tilde{H})_c$,

$$\ll \mathcal{V}(t)V_0, \mathcal{V}(t)V_0 \gg_\beta = \ll V_0, V_0 \gg_\beta, \quad \forall t \in \mathbb{R}. \tag{182}$$

Indeed, if we recall that

$$\mathcal{V}(t)V_0 \in \mathcal{C}\left(\mathbb{R}_t; D(\tilde{H})_c\right) \bigcap \mathcal{C}^1\left(\mathbb{R}_t; \tilde{\mathcal{H}}_c\right),$$

then we have

$$\ll \mathcal{V}(t)V_0, \mathcal{V}(t)V_0 \gg_\beta \in \mathcal{C}^1\left(\mathbb{R}_t\right),$$

whence we can write

$$\frac{d}{dt} \ll \mathcal{V}(t)V_0, \mathcal{V}(t)V_0 \gg_\beta$$

$$= \ll \tilde{H}\mathcal{V}(t)V_0, \mathcal{V}(t)V_0 \gg_\beta + \ll \mathcal{V}(t)V_0, \tilde{H}\mathcal{V}(t)V_0 \gg_\beta = 0$$

using (181). And by continuity of the solutions with respect to their initial data, (182) is satisfied for any $V_0 \in \tilde{\mathcal{H}}$, which is equivalent to saying that if U is a solution of (63) in $\mathcal{C}\left(\mathbb{R}_t; \mathcal{H}_c\right)$, $< U, U >_\beta$ is conserved throughout time, where $< .,. >_\beta$ is defined by (76). This completes the proof of proposition 4.1. $\square$

5 Conclusion

Thus, we have established that the Cauchy problem for the Dirac equation for a spin 3/2 massless first potential is well-posed in a natural class of functional spaces. As for the gauge quantities, they are simply solutions of the Weyl neutrino equation, which in the zero rest-mass case can be identified with the Dirac equation. Consequently, we know (see [9]) that the Cauchy problem for these quantities is also well-posed in a similar family of spaces. Therefore, at least in the Schwarzschild space-time, there doesn't seem to be anything pathological about the propagation of the spin 3/2 potential modulo gauge. This is of course only a beginning of answer to the twistorial issue we have mentionned in the introduction. But one would actually expect the situation to be similar in all Ricci-flat space-times. More precisely, we have said in the introduction that the flat-space time topological construction which allows one to define a twistor as a charge for a spin 3/2 field cannot be carried out successfully in Ricci-flat space-times. The reason for this does not seem to be of an analytic nature, i.e. some pathological behavior of the propagator of the potential modulo gauge in the cone of dependence of a topologically trivial space-like compact hypersurface. Therefore, it is more probably of a topological nature, a proper covering of S^2 (as described in [10]) might turn out not to exist at each time. However, proving it should be quite difficult. It requires to study the propagator for the spin 3/2 potential and for the Weyl field in a general Ricci-flat space-time, which really means solving the Cauchy problem for both equations in a space of minimum regularity solutions. Then, it would be necessary to work out the exact nature of the topological obstruction.

Along more usual analytic lines, there remains quite a lot to be done about spin 3/2 fields in Ricci-flat space-times. Having solved the Cauchy problem, it would seem natural to develop a time dependent scattering theory for these fields. The main problem is of course the absence of a natural self-adjointness framework. It seems however possible to overcome this difficulty by means of a gauge transformation, probably at the level of the Rarita-Schwinger equations. The Dirac formulation of the spin 3/2 equations already corresponds to a gauge choice in the Rarita-Schwinger system. A different gauge fixing could lead to a formulation where the conserved quantity is positive definite, which corresponds to a naturally self-adjoint hamiltonian. This possibility is currently under investigation in a joint work with L.J. Mason. It could also have interesting applications to the theory of integrable systems.

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